Divided differences as set function

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1 Introduction

The usual way to define divided differences is by recursion. Given pairs (f_0, x_0), (f_1, x_1),..., (f_n, x_n), such that $x_k \neq x_l$ if $k \neq l$ one constructs

$$D(x_0, x_1) = \frac{f_1 - f_0}{x_1 - x_0} \tag{1}$$

$$D(x_0, x_1, x_2) = \frac{D(x_0, x_1) - D(x_1, x_2)}{x_0 - x_2} \tag{2}$$

$$D(x_0, x_1, \ldots, x_n) = \frac{D(x_0, x_1, \ldots, x_{n-1}) - D(x_1, x_2, \ldots, x_n)}{x_0 - x_n} \tag{3}$$

If you do it this way, it is not so clear, that divided differences are really *set functions*, that is, the order in which the x_k appear is immaterial. Of course this is a theorem, and it can be (and has been) proved, but a more satisfactory way is to *define* the divided differences as set functions.

2 Definition

We define the divided differences on an arbitrary subset $\Sigma \subseteq \Omega = \{x_0, x_1, \ldots, x_n\}$ recursively:

$$D(\{x_q\}) = f_q \quad \forall x_q \in \Omega \tag{4}$$

$$D(\Sigma) = \frac{D(\Sigma \setminus x_p) - D(\Sigma \setminus x_q)}{x_q - x_p} \tag{5}$$

with $|\Sigma| > 1$, $x_p, x_q \in \Sigma$ and $x_p \neq x_q$. What remains to be shown is, that this definition does not depend on the particular choice of x_p and x_q. Clearly this is true if $|\Sigma| = 2$. We proceed with induction on the cardinality of Σ.

Theorem 1 *Let $D(\Sigma)$ be a set function for all sets $\Sigma \subseteq \Omega$ with $|\Sigma| < k$. Let $3 \leq |\Sigma| \leq k$. Let x_p, x_q and x_r be three different elements belonging to Σ. Then*

$$(x_p - x_q)D(\Sigma \setminus x_r) + (x_q - x_r)D(\Sigma \setminus x_p) + (x_r - x_p)D(\Sigma \setminus x_q) = 0 \tag{6}$$

Proof. Since $|\Sigma| \leq k$ it follows that $|\Sigma \setminus x| \leq k-1, \forall x \in \Sigma$ hence $D(\Sigma \setminus x)$ is a properly defined set function by 5. Hence the following equalities hold:

$$D(\Sigma \setminus x_r) = \frac{D(\Sigma \setminus x_r \setminus x_q) - D(\Sigma \setminus x_r \setminus x_p)}{x_p - x_q} \tag{7}$$

$$D(\Sigma \setminus x_p) = \frac{D(\Sigma \setminus x_p \setminus x_r) - D(\Sigma \setminus x_p \setminus x_q)}{x_q - x_r} \tag{8}$$

$$D(\Sigma \setminus x_q) = \frac{D(\Sigma \setminus x_q \setminus x_p) - D(\Sigma \setminus x_q \setminus x_r)}{x_r - x_p} \tag{9}$$

Substitution of these relations into the left hand side of relation 6 shows that this relation in fact is an identity. □

Theorem 2 *Let $D(\Sigma)$ be a set function for all sets $\Sigma \subseteq \Omega$ with $|\Sigma| < k$. Let $|\Sigma| = k \geq 3$. Then definition 5 does not depend on the particular choice of x_p and x_q.*

Proof. By theorem 1 we have

$$\frac{D(\Sigma \setminus x_p) - D(\Sigma \setminus x_q)}{x_q - x_p} = \frac{D(\Sigma \setminus x_p) - D(\Sigma \setminus x_r)}{x_r - x_p} \tag{10}$$

and also

$$\frac{D(\Sigma \setminus x_s) - D(\Sigma \setminus x_r)}{x_r - x_s} = \frac{D(\Sigma \setminus x_p) - D(\Sigma \setminus x_r)}{x_r - x_p} \tag{11}$$

as one may verify by multiplying out. Hence

$$\frac{D(\Sigma \setminus x_p) - D(\Sigma \setminus x_q)}{x_q - x_p} = \frac{D(\Sigma \setminus x_r) - D(\Sigma \setminus x_s)}{x_s - x_r} \tag{12}$$

and the theorem is established. □

Corollary Definitions 4 and 5 define a proper set function. Because if definition 5 properly defines a set function on sets with cardinality at most $k-1$ it also does so for sets of cardinality k.