Algebraic Multigrid for Two-dimensional Time-Harmonic Magnetic Field Computations

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Motivation

Develop fast solution procedures for the system of linear equations

$$\mathcal{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$\bowtie$ $\mathcal{A}$ complex and $\mathcal{A} = \mathcal{A}^T$

$\bowtie$ $\mathcal{A} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$

- $*$ A is large sparse complex, $A = A^T$
- $*$ B and C small and dense

Structure of the Presentation

- ⋈ Context of the Research
- ⋈ Problem Description
- ⋈ Solution Techniques
- ⋈ Step 1 : Stationary Problems
- ⋈ Step 2 : Time-Harmonic Problems
- ⋈ Step 3 : Field-Circuit Coupled Problems
- ⋈ Implementation Issues
- ⋈ Conclusions

Context of the Research

⋈ collaboration:

- ∗ Scientific Computing Group (Department of Computer Science)
- ∗ Division Electrical Energy (Department of Electrical Engineering)

⋈ aim: develop software for the numerical simulation of electromagnetic devices

⋈ main emphasis: efficient solvers for linear systems resulting from discretized partial differential equations

⋈ typical applications: electrical machines, transformers, power lines, induction furnaces, ...

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- ∗ Magnetic field problem
- ∗ Electrical circuit problem
- ∗ Coupled problem

Problem Description

Magnetic Field Problem

⋈ Field equation $\nabla \times (\nu \nabla \times \mathbf{A}) + \sigma \frac{\partial \mathbf{A}}{\partial t} = -\sigma \nabla \phi$ $\quad \mathbf{B} = \nabla \times \mathbf{A}$

$$\mathbf{E} = -\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}$$

⋈ Modelling assumptions

$*$ Time-harmonic $\quad \mathbf{A}(\mathbf{x}, t) = \text{Re}[\, \widehat{\mathbf{A}}(\mathbf{x}) \exp(j\, \omega t)\,]$

$*$ Two dimensional $\quad \widehat{\mathbf{A}}(\mathbf{x}) = (0, 0, \hat{A}_z(x, y))$

⋈ PDE

$$-\frac{\partial}{\partial x}\left(\nu \frac{\partial \hat{A}_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu \frac{\partial \hat{A}_z}{\partial y}\right) + j\, \omega\, \sigma\, \hat{A}_z = -\sigma \frac{\partial \hat{\phi}}{\partial z}$$

Problem Description (2)

Electrical Circuit Problem

- parts of the computational domain are electrically conducting
- time-varying currents induce magnetic fields
- a description of the interconnection of the conducting parts is required to model the electromagnetic interaction
- electrical circuit described by Kirchhoff Current (I) and Voltage (V) Laws

$$C \begin{pmatrix} I \\ V \end{pmatrix} = \begin{pmatrix} I_0 \\ V_0 \end{pmatrix} \quad \text{with} \quad C \quad \text{the electrical circuit matrix}$$

Problem Description (3)

Magnetic Field-Electrical Circuit Coupled Problem

- ⋈ coupling through electrically induced magnetic effects in the conductors
- ⋈ finite element discretization for the magnetic field, resulting in

$$\mathcal{A} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \quad \text{with} \quad \mathcal{A} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \quad \mathcal{A} \text{ complex} \quad \mathcal{A} = \mathcal{A}^T$$

- ∗ A discretized field problem — large, sparse
- ∗ C electrical circuit matrix — $\dim(C) \ll \dim(A)$, dense
- ∗ B coupling matrix — sparse with dense blocks or dense

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    - ∗ Multigrid
    - ∗ Algebraic Multigrid
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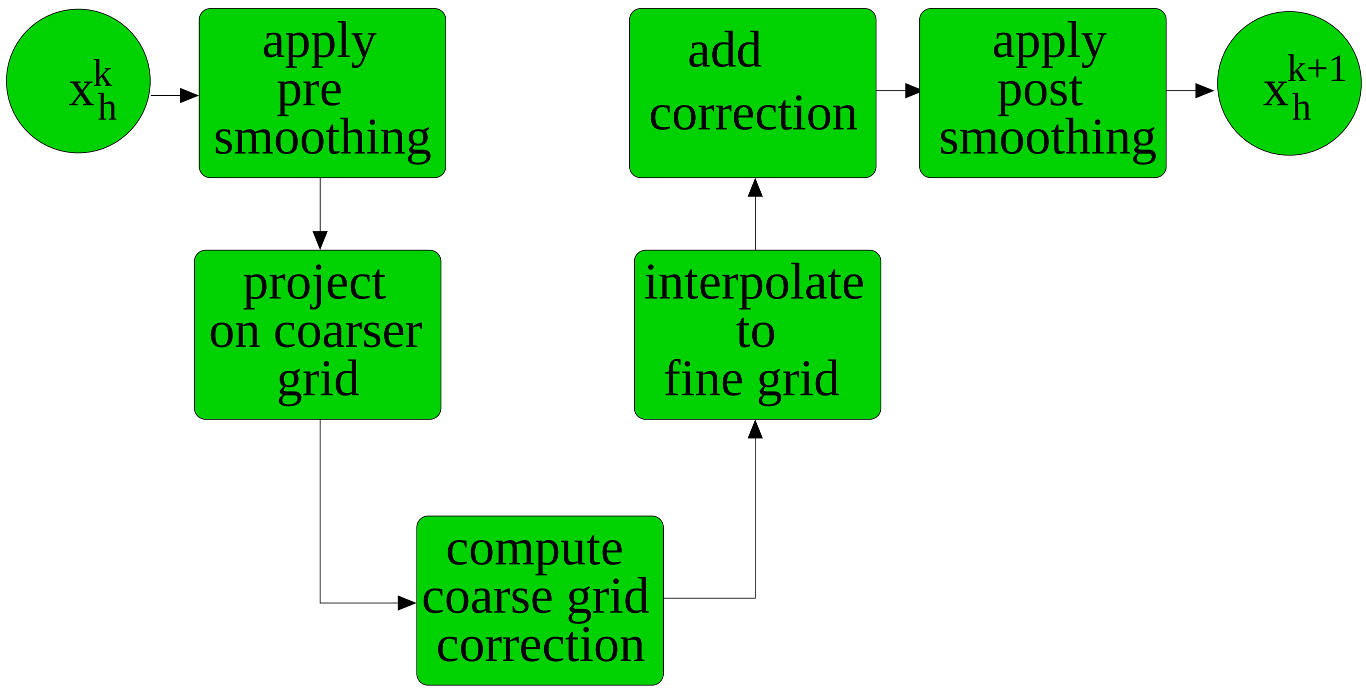
Multigrid methods

- exploit PDE background of the linear problem
- multigrid methods = smoother + coarse grid correction
- two-grid scheme



Algebraic Multigrid for $A^h x^h = b^h$

$\bowtie$ Setup phase $*$ construction of C/F splitting and interpolation

$*$ strength of coupling between nodes coded in A^h exploited

$*$ matrix dependent interpolation: $(I_h^H)_{ij} \sim A_{ij}^h / A_{ii}^h$

$*$ Galerkin coarsening: $A^H = I_h^H \, A^h \, I_H^h$

$*$ apply recursively using A^H as input

$\bowtie$ Solve phase $*$ multigrid cycling

$\Rightarrow$ only A^h required as input!

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  - ∗ Numerical Results
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Stationary Field Problems

$$A\,x = b$$

$\bowtie$ Matrix properties: $*$ A is real and sparse
$*$ A is symmetric positive definite

$\bowtie$ Algebraic multigrid: $*$ Ruge-Stüben code (RAMG) developed in the ’80s
$*$ its successor developed by Stüben in the ’90s (SAMG)

$\bowtie$ Krylov acceleration: Conjugate Gradient method

Numerical Results for Stationary Problems

Permanent Magnet Machine

∗ adaptive mesh refinement

∗ nonlinear PDE

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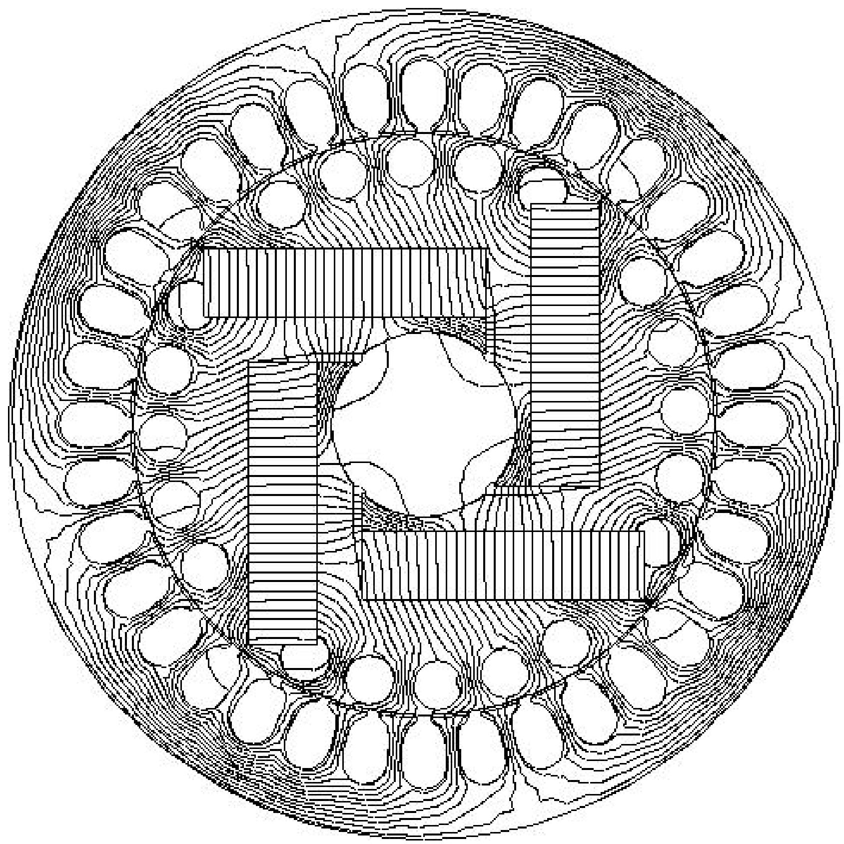
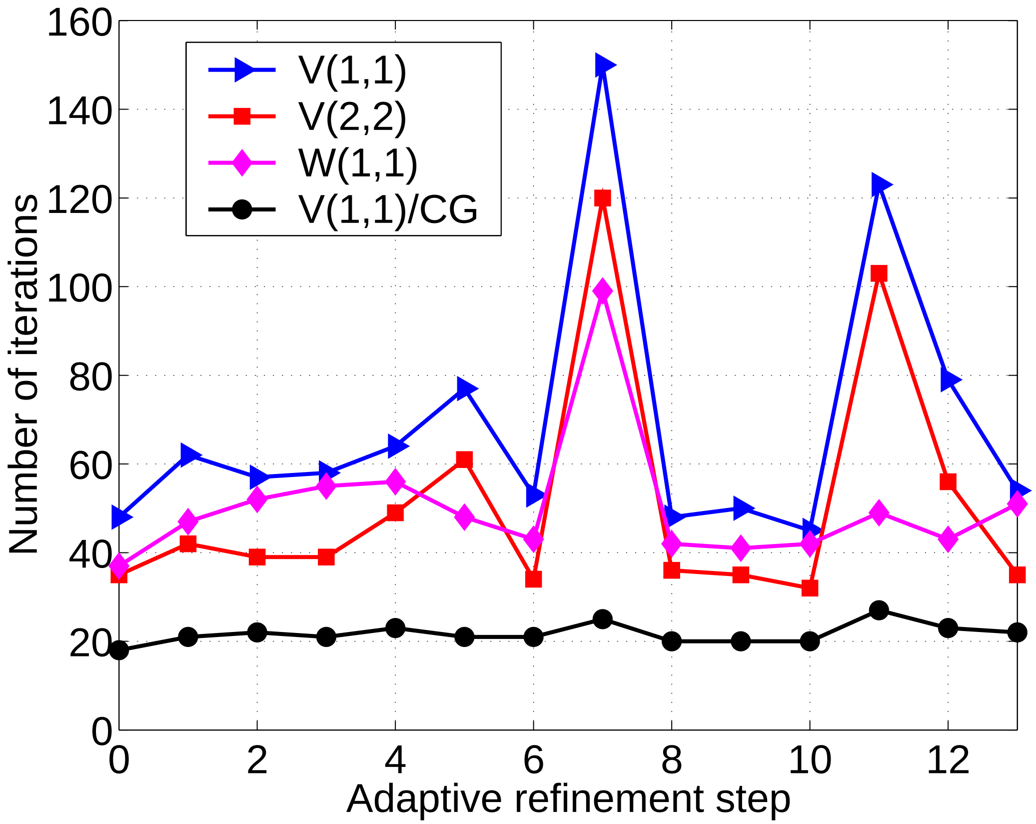


Numerical Results for Stationary Problems (2)

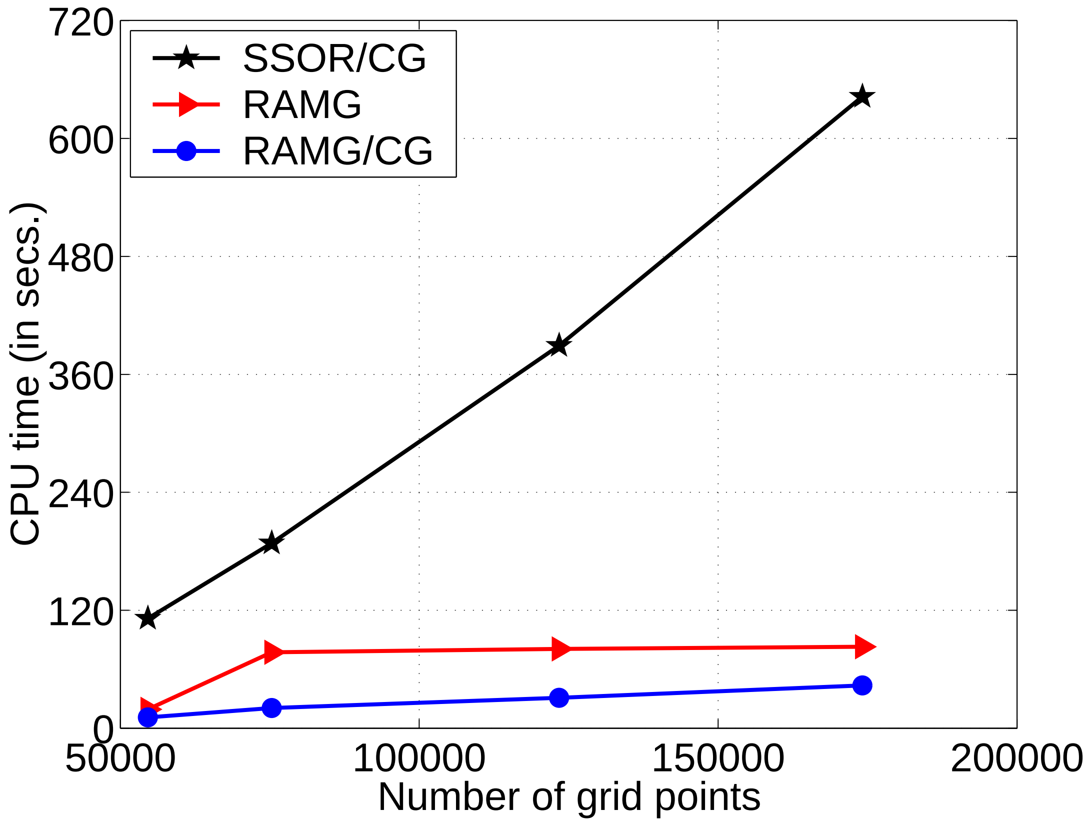
⋈ Number of iterations for RAMG



* multigrid behavior
* CG accelerates and stabilizes convergence

Numerical Results for Stationary Problems (3)

⋈ CPU-time measurements for RAMG



* RAMG speedup by factor 7.5
* RAMG/CG speedup by factor 15

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  - ∗ Extension of AMG
  - ∗ Numerical results
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Time Harmonic Field Problems

$$A\,x = b \quad \text{with} \quad A = A_R + j\,A_I \quad j^{\,2} = -1$$

⋈ Matrix properties: $*$ A is complex and symmetric

$*$ A_R has properties of system matrix in stationary problems

⋈ Krylov acceleration: $*$ CG for complex symmetric systems (COCG)

Extension of AMG to Time-Harmonic Problems

$$A^h\,x^h = b^h \quad \text{with} \quad A^h = A^h_R + j\,A^h_I \quad j^{\,2} = -1$$

⋈ Algorithm:

∗ based on real part of $A^h \Rightarrow C/F-$splitting + interpolation I_h^H
$\Rightarrow I_h^H$ is real

∗ given I_h^H, set $I_H^h = (I_h^H)^T$, and construct $A^H = I_H^h\,(A^h_R + j\,A^h_I)\,I_h^H$

⋈ Motivation: A^H has properties of coarse grid discretization
A^H inherits complex symmetry from A^h

⋈ Implementation: version of SAMG for systems of coupled differential equations

Implementation of AMG for Time-Harmonic Problems

Rewrite $A^h\, x^h = b^h$ as $\mathcal{A}^h \begin{pmatrix} x_R^h \\ x_I^h \end{pmatrix} = \begin{pmatrix} b_R^h \\ b_I^h \end{pmatrix}$ where $\mathcal{A}^h = \begin{pmatrix} A_R^h & -A_I^h \\ A_I^h & A_R^h \end{pmatrix}$

⋈ setup phase:

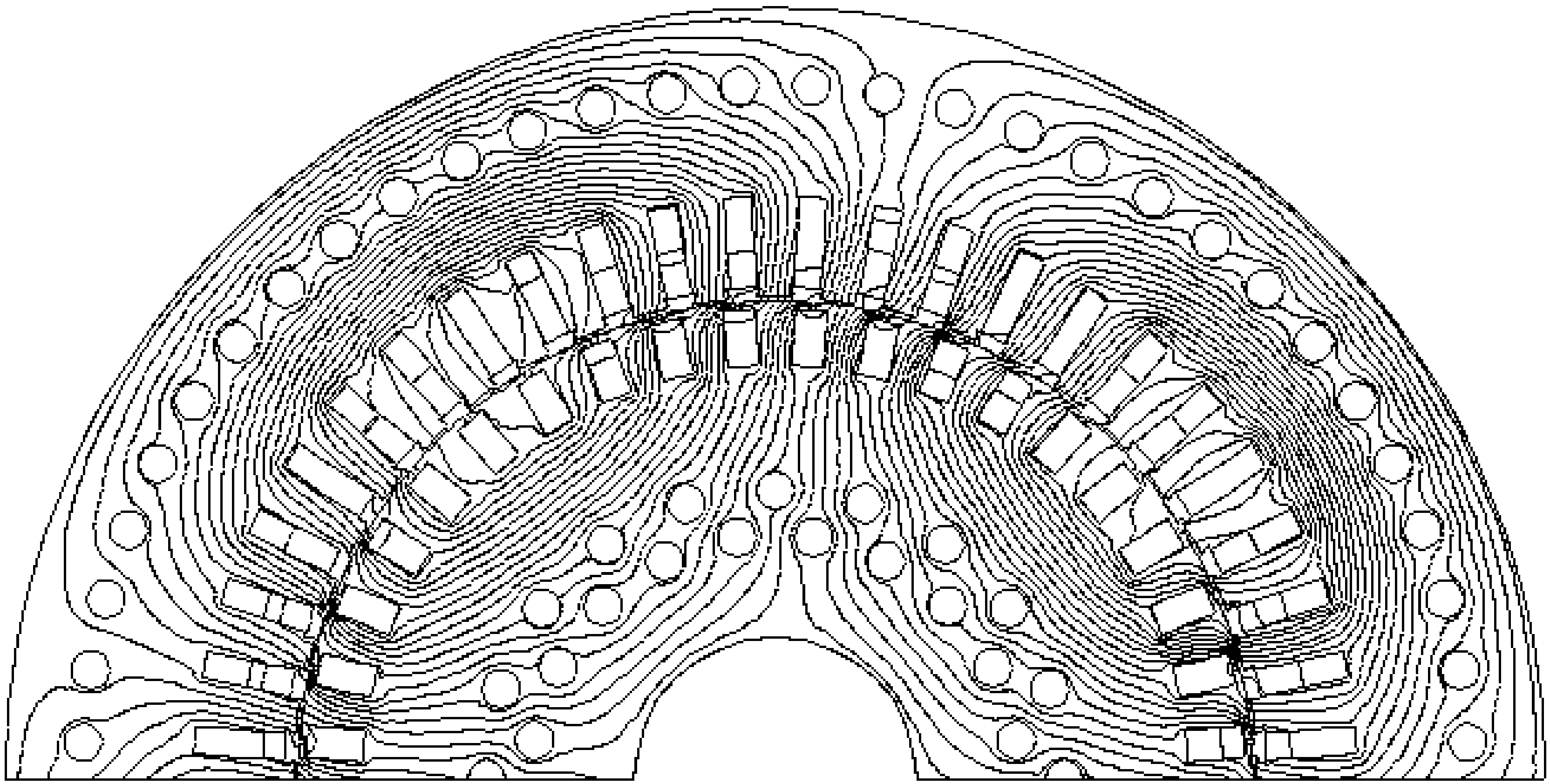
- C/F-splitting + I_h^H based on diagonal block of $\mathcal{A}^h \Rightarrow$ real part of A^h
- Galerkin coarsening: $\mathcal{I}_H^h = \begin{pmatrix} I_H^h & 0 \\ 0 & I_H^h \end{pmatrix}$ $\quad \mathcal{A}^H = \mathcal{I}_h^H\, \mathcal{A}^h\, \mathcal{I}_H^h$

⋈ solve phase:

- 2×2 block smoothers coupling real and imaginary component at each node

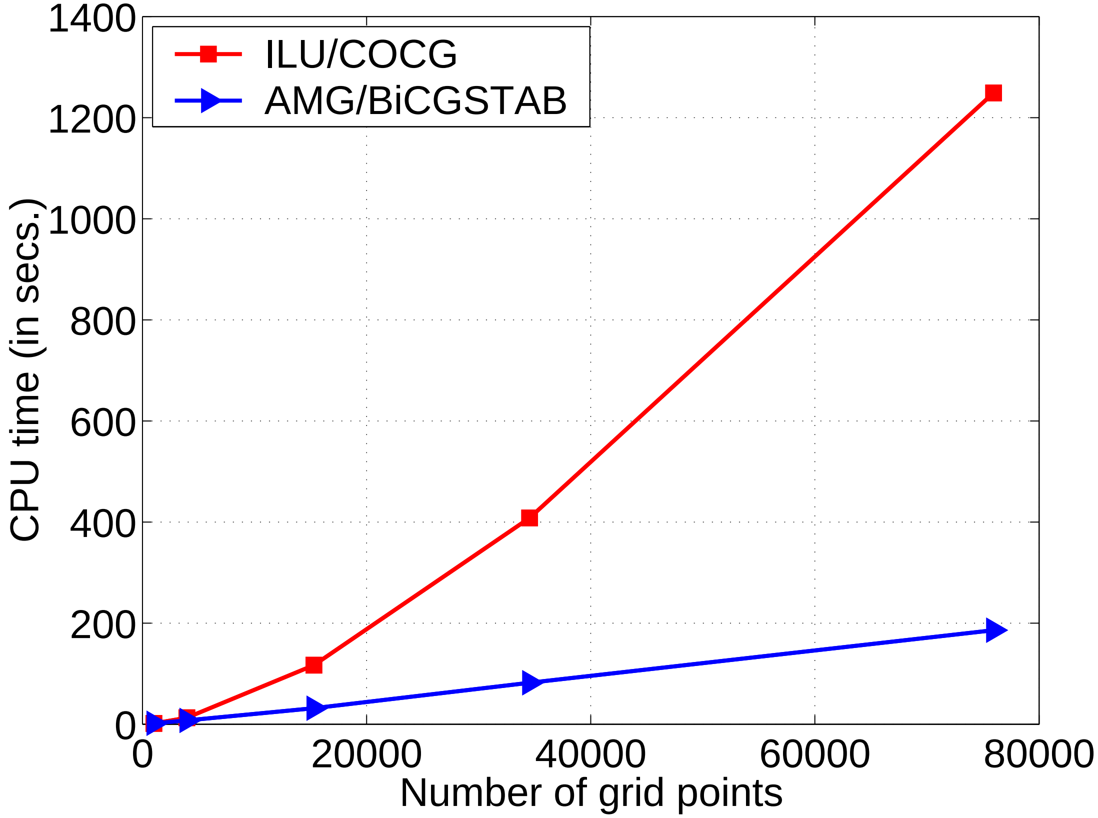
Numerical Results for Time-Harmonic Problems

Half model of an 400kW induction machine



Numerical Results for Time-Harmonic Problems (2)

⋈ CPU-time of ILU/COCG and AMG/BiCGSTAB



speedup by factor 6

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Extension of AMG for Coupled Problems

Two-grid scheme

$$\mathcal{A}^h \begin{pmatrix} x^h \\ y \end{pmatrix} = \begin{pmatrix} f^h \\ g \end{pmatrix} \quad \text{with} \quad \mathcal{A}^h = \begin{pmatrix} A^h & B^h \\ (B^h)^T & C \end{pmatrix} \quad \text{and} \quad A^h = A^h_R + j\, A^h_I$$

⋈ Setup phase

* coarsen the field variables: $A^h_R \Rightarrow C/F$ splitting $+ \, I^h_H \Rightarrow A^H = I^H_h \, A^h \, I^h_H$
* transfer all circuit variables to coarser grid
* Galerkin coarsening of the coupled problem

$$\mathcal{A}^H = \begin{pmatrix} I^H_h & 0 \\ 0 & I \end{pmatrix} \mathcal{A}^h \begin{pmatrix} I^h_H & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A^H & B^H \\ (B^H)^T & C \end{pmatrix} \quad \text{with} \quad B^H = I^H_h B^h$$

Extension of AMG for Coupled Problems (2)

Two-grid scheme

$$\mathcal{A}^h \begin{pmatrix} x^h \\ y \end{pmatrix} = \begin{pmatrix} f^h \\ g \end{pmatrix} \quad \text{with} \quad \mathcal{A}^h = \begin{pmatrix} A^h & B^h \\ (B^h)^T & C \end{pmatrix} \quad \text{and} \quad A^h = A_R^h + j\, A_I^h$$

⋈ Solve phase

- ∗ smooth only the field variables: $\mathcal{S}^h = \begin{pmatrix} S^h & 0 \\ 0 & I \end{pmatrix}$
- ∗ corrections for the field and circuit variables computed on the coarse grid

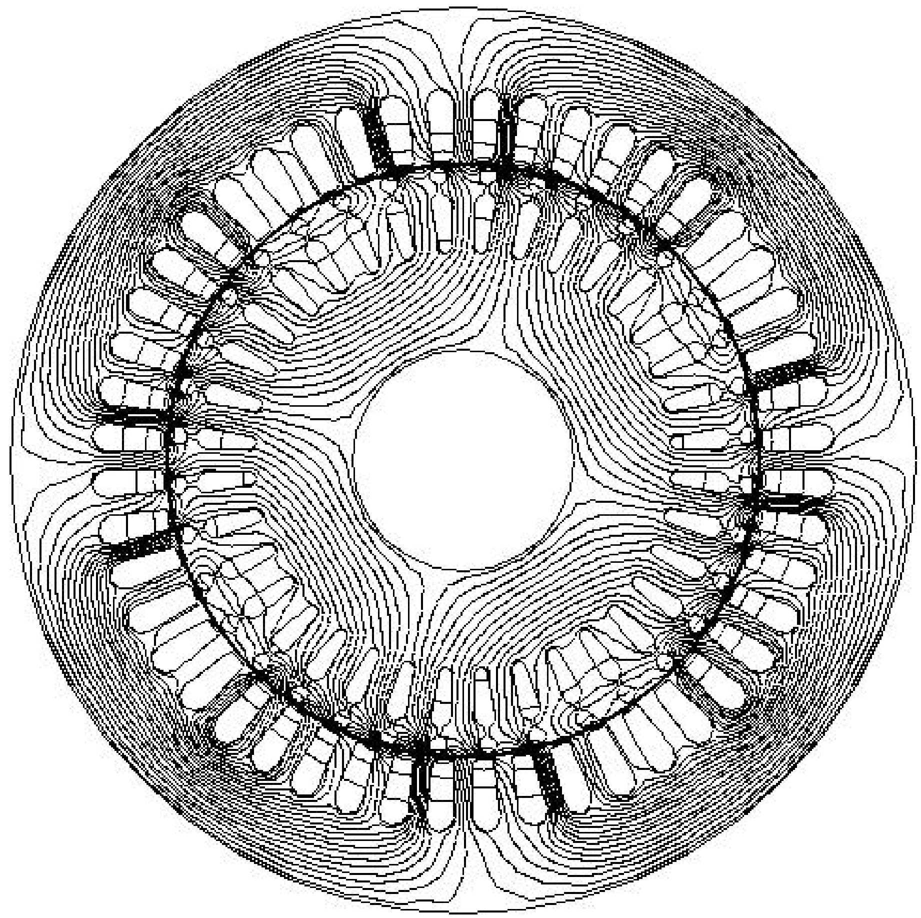
$$\mathcal{A}^H \begin{pmatrix} e_x \\ e_y \end{pmatrix} = \begin{pmatrix} r_x \\ r_y \end{pmatrix} \quad \text{with} \quad \mathcal{A}^H = \begin{pmatrix} A^H & B^H \\ (B^H)^T & C \end{pmatrix}$$

Numerical Results for Coupled Problems

Model of an 40kW induction machine

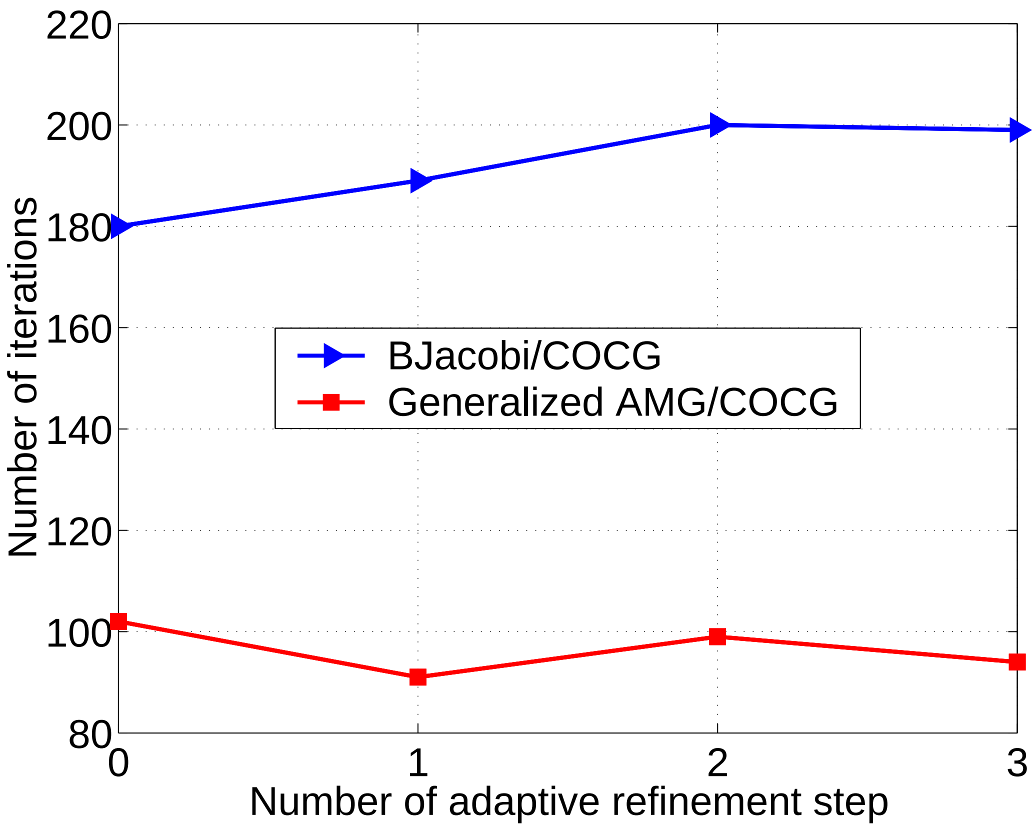
∗ 4 mesh refinement steps

∗ 148 circuit relations



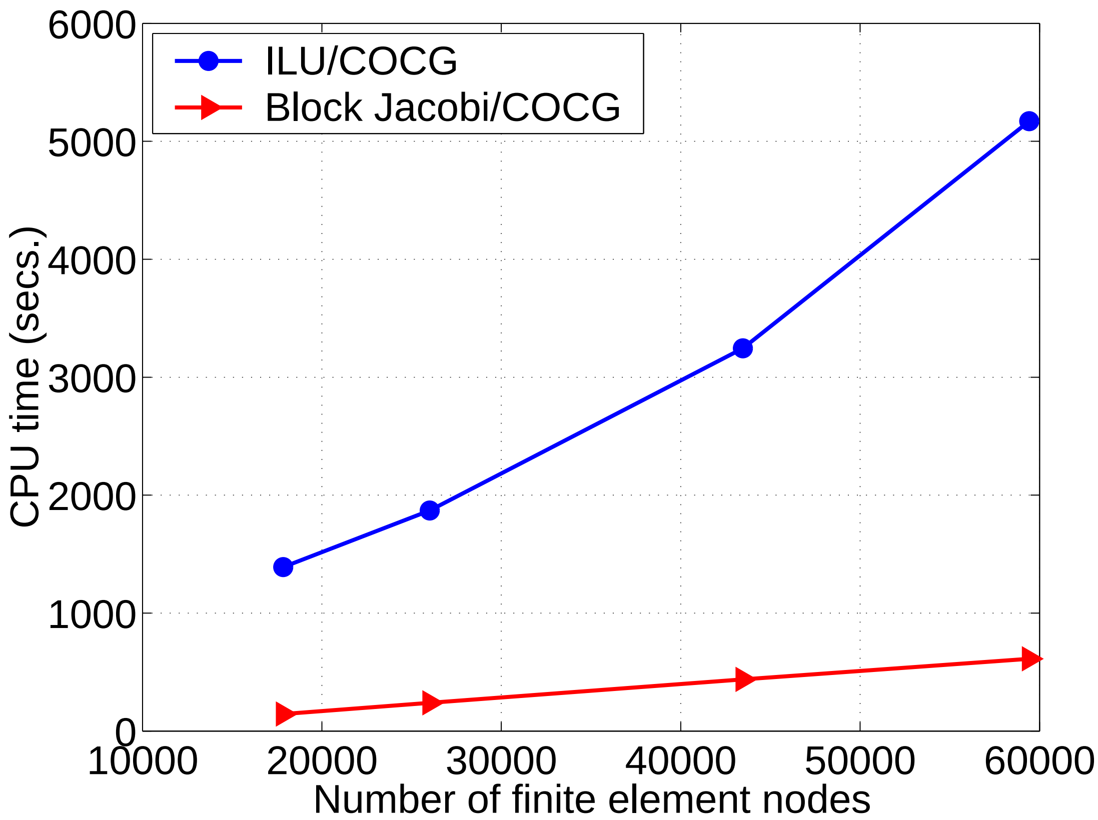
Numerical Results for Coupled Problems (2)

⋈ Number of iterations of block Jacobi and generalized AMG algorithms



Numerical Results for Coupled Problems (3)

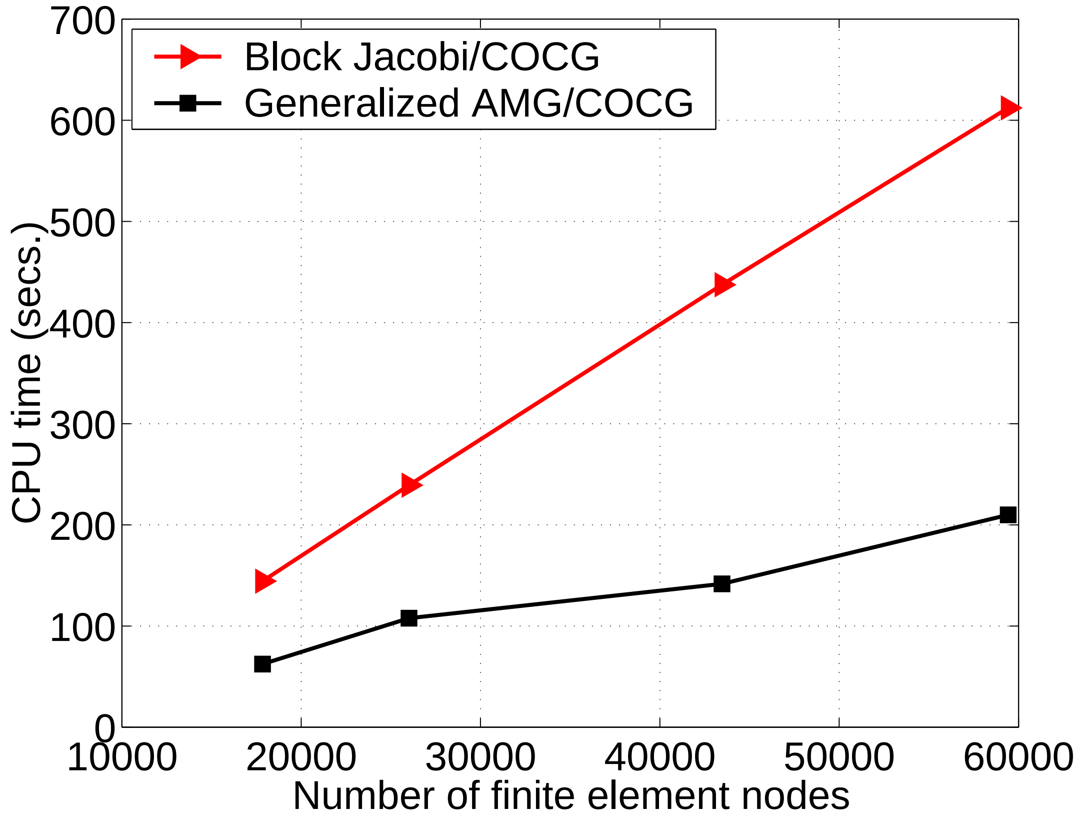
⋈ CPU-time of ILU/COCG and block-Jacobi/COCG



speedup by factor 8

Numerical Results for Coupled Problems (4)

⋈ CPU-time of block Jacobi/COCG and generalized AMG/COCG



speedup by factor 3

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Implementation Issues

The AMG-PETSc interface

$\bowtie$ level 1 interface: RAMG/SAMG available as preconditioner in PETSc
(PETSc: Smith '97)

Features $\ast$ acceleration by Krylov subspace methods available in PETSc

Level 1 interface with RAMG now available in the PETSc distribution

$\bowtie$ level 2 interface: $\ast$ SAMG constructs multigrid hierarchy
$\ast$ PETSc components do the multigrid cycling

Features $\ast$ cycling phase extensible to problem dependent requirements

Conclusions

- We presented algebraic multigrid based solvers for stationary and time-harmonic magnetic field and magnetic field-electrical circuit coupled problems

- (Algebraic) Multigrid is not a particular algorithm, but rather a general methodology suitable for a broad range of problems

- Algebraic multigrid methods deliver a speedup and outperform by far previously implemented solvers

- Algebraic multigrid solvers have been coupled with a finite element simulation package in such a way to allow their use in practical engineering problems