Isogeometric Analysis for Compressible Flow Problems in Industrial Applications

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NAG@TUD

Numerical Analysis group: Prof.dr.ir. Kees Vuik, Kees Oosterlee

- 12 assistant/associated professors, 30 PhDs, 5 guest researchers
- Industrial flows, FSI, biomath, finance, iterative solvers, HPC, ...

My research team:

- J. Hinz: *IgA-based elliptic grid generation for industrial applications*
- R. Tielen: *IgA-inspired high-order material point method*
- J. v.d. Meer: *foam enhanced oil recovery*
- A. Jaeschke (TU Lodz, PL): *IgA in turbomachinery applications*

My own research interests:

- High-order high-resolution FEM/IgA schemes and efficient solvers for compressible flow problems, hardware-oriented numerics on unconventional hardware, quantum computing

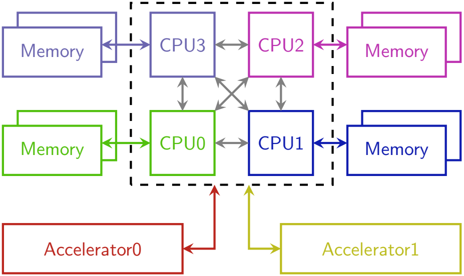
Overview

1. Introduction
   - Scientific computing from a hardware perspective
   - Isogeometric design-simulation-optimization loop
2. Elliptic 'grid' generation [J. Hinz]
   - Planar parameterizations
   - Volumetric parameterizations
3. Compressible flow solver [A. Jaeschke]
   - Problem formulation
   - Linearized FCT limiter
   - Numerical examples
4. Implementation aspects
   - Meta-programming techniques for heterogeneous HPC

Scientific computing from a hardware perspective

Today: Heterogeneous compute hardware at system and node level

- Distributed cluster systems with heterogeneous compute nodes
  - Multi-core CPUs: Intel 28, AMD 32, ARM 48, IBM 16x12
  - Many-core accelerators: GPUs, Intel MICs, FPGAs, ...
  - Memory subsystem is the bottleneck in data-intensive applications



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Future trends: 'novel' hardware architectures and paradigm shifts

- In-memory-computing for data-intensive applications (DBs, FEM?)
- Data-flow computing in space *vs.* control-flow computing in time
- Special-purpose accelerators: analogue or quantum computers

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- In-memory-computing for data-intensive applications (DBs, FEM?)
- Data-flow computing in space *vs.* control-flow computing in time
- Special-purpose accelerators: analogue or quantum computers

Philosophy: Bottom-up design of hardware-oriented numerics *for* next-gen hardware instead of squeezing existing methods *into* it.

An example of data-flow computing in space

Example: 2d Poisson equation

- IgA with tensor-product B-spline basis functions of order $p = 2$.
- Matrix-free iterative CG/BiCGStab solver

[1] CMAME 316, (2017) 606-622.

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- IgA with tensor-product B-spline basis functions of order $p = 2$.
- Matrix-free iterative CG/BiCGStab solver
- Weighted quadrature approach by Calabrò et al.[1] is used for the on-the-fly generation matrix entries in the SpMv subroutine

$$S_{i,j} = \sum_{\alpha,\beta=1}^{d} \int_{[0,1]^2} g_{\alpha,\beta}(\boldsymbol{\xi}) \partial_\beta \hat{B}_j(\boldsymbol{\xi}) \left(\partial_\alpha \hat{B}_i(\boldsymbol{\xi}) d\boldsymbol{\xi} \right)$$

$$\approx \sum_{\alpha,\beta=1}^{d} \mathfrak{Q}^{\mathrm{WQ}}_{\alpha,\beta,i} \left(g_{\alpha,\beta}(\cdot) \partial_\beta \hat{B}_j(\cdot) \right)$$

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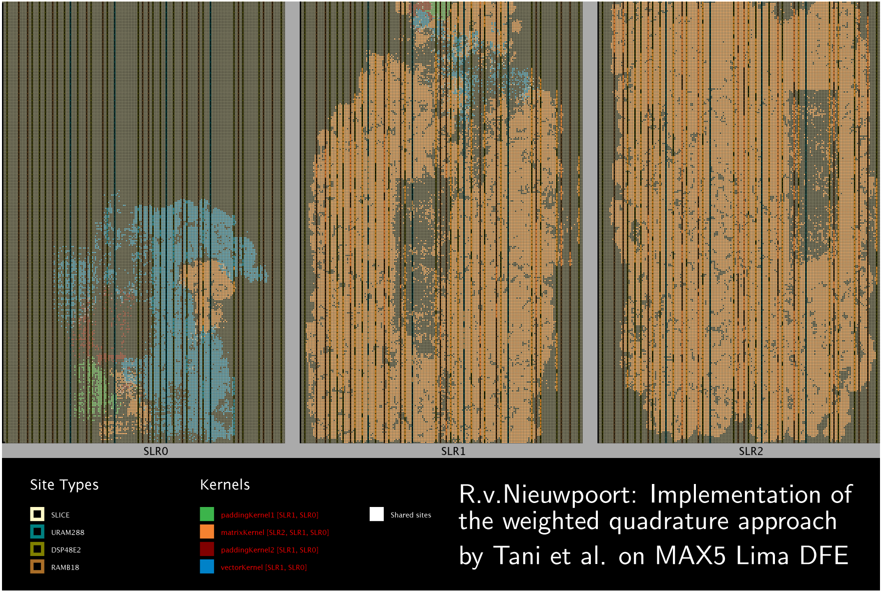
An example of data-flow computing in space



An example of analogue computing

Example: 1d Poisson equation

- Solution to the linear system $Ax = b$ can be interpreted as the steady state limit of the initial value problem

$$\frac{dx(t)}{dt} = b - Ax(t), \quad x(0) = x_0$$

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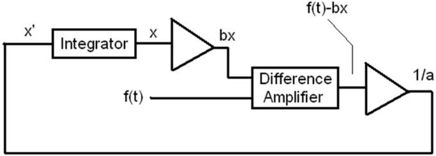
- Analogue computers are efficient in modelling differential equations using op-amps but very expensive to operate at large scales



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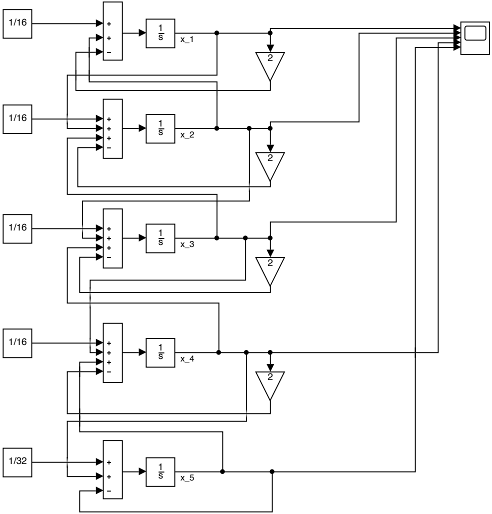
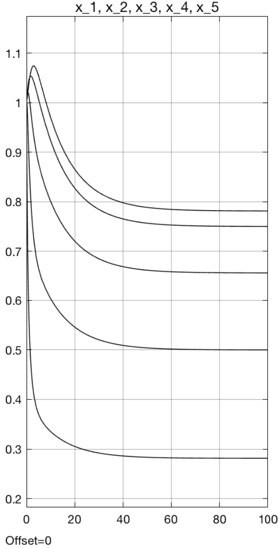
- Solution to the linear system $Ax = b$ can be interpreted as the steady state limit of the initial value problem

$$\frac{dx(t)}{dt} = b - Ax(t), \quad x(0) = x_0$$

- Analogue computers are efficient in modelling differential equations using op-amps but very expensive to operate at large scales
- Analogue computing can bring to new ideas, e.g., for data-flow computing architectures like FPGAs and quantum computers

An example of virtual analogue computing

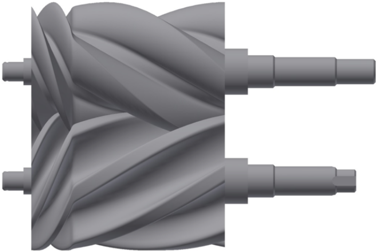




Vision: Isogeometric DSO loop

Our objective is to develop an IGA toolbox for the efficient **D**esign, **S**imulation and **O**ptimization of screw machines with variable pitches

low pressure side



high pressure side

Courtesy of Andreas Brümmer, Dortmund University of Technology.

Collaboration: TU Kaiserslautern, TU Dortmund, TU Delft, JKU Linz

Vision: Isogeometric DSO loop

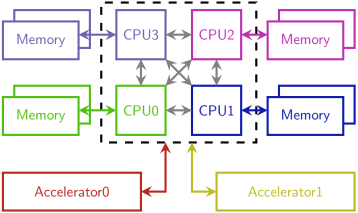
Challenges:

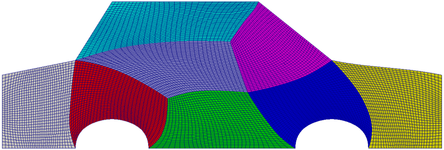
- Rotating geometries with tiny gaps (< 0.4mm) between rotors
- Multi-physics problem: compressible flows, thermal deformation, ...
- Prevent topology changes of the multi-patch structure

Design criteria:

- High-resolution capturing of shocks and discontinuities
- Support for current and future HPC platforms
- KISS (no unmaintainable hacks)

A 'novel' hardware-oriented numerics approach





Multi-patch domain parameterization for isogeometric analysis in G+Smo by Buchegger and Jüttler CAD 82, p. 2-12, 2017

Multi-patch Isogeometric Analysis:

- B-spline based iso-parametric FEM on unstructured coarse grid
- Fully structured high-order discretizations with hardware-optimized implementations on individual patches
- Multi-patch DG-coupling with minimal communication overhead

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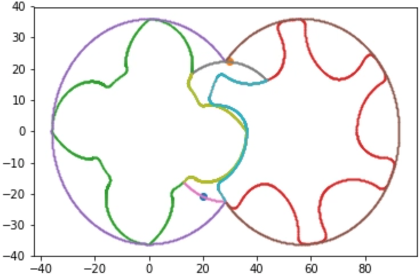
Sliding grid technology - planar case

Approach: Create multi-patch parameterizations of fluid domain (O- or C-type + separator) and let the rotors slide along the inner boundaries



Pros: no topology changes, conforming/'nested' nonconforming patches
Cons: Change of separator parameterization is not continuous in time

Elliptic 'grid' generation

Algorithm[2]**:** Given a *point cloud* describing the patch boundary create an analysis-suitable *parameterization* $\hat{G}_f : [0, 1]^2 \to \Omega_f$ as follows:

[2] J. Hinz, MM, C. Vuik, submitted to GMP 2018

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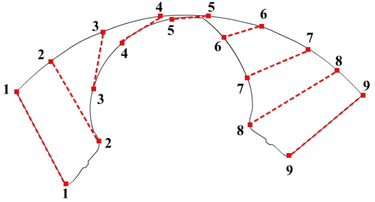
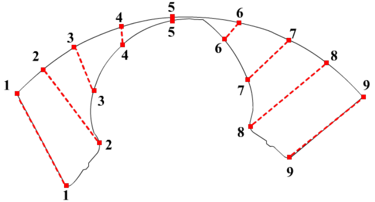
2. Reparameterize opposite boundaries at 'small-gap regions' by constrained cord-length parameterization algorithm





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4. Generate a bi-variate parameterization by solving with IgA

$$\begin{array}{rcl} \hat{g}_{22} x_{\xi\xi} - 2\hat{g}_{12} x_{\xi\eta} + \hat{g}_{11} x_{\eta\eta} & = & 0 \\ \hat{g}_{22} y_{\xi\xi} - 2\hat{g}_{12} y_{\xi\eta} + \hat{g}_{11} y_{\eta\eta} & = & 0 \end{array} \qquad s.t.\ \hat{G}_{m,f}\Big|_{\hat{\Gamma}_{m,f}} = \Gamma_{m,f}$$

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5. Adaptively refine knot spans where $\det \mathrm{D}\hat{G}_{m,f} \leq$ tol and repeat

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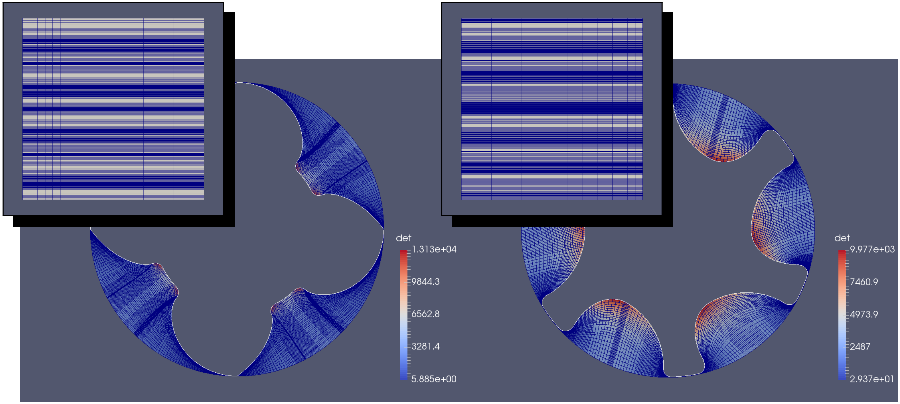
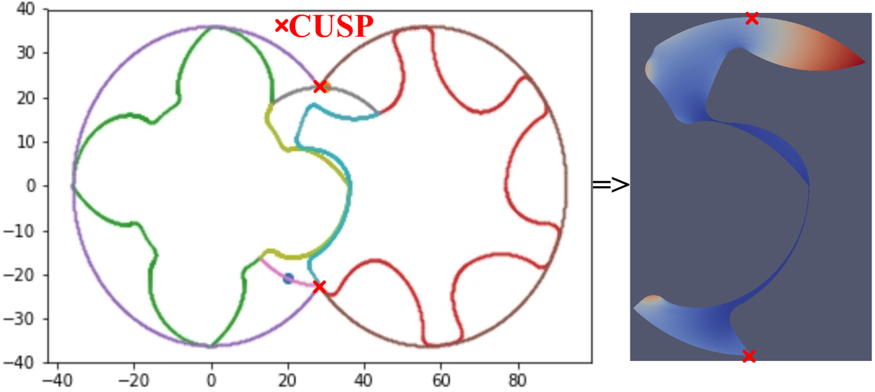
Parameterizations of male/female rotors



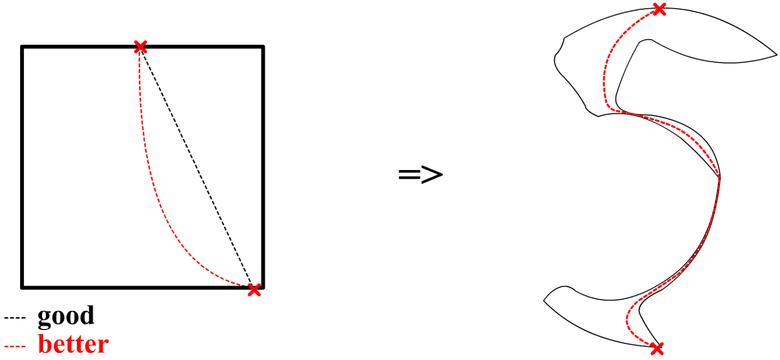
One-patch separator geometry

Approach: Create parameterization for separator geometry



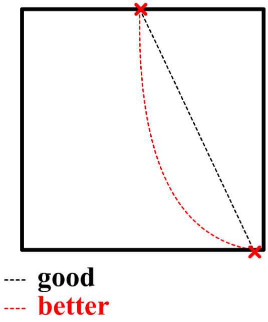
One-patch separator geometry

Approach: Create parameterization for separator geometry, compute splitting curve



One-patch separator geometry

Approach: Create parameterization for separator geometry, compute splitting curve and (a) update point cloud and regenerate O-grids, or (b) adjust O-grids to match along separator using linear elasticity.



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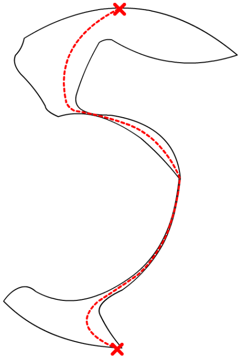
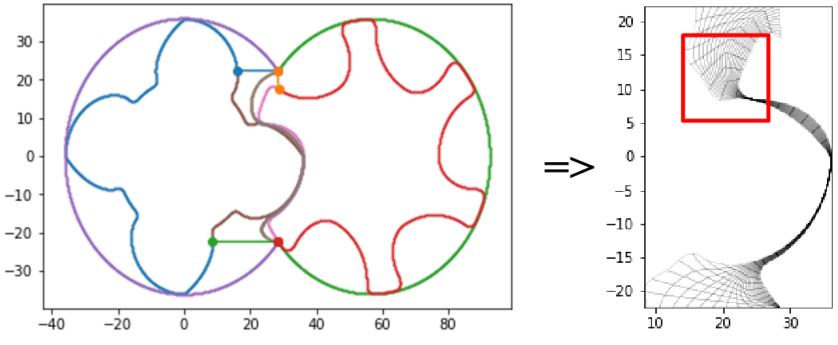


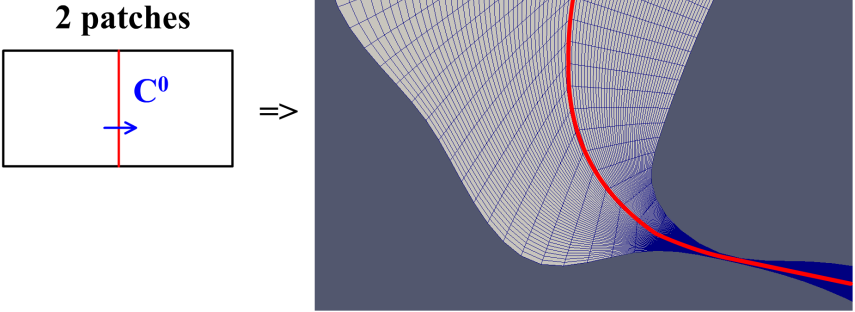
Two-patch separator geometry

Approach: Create two-patch separator



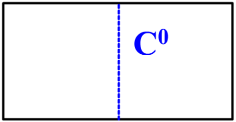
Two-patch separator geometry

Approach: Create two-patch separator with C^0 continuity along splitting curve



Two-patch separator geometry

Approach: Create two-patch separator with C^0 continuity along splitting curve, combine into one patch and run 'repair' algorithm.



run algorithm that allows for C^0 but requires bijective initial guess

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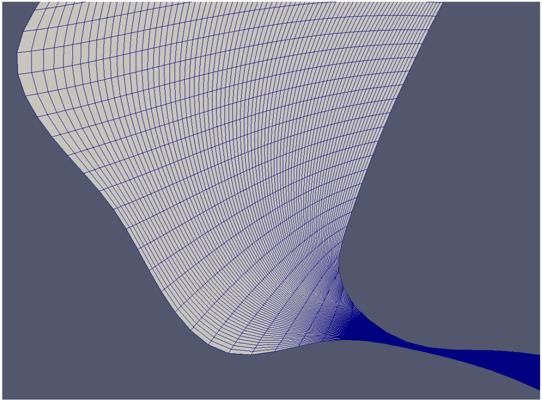

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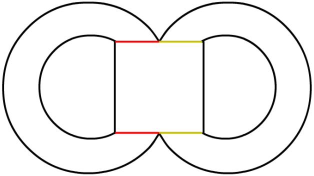
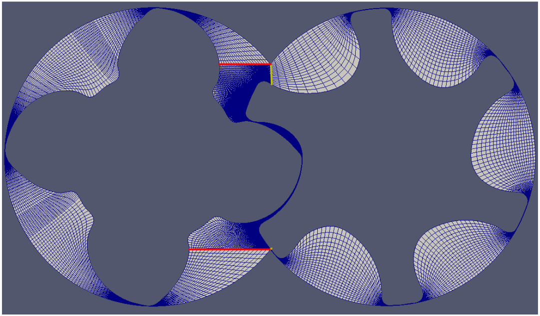
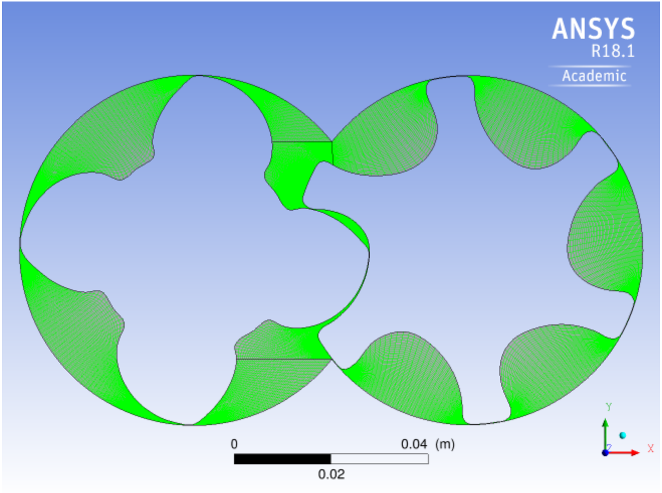




Generation of classical FEM meshes

Approach: Evaluate parameterization in parameter domain to obtain block-structured/unstructured (globally conforming) grids



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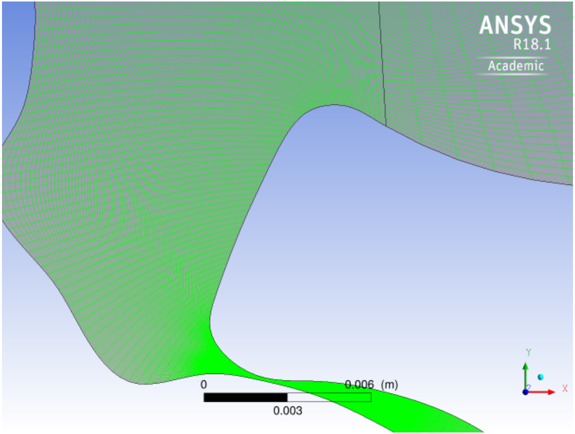
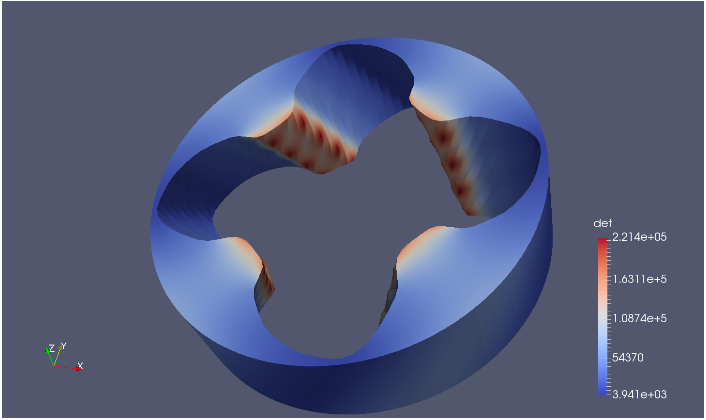


Volumetric grid generation

Approach: Interpolate (linearly) between planar parameterizations at different rotation angles to generate static volumetric parameterization



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Compressible Euler equations

Divergence form

$$U_{,t} + \nabla \cdot \mathbf{F}(U) = 0$$

Quasi-linear form

$$U_{,t} + \mathbf{A}(U) \cdot \nabla U = 0$$

Conservative[a] **variables**, inviscid **fluxes**, flux-Jacobian **matrices**

$$U = \begin{bmatrix} \rho \\ \rho\mathbf{v} \\ \rho E \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} \rho\mathbf{v} \\ \rho\mathbf{v} \otimes \mathbf{v} + \mathcal{I}p \\ \mathbf{v}(\rho E + p) \end{bmatrix}, \qquad \mathbf{A} = \frac{\partial \mathbf{F}}{\partial U}$$

Equation of state (here for an ideal gas)

$$p = (\gamma - 1)\left(\rho E - \frac{1}{2}\rho\|\mathbf{v}\|^2\right), \quad \gamma = C_p/C_v$$

[a] Similar formulations exist for primitive and entropy variables

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$$U = \begin{bmatrix} u_1 \\ \vdots \\ u_{d+2} \end{bmatrix}, \qquad \mathbf{F} = \begin{bmatrix} f_1^1 & \cdots & f_1^d \\ \vdots & \ddots & \vdots \\ f_{d+2}^1 & \cdots & f_{d+2}^d \end{bmatrix}, \qquad \mathbf{A} = \frac{\partial \mathbf{F}}{\partial U}$$

Notation

$$\mathbf{f}_k = \left[f_k^1, \ldots, f_k^d \right], \qquad \mathbf{f}^l = \begin{bmatrix} f_1^l \\ \vdots \\ f_{d+2}^l \end{bmatrix}$$

[a] Similar formulations exist for primitive and entropy variables

Galerkin ansatz

Find solution U at fixed time t s.t. for all W

$$\int_\Omega WU_{,t} - \nabla W \cdot \mathbf{F}(U)\,\mathrm{d}\Omega + \int_\Gamma WF^b(U,\cdot)\,\mathrm{d}s = 0$$

with boundary fluxes

$$F^b = \begin{cases} [0, pn_1, pn_2, pn_3, 0]^\top & \text{at solid walls} \\ \frac{1}{2}(F_n(U_-) + F_n(U_+)) - \frac{1}{2}|A_n(\mathrm{Roe}(U_-, U_+))| & \text{otherwise} \end{cases}$$

Fletcher's group formulation[3]

$$U_h = \sum_j B_j(\mathbf{x})U_j(t), \quad \mathbf{F}_h = \sum_j B_j(\mathbf{x})\mathbf{F}_j(t), \quad \mathbf{F}_j = \mathbf{F}(U_j)$$

[3] Fletcher, CMAME 37 (1983) 225–244.

SpMV-representation

Semi-discretized problem

$$\begin{bmatrix} M & & \\ & \ddots & \\ & & M \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \vdots \\ \dot{u}_{d+2} \end{bmatrix} - \begin{bmatrix} \mathbf{C} & & \\ & \ddots & \\ & & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^\top \\ \vdots \\ \mathbf{f}_{d+2}^\top \end{bmatrix} + \begin{bmatrix} \mathbf{S} & & \\ & \ddots & \\ & & \mathbf{S} \end{bmatrix} \begin{bmatrix} \mathbf{f}_1^{b\top} \\ \vdots \\ \mathbf{f}_{d+2}^{b\top} \end{bmatrix} = 0$$

Constant coefficient matrices

$$M = \left[\int_\Omega B_i B_j \,\mathrm{d}\Omega\right]_{i,j} \quad \mathbf{C} = \left[\int_\Omega \nabla B_i B_j \,\mathrm{d}\Omega\right]_{i,j} \quad \mathbf{S} = \left[\int_\Gamma B_i B_j \mathbf{n} \,\mathrm{d}s\right]_{i,j}$$

Stabilization of convective term by **Algebraic Flux Correction**[4]

[4] Kuzmin, MM, Gurris, AFC II. In: Flux-Corrected Transport, Springer, 2012

Linearized FCT[5]

1. **Perform variational mass lumping** $M \to M_l = \mathrm{diag}\{m_i\}$

$$m_i := \int_\Omega \varphi_i(\mathbf{x})\,\mathrm{d}\Omega = \int_{\hat{\Omega}} \hat{B}_i(\boldsymbol{\xi})\,|\det \mathrm{D}\hat{G}|\mathrm{d}\boldsymbol{\xi} > 0$$

Since

- B-spline basis functions satisfy PU property $\sum_j \hat{B}_j(\xi) \equiv 1$
- $\det \mathrm{D}\hat{G} > \mathrm{tol} > 0$ by design of the grid generation algorithm
- B-spline basis functions $\hat{\varphi}_i(\xi) > 0$ over their entire support

[5] Kuzmin, MM, Shadid, Shashkov, JCP 229 (2010) 8766-8779.

Linearized FCT[5]

1. Perform variational mass lumping $M \to M_l = \mathrm{diag}\{m_i\}$
2. **Eliminate all negative eigenvalues from the flux-Jacobian**

$$R_i^{\text{high}} := \sum_j \mathbf{c}_{ij} \cdot \mathbf{F}_j = \sum_{j \neq i} \mathbf{e}_{ij} \cdot \mathbf{A}_{ij}^{\text{Roe}} (U_j - U_i), \quad \mathbf{e}_{ij} = 0.5(\mathbf{c}_{ij} - \mathbf{c}_{ji})$$

by adding artificial viscosities $D_{ij} := \|\mathbf{e}_{ij}\| \, R_{ij} \, |\Lambda_{ij}| \, R_{ij}^{-1}$

$$R_i^{\text{low}} := \sum_{j \neq i} [\mathbf{e}_{ij} \cdot \mathbf{A}_{ij}^{\text{Roe}} + D_{ij}](U_j - U_i)$$

[5] Kuzmin, MM, Shadid, Shashkov, JCP 229 (2010) 8766-8779.

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2. Eliminate all negative eigenvalues from the flux-Jacobian
3. **Solve $M_l \dot{U}^{\mathrm{low}} = R^{\mathrm{low}} + S^{\mathrm{low}}$ by an explicit SSP-RK method**

$$M_l U^{(1)} = M_l U^n + \Delta t [R^n + S^n]$$
$$M_l U^{\mathrm{low}} = \frac{1}{2} M_l U^n + \frac{1}{2} \left(M_l U^{(1)} + \Delta t [R^{(1)} + S^{(1)}] \right)$$

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3. Solve $M_l \dot{U}^{\mathrm{low}} = R^{\mathrm{low}} + S^{\mathrm{low}}$ by an explicit SSP-RK method
4. **Linearize antidiffusive fluxes about the low-order predictor**

$$F_{ij} := m_{ij}(\dot{U}_i^{\mathrm{low}} - \dot{U}_j^{\mathrm{low}}) + D_{ij}^{\mathrm{low}}(U_i^{\mathrm{low}} - U_j^{\mathrm{low}})$$

[5] Kuzmin, MM, Shadid, Shashkov, JCP 229 (2010) 8766-8779.

Linearized FCT[5]

1. Perform variational mass lumping $M \to M_l = \mathrm{diag}\{m_i\}$
2. Eliminate all negative eigenvalues from the flux-Jacobian
3. Solve $M_l \dot{U}^{\mathrm{low}} = R^{\mathrm{low}} + S^{\mathrm{low}}$ by an explicit SSP-RK method
4. Linearize antidiffusive fluxes about the low-order predictor
5. **Apply a generalized version of Zalesak's multidimensional 'nodal' flux limiter to scalar indicator variables ρ and p ($p > 0 \Rightarrow \rho E > 0$) and add a limited antidiffusive correction**

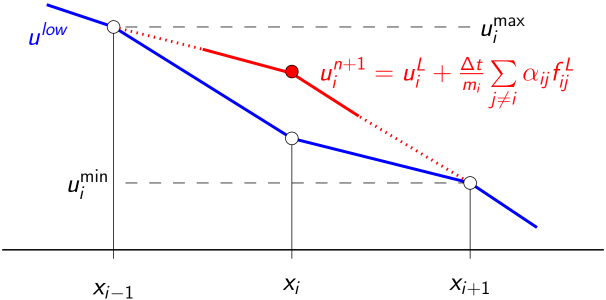
$$U_i^{n+1} = U_i^{\mathrm{low}} + \frac{\Delta t}{m_i} \sum_{j \neq i} \min\{\alpha_{ij}^{\rho}, \alpha_{ij}^{p}\} F_{ij}$$

[5] Kuzmin, MM, Shadid, Shashkov, JCP 229 (2010) 8766-8779.

Zalesak's nodal FCT limiter[6]

'Nodal' flux limiter for $u \in \{\rho, p\}$ is designed to ensure that for all i

$$\min_{j:m_{ij}\neq 0} u_j^{\mathrm{low}} =: u_i^{\min} \leq u_i^{n+1} \leq u_i^{\max} := \max_{j:m_{ij}\neq 0} u_j^{\mathrm{low}}$$



[6] S. Zalesak, JCP 1979, 31(3), pp. 33 5–362

Positivity proof for B-spline based IgA

Theorem

When the same constraints are imposed on the *weights* of the B-spline function then the end-of-step solution stays within the upper and lower bounds set up by the low-order predictor.

<u>Proof</u>: Note that it is not the coefficients $u_i^{\overset{\max}{\min}}$ that are the local bounds (as in nodal FEM) but the functions $u^{\overset{max}{\min}}(\mathbf{x}) = \sum_j u_j^{\overset{\max}{\min}} \varphi(\mathbf{x})$.

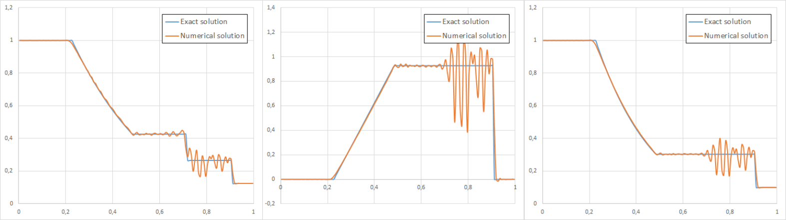
Assume that for some $\mathbf{x}^* \in \Omega$ we have $u^{n+1}(\mathbf{x}^*) > u^{\max}(\mathbf{x}^*)$:

$$0 > u^{\max}(\mathbf{x}^*) - u^{n+1}(\mathbf{x}^*) = \sum_j \underbrace{[u_j^{\max} - u_j^{n+1}]}_{>0} \underbrace{\varphi_j(\mathbf{x}^*)}_{>0} > 0$$

q.e.d.

Numerical examples

Sod's shock tube problem[7]



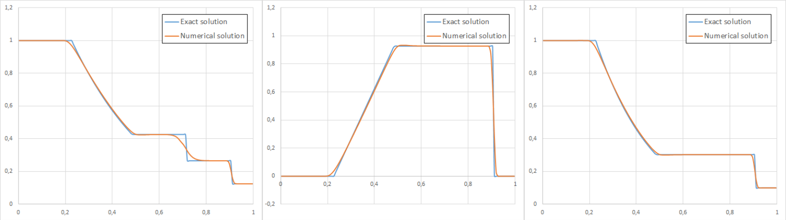
ρ v_x p

Quadratic bi-variate B-spline basis functions with expl. SSP-RK(2).

[7] G.A. Sod, JCP 27 (1978) 1–31.

Numerical examples

Sod's shock tube problem[7]



ρ v_x p

Quadratic bi-variate B-spline basis functions with expl. SSP-RK(2).

[7] G.A. Sod, JCP 27 (1978) 1–31.

Overview

Meta-programming techniques for heterogeneous HPC

Strategy: Implement **one** device-independent single-patch compute kernel and just-in-time compile it into hardware-, formulation- and algorithm-optimized dynamically loadable libraries per patch

Computational building blocks

- Open-Source IGA library G+Smo[8] (JKU, RICAM)
- 3rd party linear algebra libraries: ArrayFire, Blaze, Eigen, VexCL, ...
- Fluid Dynamics Building Blocks expression templates library[9]

---

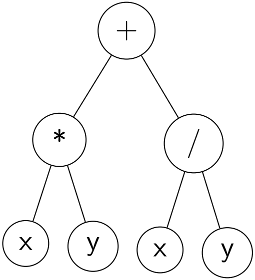
[8] `https://gs.jku.at/trac/gismo`
[9] `https://gitlab.com/mmoelle1/FDBB`

Expression templates

Code that you write

```
vex::vector<float> x,y,z;
auto expr = x*y + x/y;

z   = expr;
```



All arithmetic operations are kept symbolically as expression tree whose evaluation is delayed until the assignment to the vector z. All arithmetic operation are then fused into a single evaluation (=loop).

Expression templates, cont'd

Compute kernel from VexCL

```
kernel void vexcl_vector_kernel(...)
{
  for(size_t idx = get_global_id(0);
             idx < n;
             idx += get_global_size(0))
  {
    prm_1[idx] = prm_2[idx] * prm_3[idx]
               + prm_2[idx] / prm_3[idx];
  }
}
```

Fluid Dynamics Building Blocks

High-level

ET's for conservative/primitive variables, EOS, inviscid/ viscous fluxes, flux Jacobians, and Riemann invariants

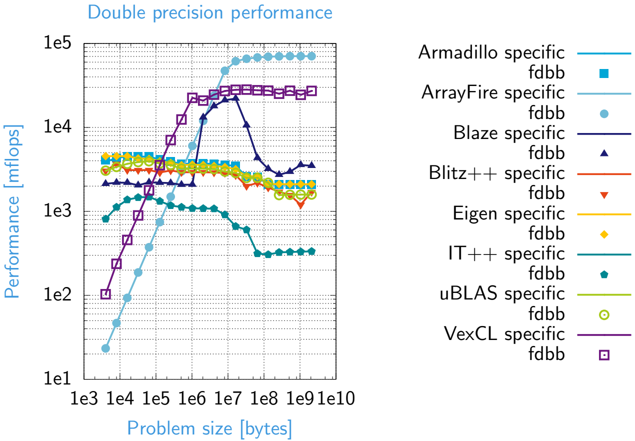
Low-level

Unified wrapper function API to core functionality of ETL's: make_temp, tag, tie, +, -, *, /, abs, sqrt, ...

Armadillo	ArrayFire	Blaze	Blitz++	Eigen	IT++	MTL4	uBLAS	VexCL	ViennaCL	...

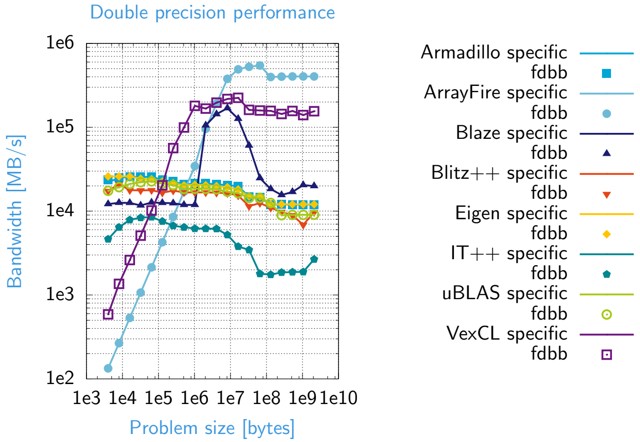
FDBB μ-benchmark

$$y \leftarrow (m_x .* m_x + m_y .* m_y + m_z .* m_z)./(\rho .* \rho) \qquad \text{7 flop}$$



FDBB μ-benchmark

$$y \leftarrow (m_x .* m_x + m_y .* m_y + m_z .* m_z)./(\rho .* \rho) \qquad \text{7 flop}$$



Factory method design pattern

Code example: *single-patch inviscid fluxes*

```
1 using var = Variables<eos::idealGas<T>, dim,
2                                EnumForm::conservative>;
3 using fty = Factory<config,device>;
4 auto    U = fty::stateVector<var>();
5 auto   dF = fty::inviscidFluxes<var>();
```

1-2 The Variable typedef specifies all internals of the formulation like the equation of state, the spatial dimension, and the type of state vector variables (thereby defining fluxes and flux-Jacobians)

Factory method design pattern

Code example: *single-patch inviscid fluxes*

```
1  using var = Variables<eos::idealGas<T>, dim,
2                           EnumForm::conservative>;
3  using fty = Factory<config,device>;
4  auto    U = fty::stateVector<var>();
5  auto   dF = fty::inviscidFluxes<var>();
```

3 The Factory typedef selects optimal LA backend (Eigen, VexCL), data structures (SoA, AoS) and matrix formats (CSR, ELL, Band) for the particular device and problem config.

Factory method design pattern

Code example: *single-patch inviscid fluxes*

```
1 using var = Variables<eos::idealGas<T>, dim,
2                                    EnumForm::conservative>;
3 using fty = Factory<config,device>;
4 auto    U = fty::stateVector<var>();
5 auto   dF = fty::inviscidFluxes<var>();
```

4-5 Vector and operator objects are created form the Factory typedef delegating memory allocation, etc. to the underlying LA backend

Factory method design pattern

Code example: *single-patch inviscid fluxes*

```
using var = Variables<eos::idealGas<T>, dim,
                      EnumForm::conservative>;
using fty = Factory<config,device>;
auto    U = fty::stateVector<var>();
auto   dF = fty::inviscidFluxes<var>();
```

The operator* is overloaded such that dF*U unfolds to the SpMV representation of the discretized $\nabla \cdot \mathbf{F}(U)$ term, whereby fluxes, eos, etc. are implemented as lightweight expression templates in FDBB.

Factory method design pattern

Code example: *single-patch inviscid fluxes*

```
using var = Variables<eos::idealGas<T>, dim,
                      EnumForm::conservative>;
using fty = Factory<config,device>;
auto     U = fty::stateVector<var>();
auto    dF = fty::invicidFluxes<var>();
```

Is it worth the effort? Yes, because you can tune the implementation 'behind the scenes', e.g., by switching from AoS to SoA, i.e.

```
SpMatrix<T> C[dim];     --> SpMatrix<CST<T,1,dim> >     C;
  Vector<T> U[dim+2];   -->   Vector<CST<T,dim+2,1> > U;
```

to save memory and computing time by factor $O(\mathrm{dim})$.

Topics discussed:

- Parameterizations for screw machines with variable pitch
- Positivity-preserving IGA-solver for compressible flows
- Efficient implementation for heterogeneous HPC systems

Future work:

- THB-splines for solution-adapted parameterizations

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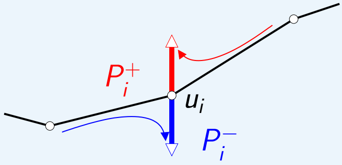
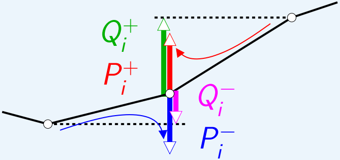
Zalesak's flux limiter[10]



- Consider sums of positive/negative antidiffusive fluxes into each node

[10] S. Zalesak, JCP 1979, 31(3), pp. 33 5–362

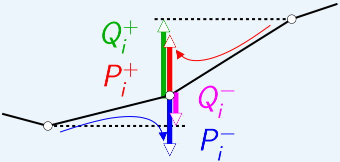
Zalesak's flux limiter[10]



- Consider sums of positive/negative antidiffusive fluxes into each node
- Limit antidiffusive flux if it exceeds the distance to upper/lower bounds

[10] S. Zalesak, JCP 1979, 31(3), pp. 33 5–362

Zalesak's flux limiter[10]



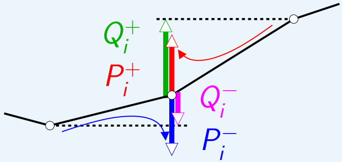
- Consider sums of positive/negative antidiffusive fluxes into each node
- Limit antidiffusive flux if it exceeds the distance to upper/lower bounds

- Compute nodal correction factors

$$R_i^+ = \min\{1, Q_i^+/P_i^+\} \quad \text{and} \quad R_i^- = \min\{1, Q_i^-/P_i^-\}$$

[10] S. Zalesak, JCP 1979, 31(3), pp. 33 5–362

Zalesak's flux limiter[10]



- Consider sums of positive/negative antidiffusive fluxes into each node
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- Compute nodal correction factors

$$R_i^+ = \min\{1, Q_i^+/P_i^+\} \quad \text{and} \quad R_i^- = \min\{1, Q_i^-/P_i^-\}$$

- Limit antidiffusive flux for edge ij by

$$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\} & \text{for positive fluxes} \\ \min\{R_i^-, R_j^+\} & \text{for negative fluxes} \end{cases}$$

[10] S. Zalesak, JCP 1979, 31(3), pp. 33 5–362

Extended version of Zalesak's FCT limiter

Input: predictor u^L and antidiffusive fluxes f_{ij}^u, where $f_{ji}^u \neq f_{ij}^u$

1. Sums of positive/negative antidiffusive fluxes into node i

$$P_i^+ = \sum_{j \neq i} \max\{0, f_{ij}^u\}, \qquad P_i^- = \sum_{j \neq i} \min\{0, f_{ij}^u\}$$

2. Upper/lower bounds based on the local extrema of u^L

$$Q_i^+ = m_i(u_i^{\max} - u_i^L), \qquad Q_i^- = m_i(u_i^{\min} - u_i^L)$$

3. Correction factors $\alpha_{ij}^u = \alpha_{ji}^u$ to satisfy the FCT constraints

$$\alpha_{ij}^u = \min\{R_{ij}, R_{ji}\}, \quad R_{ij} = \begin{cases} \min\{1, Q_i^+/P_i^+\} & \text{if } f_{ij}^u \geq 0 \\ \min\{1, Q_i^-/P_i^-\} & \text{if } f_{ij}^u < 0 \end{cases}$$

Node-based transformation of control variables[11]

- **Conservative variables:** density, momentum, total energy

$$U_i = [\rho_i, (\rho\mathbf{v})_i, (\rho E)_i], \qquad F_{ij} = \left[f_{ij}^{\rho}, \mathbf{f}_{ij}^{\rho v}, f_{ij}^{\rho E}\right], \qquad F_{ji} = -F_{ij}$$

- **Primitive variables** $V = TU$**:** density, velocity, pressure

$$V_i = [\rho_i, \mathbf{v}_i, p_i], \qquad \mathbf{v}_i = \frac{(\rho\mathbf{v})_i}{\rho_i}, \qquad p_i = (\gamma - 1)\left[(\rho E)_i - \frac{|(\rho\mathbf{v})_i|^2}{2\rho_i}\right]$$

$$G_{ij} = \left[f_{ij}^{\rho}, \mathbf{f}_{ij}^{v}, f_{ij}^{p}\right] = T(U_i)F_{ij}, \qquad T(U_j)F_{ji} = G_{ji} \neq -G_{ij}$$

- Raw antidiffusive fluxes for the velocity and pressure

$$\mathbf{f}_{ij}^{v} = \frac{\mathbf{f}_{ij}^{\rho v} - \mathbf{v}_i f_{ij}^{\rho}}{\rho_i}, \qquad f_{ij}^{p} = (\gamma - 1)\left[\frac{|\mathbf{v}_i|^2}{2} f_{ij}^{\rho} - \mathbf{v}_i \cdot \mathbf{f}_{ij}^{\rho v} + f_{ij}^{\rho E}\right]$$

[11] Kuzmin, MM, Shadid, Shashkov, JCP 229 (2010) 8766-8779.