Computer Assignment EE4550: Block 1

Finite Difference Solver of a Poisson Equation in One Dimension

The objective of this assignment is to guide the student to the development of a finite difference method (FDM) solver of a Poisson Equation in one dimension from scratch. This assignment consists of both pen-and-paper and implementation exercises.

We request the students to prepare a report on these assignments. We appreciate receiving a clearly structured report with an introduction, body and conclusions.

1 Finite Difference Method in 1D

In the first part of this assignment we aim at solving the Poisson equation on the open interval $\Omega = (0, 1)$. Our objective is to numerically approximate the function $u(x)$ that is the solution of the following problem: given the source function $f(x)$ and the number α,find the function $u(x)$ such that $u(x)$ is the solution of the differential equation

$$-\frac{d^2\, u}{dx^2} = f(x) \tag{1}$$

supplied with the following homogeneous Dirichlet boundary condition in $x = 0$ and the non-homogeneous Neumann boundary conditions in $x = 1$.

$$u(x = 0) = 0 \quad \text{and} \quad \frac{du}{dx}(x = 1) = \alpha. \tag{2}$$

Compulsory Analytical Part

Assignment 1 Choose $f(x) = x$ and $\alpha = 0.5$ and show that the function $u(x) = -\frac{1}{6}x^3 + x$ is the exact solution of the problem. Do so by verifying using pen and paper that $u(x)$ satisfies the differential equation as well as both boundary condition. There is no need to use symbolic differentiation, not to use integration to construct $u(x)$. We will use the short hand notation $u'(x) = \frac{d\,u}{dx}$ and $u''(x) = \frac{d^2\,u}{dx^2}$.

Assignment 2 Plot the function $u(x)$ as a function of x for $0 \leq x \leq 1$ by avoiding for-loops in Matlab.

Compulsory Numerical Part

Assume that the interval Ω is discretized by an uniform mesh consisting of N elements with mesh width $h = 1/N$ and vertices $x_i = (i - 1)\, h$, where i runs from 1 to $N + 1$. This enumeration includes the end points of Ω, that is, $x_1 = 0$ and $x_{N+1} = 1$. The grid nodes can then be denotes by

$$G_h = \left\{ x_i \,|\, x_i = (i-1)\, h; h = \frac{1}{N}, 0 \leq i \leq N+1 \right\} . \tag{3}$$

Assignment 3 (Discretization in the interior nodes) The differential equation holds in particular for all of the internal nodes on Ω, that is, we have that

$$-\frac{d^2 u}{dx^2}\Big|_{x=x_i} = f(x_i) \text{ for all } 2 \leq x_i \leq N\,. \tag{4}$$

Use a finite difference formula twice to show that the second derivative $\frac{d^2 u}{dx^2}\big|_{x=x_i} = u''(x_i)$ can be discretized as follows

$$\begin{aligned} u''(x_i) &\approx \frac{u'(x_i + h/2) - u'(x_i - h/2)}{h} \\ &= \frac{u(x_{i+1}) - 2\,u(x_i) + u(x_{i-1})}{h^2}\,. \end{aligned} \tag{5}$$

(Using Taylor polynomials is can be shown that the local truncation error is of second order in h). The finite difference discretization thus leads to the following **stencil** for the approximation of $-\frac{d^2u}{dx^2}\big|_{x=x_i}$ (beware of the minus-sign)

$$\frac{1}{h^2}\left[\begin{array}{ccc} \underset{x_{i-1}}{-1} & \underset{x_i}{2} & \underset{x_{i+1}}{-1} \end{array}\right] \tag{6}$$

This stencil implies that each node x_i is coupled to its left (x_{i-1}) and right neighbour (x_{i+1}) with a weight of $-\frac{1}{h^2}$.

Assignment 4 **(Discretization in the left end point)** Verify that the Dirichlet boundary condition in the left end point can be enforced by requiring that

$$u_1 = 0\,. \tag{7}$$

The finite difference stencil in the left end point thus reduces to

$$\left[\begin{array}{ccc} \underset{x_1}{1} & \underset{x_2}{0} & \underset{x_3}{0} \end{array}\right]. \tag{8}$$

Assignment 5 **(Discretization in the right end point)** Verify that the Neumann boundary condition in the right end point can be enforced by requiring that

$$\frac{du}{dx}(x = 1) = \frac{u_{N+1} - u_N}{h} = \alpha\,. \tag{9}$$

The finite difference stencil in the right end point thus reduces to

$$\frac{1}{h}\left[\begin{array}{ccc} \underset{x_{N-1}}{0} & \underset{x_N}{-1} & \underset{x_{N+1}}{1} \end{array}\right]. \tag{10}$$

Assignment 6 Assume $h = 1/3$ (and thus N=3). Determine the size of the global matrix A^h and the global right-hand vector $\mathbf{f}$. Give all the elements of this matrix and vector with pen (or pencil) on paper.

Assignment 7 Assume $h = 1/4, 1/8, 1/16, \ldots$ and assemble for all these values the global matrix A^h and the global right-hand vector $\mathbf{f}^h$. Solve the linear system

$$A\,\mathbf{u}^h = \mathbf{f}^h \tag{11}$$

using the Matlab `backslash (\)` command. Plot the various solution for $\mathbf{u}^h$ found and compare this plot with the plot of the exact solution in the first assignment.

Elective Assignments

Assignment 8 Redo **Assignment 7** for other choices for $f(x)$ and/or α.

Assignment 9 Verify that the numerical scheme is indeed second order accurate by investigating how the max-norm of the discretization error

$$E = \|u(x) - u^h(x)\|_\infty = \max_{1\le i\le N+1} |u(x_i) - u^h(x_i)| \tag{12}$$

scales with the meshwidth h as expected.

Assignment 10 Assemble the matrix A^h and the vector $\mathbf{f}^h$ avoiding for-loops in Matlab.

Assignment 11 Extend your implementation is such a way to be able to treat a variable diffusion coefficient $c(x)$, i.e, the differential equation

$$-\frac{d}{dx}\left[c(x)\frac{du}{dx}\right] = f(x)\,. \tag{13}$$