# Introduction into Finite Elements

## FEM for Convection-Diffusion Problems

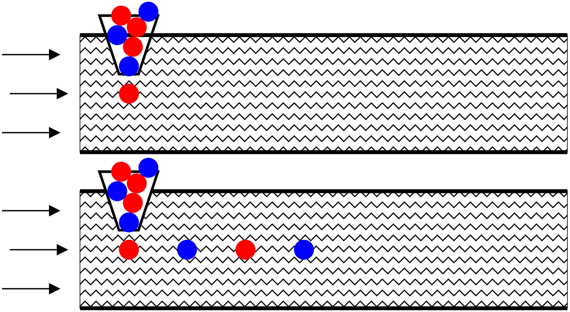
Matthias Möller, DIAM TU Delft
November 21, 2013

TUDelft

# Transport phenomena: Convection

**Convection** (alias **advection**) is the transport of a conserved quantity of interest by a vector field, e.g., the velocity field.

Injection of tracer particles in a moving fluid.

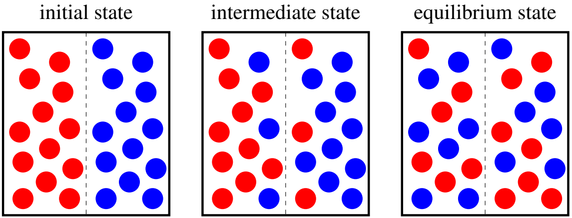


TU Delft

# Transport phenomena: Diffusion

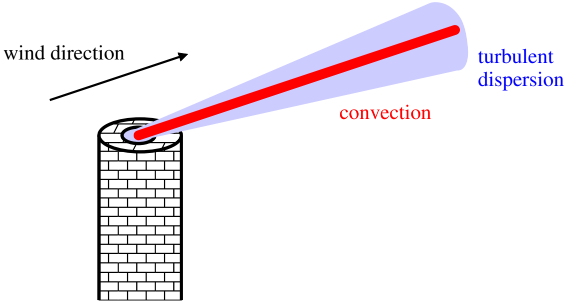
**Diffusion** is the transport of a conserved quantity from a region of high concentration to a region of low concentration, e.g., due to Brownian random molecular motion or heat conduction.

Transport of particles due to random molecular motion.



# Examples of transport phenomena

- Flow processes in our body (blood flow, drug delivery)
- Heating and air conditioning in rooms, cars, aircrafts
- Transport of pollutants in air (with turbulent effects)



# Governing equations

Transient convection-diffusion equation in *conservative form*

$$\partial_t u + \partial_x(vu) - \partial_x(d\partial_x u) = f$$

with

- transient term $\partial_t u = \partial_t u(x,t)$
- velocity field $v = v(x,t)$
- diffusion coefficient $d = d(x) \geq 0$
- load vector $f = f(x,t)$

Simplification for constant uniform diffusion coefficient $d$

$$\partial_t u + \partial_x(vu) - d\partial_{xx} u = f$$

# Governing equations, cont'd

Application of the chain rule to the convective term yields

$$\partial_x(vu) = v(\partial_x u) + (\partial_x v)u$$

In case of a so-called divergence-free velocity field

$$\mathrm{div}\mathbf{v} = \partial_x v^x + \partial_y v^y + \cdots = 0$$
$$\partial_x v = 0 \quad \Leftrightarrow \quad v = \mathrm{const} \quad \text{in 1D}$$

this leads to the *non-conservative form*

$$\partial_t u + v\partial_x u - d\partial_{xx} u = f$$

TUDelft

# Model problems

Time-dependent convection-diffusion problem

$$\partial_t u + \partial_x(vu) - \partial_x(d\partial_x u) = f \qquad \text{in} \quad [a, b]$$

is complemented by initial conditions at time $t = 0$

$$u = u_0 \quad \text{in} \quad [a, b]$$

and boundary conditions at $a = x$ and $x = b$:

- Dirichlet bc's: $u = u_D$
- Neumann bc's: $u' = g_N$
- Flux bc's: $(vu - d\partial_x u)' = g_F$

# Model problems, cont'd

Time-dependent convection problem (hyperbolic)

$$\partial_t u + \partial_x(vu) = f \qquad \text{in} \quad [a, b]$$

is complemented by initial conditions at time $t = 0$

$$u = u_0 \quad \text{in} \quad [a, b]$$

and boundary conditions at $x = a$ and/or $x = b$ if and only if the (normal) flow velocity is directed into the domain

**Example:** If $v \equiv \text{const} > 0$ ($\hat{=}$ translation of $u_0$ to the right) then $u(x = a) = u_D$ is prescribed at the inflow boundary part at $x = a$ but no boundary condition is imposed at the outflow part at $x = b$.

# *Steady* convection-diffusion problem

**Boundary value problem:** Given $v$ and $d > 0$ find $u$

$$\text{s.t.} \quad \begin{cases} \partial_x(vu) - \partial_x(d\partial_x u) & = & f & \text{in } [a,b] \\ u & = & u_a & \text{at } x = a \\ u & = & u_b & \text{at } x = b \end{cases}$$

**Weak form:** Find $u \in S = \{u \in H^1 : u(a) = u_a \wedge u(b) = u_b\}$

$$\text{s.t.} \quad \int_a^b w\left[\partial_x(vu) - \partial_x(d\partial_x u)\right] \mathrm{d}x = \int_a^b wf \, \mathrm{d}x$$

$$\overset{\text{i.b.p}}{\Leftrightarrow} \int_a^b w\partial_x(vu) + \partial_x w(d\partial_x u) \, \mathrm{d}x - \underbrace{w(d\partial_x u)\big|_a^b}_{=0} = \int_a^b wf \, \mathrm{d}x$$

for all $w \in W = \{u \in H^1 : w(a) = 0 \wedge w(b) = 0\}$

# *Steady* convection-diffusion problem, cont'd

**Boundary value problem:** Given $v$ and $d > 0$ find $u$

$$\text{s.t.} \quad \begin{cases} \partial_x(vu) - \partial_x(d\partial_x u) = f & \text{in } [a,b] \\ u = u_a & \text{at } x = a \\ u' = g_b & \text{at } x = b \end{cases}$$

**Weak form:** Find $u \in S = \{u \in H^1 : u(a) = u_a\}$

$$\text{s.t.} \quad \int_a^b w\left[\partial_x(vu) - \partial_x(d\partial_x u)\right] \mathrm{d}x = \int_a^b wf \,\mathrm{d}x$$

$$\overset{\text{i.b.p.}}{\Leftrightarrow} \int_a^b w\partial_x(vu) + \partial_x w(d\partial_x u)\,\mathrm{d}x - \underbrace{w(b)}_{\neq 0}(dg_b) = \int_a^b wf\,\mathrm{d}x$$

for all $w \in W = \{u \in H^1 : w(a) = 0\}$

# *Steady* convection-diffusion problem, cont'd

**Boundary value problem:** Given $v$ and $d > 0$ find $u$

$$\text{s.t.} \quad \begin{cases} \partial_x(vu) - \partial_x(d\partial_x u) = f & \text{in } [a, b] \\ u = u_a & \text{at } x = a \\ (vu - d\partial_x u)' = g_b & \text{at } x = b \end{cases}$$

**Weak form:** Find $u \in S = \{u \in H^1 : u(a) = u_a\}$

$$\text{s.t.} \quad \int_a^b w\,[\partial_x(vu) - \partial_x(d\partial_x u)]\,\mathrm{d}x = \int_a^b wf\,\mathrm{d}x$$

$$\overset{\text{i.b.p.}}{\Leftrightarrow} \int_a^b -\partial_x w\,[vu - d\partial_x u]\,\mathrm{d}x + \underbrace{w(b)}_{\neq 0}\,g_b = \int_a^b wf\,\mathrm{d}x$$

for all $w \in W = \{u \in H^1 : w(a) = 0\}$

TU Delft

# Galerkin finite element method

Generic weak form for the problem at hand

$$\text{Find } u \in S: \quad a(u,w) = b(w) \quad \text{for all } w \in W$$

with non-symmetric bilinear form (i.e. $a(u,w) \neq a(w,u)$)

$$a(u,w) = \int_a^b w\partial_x(vu) + \partial_x w(d\partial_x u)\mathrm{d}x$$

$$\text{or} \quad a(u,w) = \int_a^b -\partial_x w(vu - d\partial_x u)\mathrm{d}x$$

and linear form with or without boundary contributions

$$b(w) = \int_a^b wf + w(b)(dg_b)\,\mathrm{d}x \quad \text{or} \quad b(w) = \int_a^b wf\,\mathrm{d}x - w(b)g_b$$

# Galerkin finite element method, cont'd

Approximate trial and test spaces by finite approximations

$$u_h = \sum_{j=1}^{N} \varphi_j u_j \in S_h = \text{span}\langle \varphi_1, \ldots, \varphi_N \rangle \subset S$$

$$w_h = \sum_{i=1}^{N} \phi_i w_i \in W_h = \text{span}\langle \phi_1, \ldots, \phi_N \rangle \subset W$$

and solve the discrete problem

$$\text{Find } u_h \in S_h : \quad a(u_h, w_h) = b(w_h) \quad \text{for all } w_h \in W_h$$

# Galerkin finite element method, cont'd

Assemble the system matrix $A$ and the right-hand side vector $b$

$$A = \begin{pmatrix} a(\varphi_1,\phi_1) & \dots & a(\varphi_N,\phi_1) \\ \vdots & \ddots & \vdots \\ a(\varphi_1,\phi_N) & \dots & a(\varphi_N,\phi_N) \end{pmatrix} \qquad b = \begin{pmatrix} b(\phi_1) \\ \vdots \\ b(\phi_N) \end{pmatrix}$$

and impose Dirichlet boundary conditions, e.g. $u(x = a) = u_a$

$$A = \begin{pmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ a(\varphi_1,\phi_N) & \dots & a(\varphi_N,\phi_N) \end{pmatrix} \qquad b = \begin{pmatrix} u_a \\ \vdots \\ b(\phi_N) \end{pmatrix}$$

Solve the linear system $Au = b$ for the vector of unknowns

TUDelft

# Numerical example

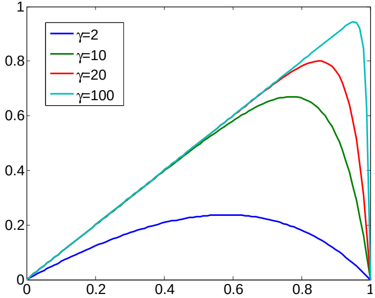
**Boundary value problem:** Given $v$ and $d > 0$ find $u$

$$\text{s.t.} \quad \begin{cases} v\partial_x u - d\partial_{xx} u &= 1 \quad \text{in } [0,1] \\ u &= 0 \quad \text{at } x = 0 \text{ and } x = 1 \end{cases}$$

with known exact solution

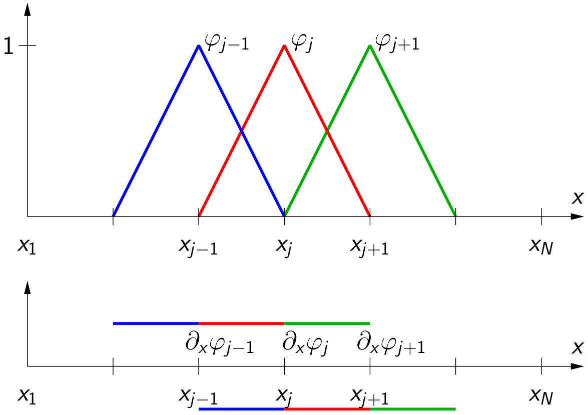
$$u_{\text{ex}}(x) = \frac{1}{v}\left(x - \frac{1 - e^{\gamma x}}{1 - e^{\gamma}}\right)$$

where $\gamma = \frac{v}{d}$. If $\gamma \gg 1$ the problem is termed convection-dominated. Numerical methods have problems in resolving the boundary layer at $x = b$.



# Numerical example, cont'd

Let $\varphi_j = \phi_j$, $j = 1, \ldots, N$ and choose linear finite elements




TUDelft

# Numerical example, cont'd

Resulting system matrix and right-hand side vector

$$A = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & 0 \\ -\frac{v}{2}-\frac{d}{h} & \frac{2d}{h} & \frac{v}{2}-\frac{d}{h} & & & & \\ & -\frac{v}{2}-\frac{d}{h} & \frac{2d}{h} & \frac{v}{2}-\frac{d}{h} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -\frac{v}{2}-\frac{d}{h} & \frac{2d}{h} & \frac{v}{2}-\frac{d}{h} & \\ 0 & \dots & \dots & \dots & \dots & 0 & 1 \end{pmatrix}$$

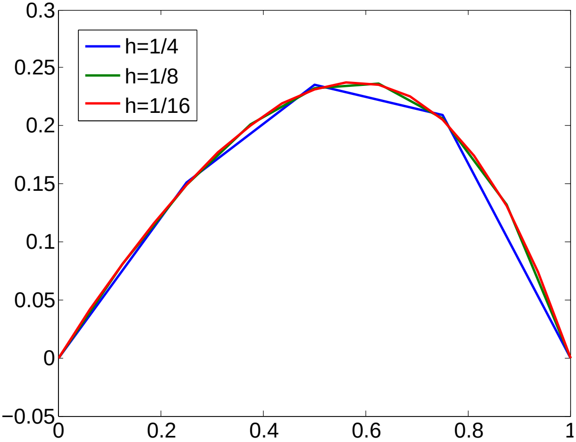
$$b = \begin{pmatrix} 0 & h & \dots & \dots & \dots & \dots & h & 0 \end{pmatrix}^T$$

Galerkin FEM for an internal node $i$ ($\widehat{=}$ central FD scheme)

$$v\frac{u_{i+1}-u_{i-1}}{2h} - d\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = 1$$

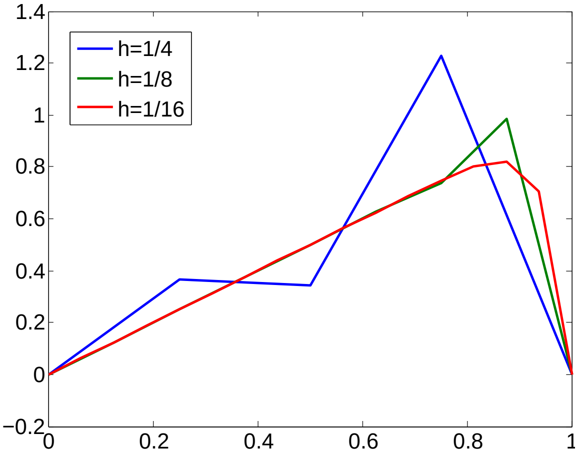
# Numerical example, cont'd

FEM yields good approximations for $\gamma = 2$.



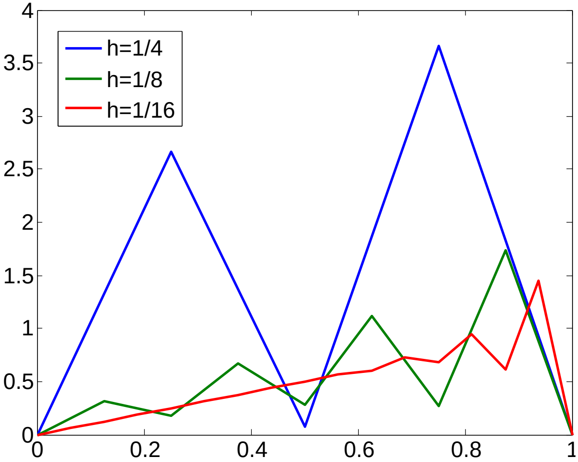
# Numerical example, cont'd

FEM yields poor approximations for $\gamma = 20$ unless $h$ is small.



# Numerical example, cont'd

FEM yields oscillatory approximation for $\gamma = 100$ even for small $h$.



# Analysis of the discrete problem

**Observation:** oscillatory behavior depends on the size of $\gamma$ and on the mesh width $h$. A useful measure is the mesh Péclet number

$$\text{Pe} = \frac{\gamma h}{2} = \frac{vh}{2d}$$

Galerkin FEM for an internal node $i$ in terms of Pe reads

$$v\frac{u_{i+1}-u_{i-1}}{2h} - d\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = 1$$

$$\Leftrightarrow \quad \left(\frac{v}{2h}-\frac{d}{h^2}\right)u_{i+1} + \frac{2d}{h^2}u_i - \left(\frac{v}{2h}+\frac{d}{h^2}\right)u_{i-1} = 1$$

$$\Leftrightarrow \quad \frac{v}{2h}\left(\frac{\text{Pe}-1}{\text{Pe}}u_{i+1} + \frac{2}{\text{Pe}}u_i - \frac{\text{Pe}+1}{\text{Pe}}u_{i-1}\right) = 1$$

TU Delft

# Analysis of the discrete problem, cont'd

**Aim:** to construct an alternative three-point formula

$$\alpha_1 u_{i-1} + \alpha_2 u_i + \alpha_3 u_{i+1} = 1$$

which reproduces the exact solution at the mesh nodes

$$u_{i-1} = \frac{1}{v}\left(x_i - h - \frac{1 - e^{\gamma x_i} e^{-2\mathrm{Pe}}}{1 - e^{\gamma}}\right)$$

$$u_i = \frac{1}{v}\left(x_i - \frac{1 - e^{\gamma x_i}}{1 - e^{\gamma}}\right)$$

$$u_{i+1} = \frac{1}{v}\left(x_i + h - \frac{1 - e^{\gamma x_i} e^{2\mathrm{Pe}}}{1 - e^{\gamma}}\right)$$

TUDelft

# Analysis of the discrete problem, cont'd

Substitute expressions for $u_i$ and $u_{i\pm1}$ into three-point formula and derive sufficient conditions for the unknown coefficients

$$\underbrace{(\alpha_1+\alpha_2+\alpha_3)}_{=0}x_i \underbrace{-(\alpha_1-\alpha_3)}_{v/h}h - \underbrace{\left(\alpha_1 e^{-2\mathrm{Pe}}+\alpha_2+\alpha_3 e^{2\mathrm{Pe}}\right)}_{=0}\frac{1-e^{\gamma x_i}}{1-e^{\gamma}} = v$$

Solution of the $3\times3$ system for the coefficients $\alpha_1,\alpha_2,\alpha_3$ yields

$$\alpha_1 = -v\frac{1+\coth\mathrm{Pe}}{2h},\quad \alpha_2 = v\frac{\coth\mathrm{Pe}}{h},\quad \alpha_3 = v\frac{1-\coth\mathrm{Pe}}{2h}$$

# Analysis of the discrete problem, cont'd

**Conclusion:** given $\gamma = v/d$ and $h$ the exact solution at the nodes is reproduced by the alternative discrete method

$$\frac{v}{2h}\left((1-\coth\mathrm{Pe})u_{i+1} + (2\coth\mathrm{Pe})u_i - (1+\coth\mathrm{Pe})u_{i-1}\right) = 1$$

$$\Leftrightarrow \quad v\frac{u_{i+1}-u_{i-1}}{2h} - (d+\hat{d})\frac{u_{i+1}-2u_i+u_{i-1}}{h^2} = 1$$

with stabilizing *artificial/numerical diffusion*

$$\hat{d} = \beta\frac{vh}{2} = \beta d\,\mathrm{Pe}, \quad \beta = \coth\mathrm{Pe} - \frac{1}{\mathrm{Pe}}$$

TUDelft

# Conclusions on Galerkin FEM

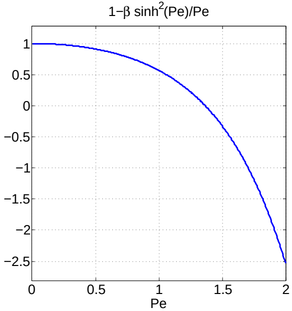
- Galerkin FEM tends to produce oscillations if $\text{Pe} \gg 1$ but it can be stabilized by adding artificial diffusion, e.g.

$$v\frac{u_{i+1}-u_{i-1}}{2h}-(d+\hat{d})\frac{u_{i+1}-2u_i+u_{i-1}}{h^2}=1$$

- Galerkin FEM without stabilization produces nodally exact solution to the *modified equation*

$$v\partial_x u - d\left(1-\beta\frac{\sinh^2}{\text{Pe}}\right)\partial_{xx}u = 1$$

with negative net diffusion for $\text{Pe} > 1$



# Systematic approach towards stabilization for FEM

Given the residual of the original PDE, e.g.,

$$\mathcal{R}[u] = \partial_x(vu) - \partial_x(d\partial_x u) - f$$

an element-wise contribution is added to the standard weak form

$$\int_a^b w\mathcal{R}[u]\,\mathrm{d}x + \sum_{k=1}^{N-1}\int_{x_k}^{x_{k+1}} \tau_k \mathcal{P}[w]\mathcal{R}[u]\,\mathrm{d}x = 0$$

$$\Leftrightarrow \quad \sum_{k=1}^{N-1}\int_{x_k}^{x_{k+1}} (w + \tau_k\mathcal{P}[w])\,\mathcal{R}[u]\,\mathrm{d}x = 0 \quad \forall w \in W$$

This stabilization is *consistent*, that is, all terms on the left-hand side vanish if $u$ equals the exact solution since $\mathcal{R}[u_{\text{ex}}] \equiv 0$

TUDelft

# Making it work in practice

- Stabilization parameter $\tau_k$ may be defined per element

$$\tau_k = \beta \frac{h_k}{|v|}, \quad h_k = x_{k+1} - x_k$$

- Streamline-Upwind Petrov-Galerkin (SUPG) method

$$\mathcal{P}[w] = v(\partial_x w)$$

- Galerkin Least-Squares (GLS)/Subgrid Scale (SGS) method

$$\mathcal{P}[w] = v(\partial_x w) \mp \partial_x(d\partial_x w) \pm (\partial_x v)w$$

Since i.b.p is not performed for the stabilizing term the second-order derivative vanishes for linear finite elements

TUDelft

# Working out the (bi-)linear forms

Definition of bilinear and linear forms with SUPG-stabilization, e.g.,

$$a(u,w) = \sum_{k=1}^{N-1} a_k(u,w), \quad b(w) = w(b)(d g_b) + \sum_{k=1}^{N-1} b_k(w)$$

with element-wise counterparts defined as follows

$$a_k(u,w) = \int_{x_k}^{x_{k+1}} w\partial_x(vu) + \partial_x w(d\partial_x u) + \tau_k(v\partial_x w)(\partial_x(vu)+\partial_x(d\partial_x u))\,\mathrm{d}x$$

$$b_k(w) = \int_{x_k}^{x_{k+1}} wf + \tau_k(v\partial_x w)f\,\mathrm{d}x$$

# Shock capturing methods

**Observation:** *linear* stabilization methods such as SUPG, GLS, and SGS may fail ti suppress oscillations in the vicinity of steep gradients or discontinuities (e.g., shock waves)

**Remedy:** replace $\mathcal{P}[w]$ by a *nonlinear* stabilization operator

$$\int_a^b w\mathcal{R}[u]\,\mathrm{d}x + \sum_{k=1}^{N-1} \int_{x_k}^{x_{k+1}} \tau_k \hat{\mathcal{P}}[u,w]\mathcal{R}[u]\,\mathrm{d}x = 0$$

where

$$\hat{\mathcal{P}}[u,w] = \begin{cases} \hat{v}\partial_x w & \text{if } |u| \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \hat{v} = \left(\frac{\mathcal{R}[u]}{|\partial_x u|^2}\right)\partial_x u$$

# Extension to time-dependent problems

Redefine the residual of the PDE to include the transient term

$$\mathcal{R}[u] = \partial_t u + v\partial_x u - d\partial_{xx} u - f$$

Redefine the stabilization operator

SUPG: $\mathcal{P}[w] = v\partial_x w$

GLS/SGS: $\mathcal{P}[w] = \pm\frac{w}{\Delta t} + v(\partial_x w) \mp \partial_x(d\partial_x w) \pm (\partial_x v)w$

and work out the (bi-)linear forms so that the weak problem reads

$$\int_a^b w\frac{du}{dt}\,\mathrm{d}x + a(u,w) = b(w) \quad \text{for all } w \in W$$

# Extension to time-dependent problems, cont'd

Discretization in space by FEM yields the semi-discrete problem

$$M\frac{\mathrm{d}u}{\mathrm{d}t} + Au = b$$

Application of the $\theta$-scheme yields the fully discrete problem

$$M\frac{u^{n+1} - u^n}{\Delta t} + \theta A u^{n+1} + (1-\theta)Au^n = \theta b^{n+1} + (1-\theta)b^n$$

- $\theta = 0$: explicit forward Euler method
- $\theta = \frac{1}{2}$: implicit Crank-Nicolson method
- $\theta = 1$: implicit backward Euler method

TUDelft