Short note on the Biot soil consolidation

Jos van Kan

March 1, 2005

Abstract

We introduce briefly a computational scheme for the time dependent Biot soil consolidation model. The scheme is equivalent to solving 4 laplacian equations per time step. We argue that the schem is $O(\Delta t^2)$ in the pore pressure p and the *dilatation* div $\mathbf{u}$ in which $\mathbf{u}$ is the soil displacement vector.

1 Introduction

The Biot soil consolidation model is given by

$$-\mu\tilde{\Delta}\mathbf{u} - (\lambda+\mu)\text{grad div }\mathbf{u} + \text{grad }p = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(\text{div }\mathbf{u}) - \frac{\kappa}{\eta}\Delta p = f(\mathbf{x},t) \tag{2}$$

with boundary conditions $p = 0$ on Γ and $\mathbf{u} = 0$ on Γ. Initial conditions are given for the dilatation div $\mathbf{u}$. To facilitate the calculations we introduce a special symbol φ for the dilatation div $\mathbf{u}$ which transforms the system to:

$$\frac{\partial\varphi}{\partial t} - \frac{\kappa}{\eta}\Delta p = f(\mathbf{x},t) \tag{3}$$

$$-\mu\tilde{\Delta}\mathbf{u} - (\lambda+\mu)\text{grad }\varphi + \text{grad }p = 0 \tag{4}$$

$$\text{div }\mathbf{u} = \varphi \tag{5}$$

This also introduces an extra boundary condition

$$\varphi = \frac{\partial u_n}{\partial n} \quad \text{on } \Gamma. \tag{6}$$

2 Numerical model

We assume that we have a natural staggered space discretization. (φ- and p-points must coincide). We also assume that we have the quantities p, $\mathbf{u}$ and φ on time level n with sufficient accuracy and the straightforward way to calculate an update δp, $\delta\mathbf{u}$ and $\delta\varphi$ would be the solution of the following system of simultaneous linear equations:

$$\delta\varphi - \frac{1}{2}\Delta t\frac{\kappa}{\eta}\Delta\ \delta p = \Delta t[\frac{\kappa}{\eta}\Delta\ p^n + f^{n+1/2}] \tag{7}$$

$$-\mu\tilde{\Delta}\ \delta\mathbf{u} - (\lambda+\mu)\mathrm{grad}\ \delta\varphi + \mathrm{grad}\ \delta p = 0 \tag{8}$$

$$\delta\varphi - \mathrm{div}\ \delta\mathbf{u} = 0 \tag{9}$$

The accuracy of such an update would be $O(\Delta t^3)$ at best (that is the error of the mid point integration) and for that to happen the error in p^n should be no worse than $O(\Delta t^2)$. Fortunately these two characteristics are compatible.

This simultaneous system is a bit unwieldy and we therefore try a simplification that won't lose us the overall accuracy. First of all we note, by taking the divergence of the second equation that

$$-(\lambda+2\mu)\Delta\delta\varphi + \Delta\delta p = 0. \tag{10}$$

Therefore we may write the *first* equation of the system as:

$$\delta\varphi - \frac{1}{2}\Delta t\frac{\kappa(\lambda+2\mu)}{\eta}\Delta\ \delta\varphi = \Delta t[\frac{\kappa}{\eta}\Delta\ p^n + f^{n+1/2}] \tag{11}$$

with boundary condition

$$\delta\varphi = \frac{\partial\delta\mathbf{u}_n}{\partial n} \quad \text{on } \Gamma. \tag{12}$$

Unfortunately, we do not know $\delta\mathbf{u}_n$ yet, but we can be sure, that is is of order $O(\Delta t)$. So if we take $\delta\varphi^* = 0$ on Γ instead, we have an error of $O(\Delta t)$ in the boundary conditions, but only an error $O(\Delta t^2)$ in the solution $\delta\varphi^*$ of system (11) on the interior points, since those boundary condtions are multiplied by Δt in the equations.

From equation (10) we now could calulate δp^* with an accuracy of $O(\Delta t^2)$, provided we had boundary values to $\delta\varphi^*$ with that accuracy. We know however, $\delta\varphi^*$ in the interior with an accuracy of $O(\Delta t^2)$, hence we can extrapolate to the boundary without loss of accuracy. We plug this into equation (8) to obtain $\delta\mathbf{u}^*$ with $O(\Delta t^2)$ accuracy. Note that this system is uncoupled in the $\mathbf{u}$ components. This yields us $O(\Delta t^2)$ accurate boundary values for equation (11), and hence an $O(\Delta t^3)$ accurate solution $\delta\varphi$. From this we extrapolate the boundary contions for $\delta\varphi$ once more to obtain $O(\Delta t^3)$ accurate boundary conditions and an $O(\Delta t^3)$ accurate solution δp to (10).

There may be a stability problem.