Tutorial ANM: Week 5

Friday, October 6th, 2016

Finite Difference Solver of a Poisson Equation in One Dimension

The Poisson equation is a model equation used in engineering application to model diffusion processes. Examples include the diffusion of heat in a chemical reactor or the diffusion of chemical substances in a substrate. The numerical solution of this equation has a history that is as old as the development of computers.

The objective of this assignment is to guide the student to the development of a finite difference method (FDM) solver of a Poisson Equation in one dimension from scratch. The assignment consists of both pen-and-paper and implementation exercises.

We strongly encourage the students to maintain a set of notes (pen and paper or digital) on this assignment. Such notes will serve as an aid in preparing the homework exercises and the final exam.

We aim at solving the Poisson equation on the open interval $\Omega = (0, 1)$. Our objective is to numerically approximate the function $u(x)$ that is the solution of the following problem:
given the source function $f(x)$ and the real numbers α and β, find the function $u(x)$ such that $u(x)$ is the solution of the differential equation

$$-\frac{d^2\,u}{dx^2} = f(x) \tag{1}$$

(beware of the minus sign) supplied with the following homogeneous Dirichlet boundary condition in $x = 0$ and in $x = 1$

$$u(x = 0) = \alpha \quad \text{and} \quad u(x = 1) = \beta. \tag{2}$$

Compulsory Analytical Part

Assignment 1 Choose $f(x) = x$, $\alpha = 0$ and $\beta = 2$ and show that the function $u(x) = x^3 + x$ is the exact solution of the problem. Do so by verifying using pen and paper that $u(x)$ satisfies the differential equation as well as both boundary condition. There is no need to use symbolic differentiation, nor to use integration to construct $u(x)$. We will use the short hand notation $u'(x) = \frac{d\,u}{dx}$ and $u''(x) = \frac{d^2\,u}{dx^2}$.

Assignment 2 Plot the function $u(x)$ as a function of x for $0 \leq x \leq 1$ by avoiding for-loops in Matlab.

Compulsory Numerical Part

Assume that the interval Ω is discretized by an uniform mesh consisting of N elements with mesh width $h = 1/N$ and vertices $x_i = (i - 1)\,h$, where i runs from 1 to $N + 1$. This enumeration includes the end points of Ω, that is, $x_1 = 0$ and $x_{N+1} = 1$. The grid nodes can then be denotes by

$$G_h = \left\{ x_i \,|\, x_i = (i-1)\,h; h = \frac{1}{N}, 1 \leq i \leq N+1 \right\} . \tag{3}$$

Assignment 3 (Discretization in the interior nodes) The differential equation holds in particular for all of the internal nodes on Ω, that is, we have that

$$-\frac{d^2 u}{dx^2}\Big|_{x=x_i} = f(x_i) \text{ for all } 2 \leq i \leq N\,. \tag{4}$$

Use a finite difference formula twice to show that the second derivative $\frac{d^2u}{dx^2}\big|_{x=x_i} = u''(x_i)$ can be discretized as follows

$$\begin{aligned} u''(x_i) &\approx \frac{u'(x_i + h/2) - u'(x_i - h/2)}{h} \\ &= \frac{u(x_{i+1}) - 2\,u(x_i) + u(x_{i-1})}{h^2}\,. \end{aligned} \tag{5}$$

(Using Taylor polynomials it can be shown that the local truncation error is of second order in h). The finite difference discretization thus leads to the following **stencil** for the approximation of $-\frac{d^2u}{dx^2}\Big|_{x=x_i}$ (beware of the minus-sign)

$$\frac{1}{h^2}\left[\begin{array}{ccc} -1 & 2 & -1 \\ {\scriptstyle x_{i-1}} & {\scriptstyle x_i} & {\scriptstyle x_{i+1}} \end{array}\right] \tag{6}$$

This stencil implies that each node x_i is coupled to its left (x_{i-1}) and right neighbour (x_{i+1}) with a weight of $-\frac{1}{h^2}$.

Assignment 4 (Discretization in the left end point) Verify that the Dirichlet boundary condition in the left end point can be enforced by requiring that

$$u_1 = \alpha\,. \tag{7}$$

The finite difference stencil in the left end point thus reduces to

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ {\scriptstyle x_1} & {\scriptstyle x_2} & {\scriptstyle x_3} \end{array}\right]. \tag{8}$$

Assignment 5 (Discretization in the right end point) Verify that the Dirichlet boundary condition in the right end point can be enforced by requiring that

$$u_{N+1} = \beta\,. \tag{9}$$

The finite difference stencil in the left end point thus reduces to

$$\left[\begin{array}{ccc} 0 & 0 & 1 \\ {\scriptstyle x_{N-1}} & {\scriptstyle x_N} & {\scriptstyle x_{N+1}} \end{array}\right]. \tag{10}$$

Assignment 6 Assume $h = 1/3$ (and thus N=3). Determine the size of the global matrix A^h and the global right-hand vector $\mathbf{f}$. Give all the elements of this matrix and vector with pen (or pencil) on paper.

Assignment 7 Assume $h = 1/4, 1/8, 1/16, \ldots$ and assemble for all these values the global matrix A^h and the global right-hand vector $\mathbf{f}^h$. Try to form the matrix A^h without using for-loops using the Matlab `diag` command. Solve the linear system

$$A\,\mathbf{u}^h = \mathbf{f}^h \tag{11}$$

using the Matlab `backslash (\)` command. Plot the various solution for $\mathbf{u}^h$ found and compare this plot with the plot of the exact solution in the first assignment.

Assignment 8 Assume that $h = 1/16$, recompute and plot the numerical solution for various values of α and β. Explain the plots you obtain.

Congratulations for developing your very first finite difference solver!