technische universität dortmund

fakultät für mathematik

A Comparative Study of Conforming and Nonconforming High-Resolution Finite Element Schemes

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European Seminar on Computing
Pilsen, Czech Republic
June 25, 2012

Family of AFC schemes *Kuzmin et al.*

$$\mathbf{M_L}\dot{\mathbf{u}} = [\mathbf{K} + \mathbf{D}]\mathbf{u} + \mathbf{F}(\dot{\mathbf{u}}, \mathbf{u})$$

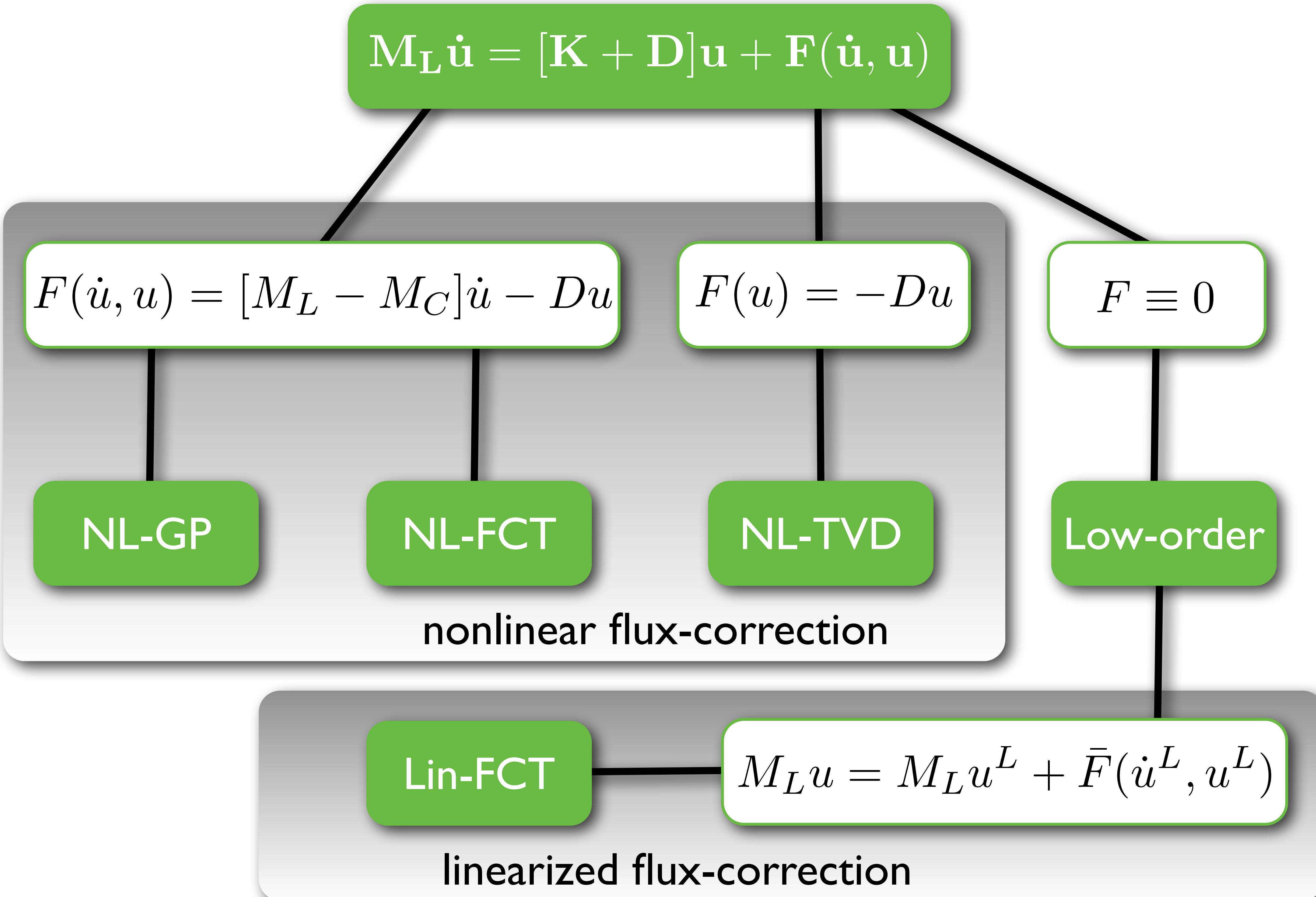
lumped mass matrix

antidiffusive correction

artificial diffusion operator

Family of AFC schemes

Kuzmin et al.



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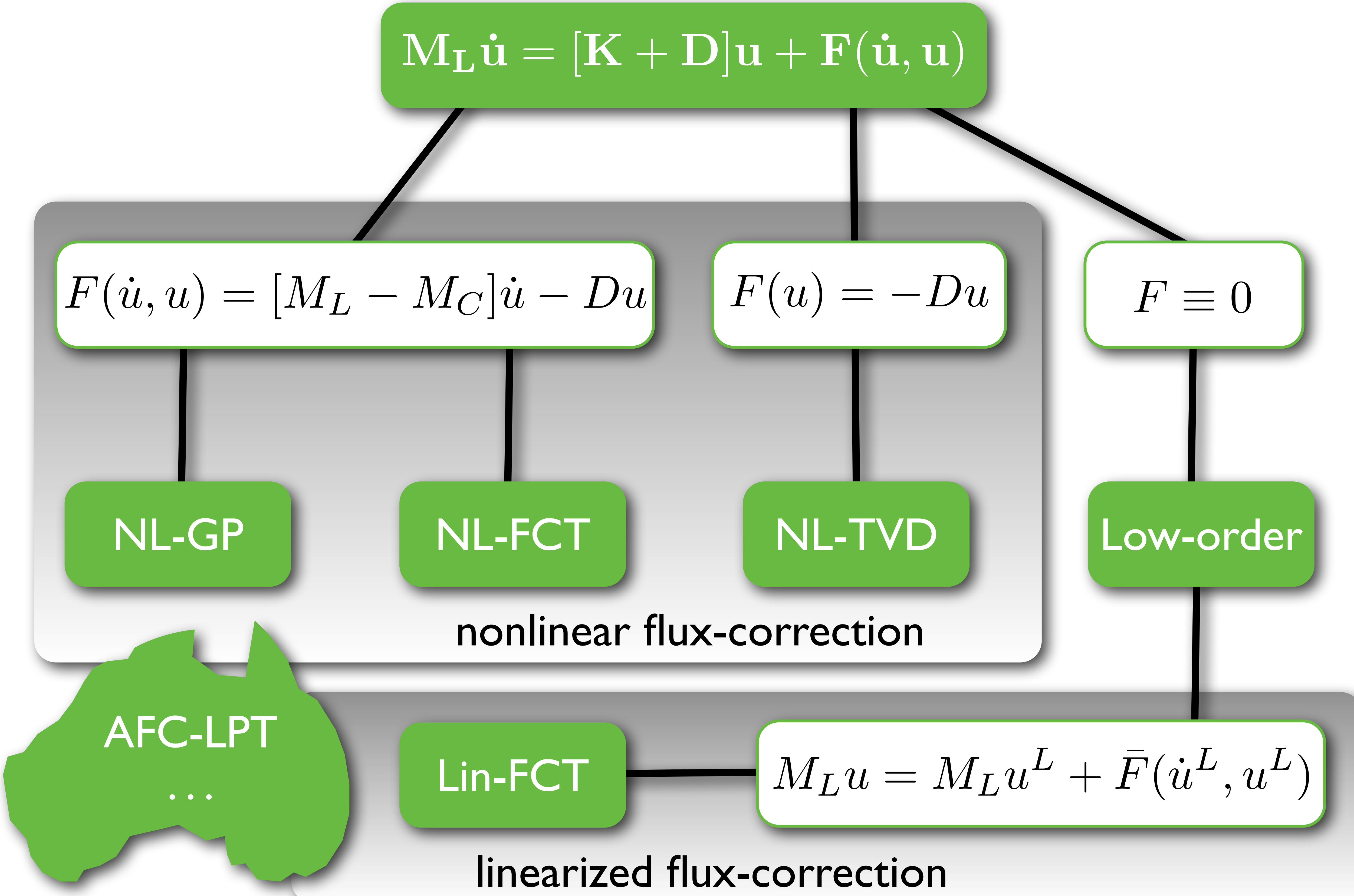

Review of design principles

- Jameson's Local Extremum Diminishing criterion

IF $$m_i \dot{u}_i = \sum_{j \neq i} \sigma_{ij} (u_j - u_i)$$

m_i: positive; σ_{ij}: not negative

THEN local solution maxima/minima do not increase/decrease

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- Semi-discrete high-resolution scheme

$$m_i \dot{u}_i = \sum_{j \neq i} (k_{ij} + d_{ij})(u_j - u_i) + \delta_i u_i + \sum_{j \neq i} \alpha_{ij} f_{ij}$$

$m_i = \sum_j m_{ij}$

$k_{ij} + d_{ij}$: not negative by construction of diffusion coefficient

$\alpha_{ij} f_{ij}$: controlled by flux limiter

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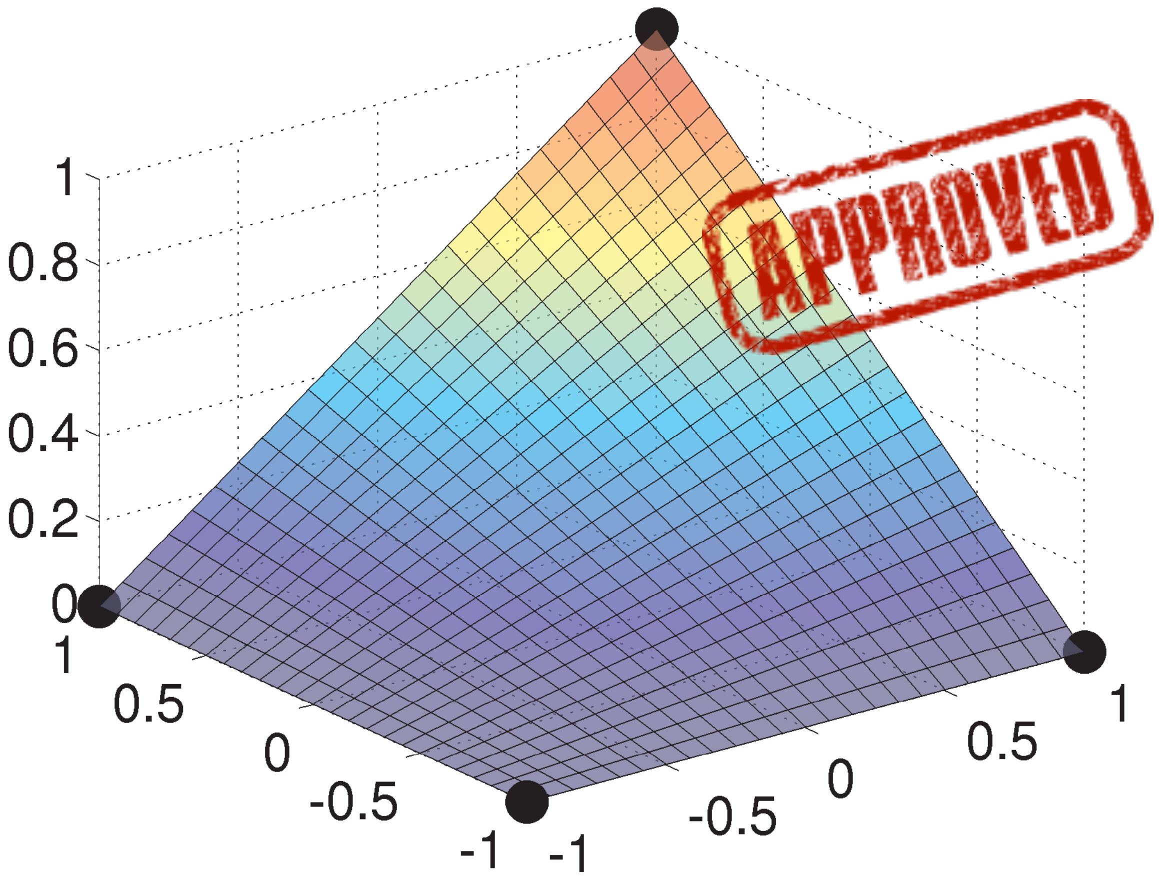
$$m_i = \sum_j m_{ij}$$

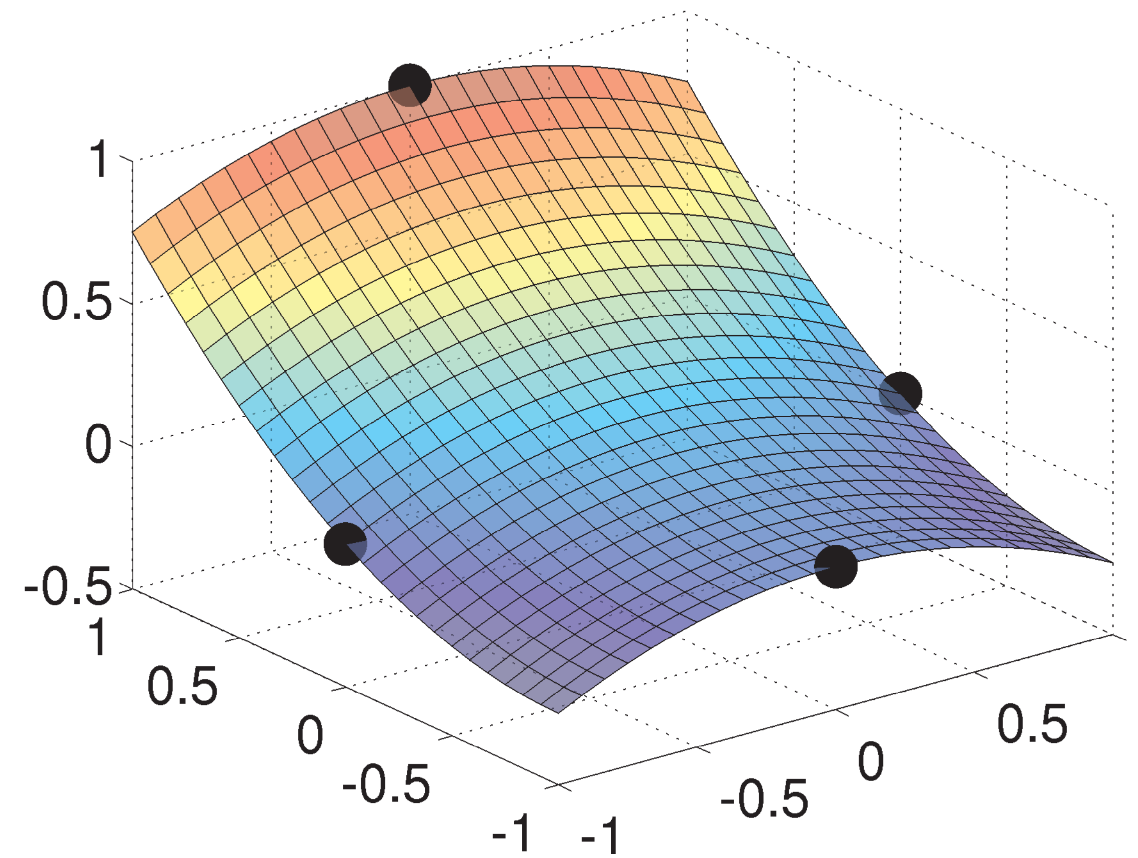
not negative by construction of diffusion coefficient

controlled by flux limiter

- Edge-based assembly of operators/vectors is feasible and efficient!

Finite Elements



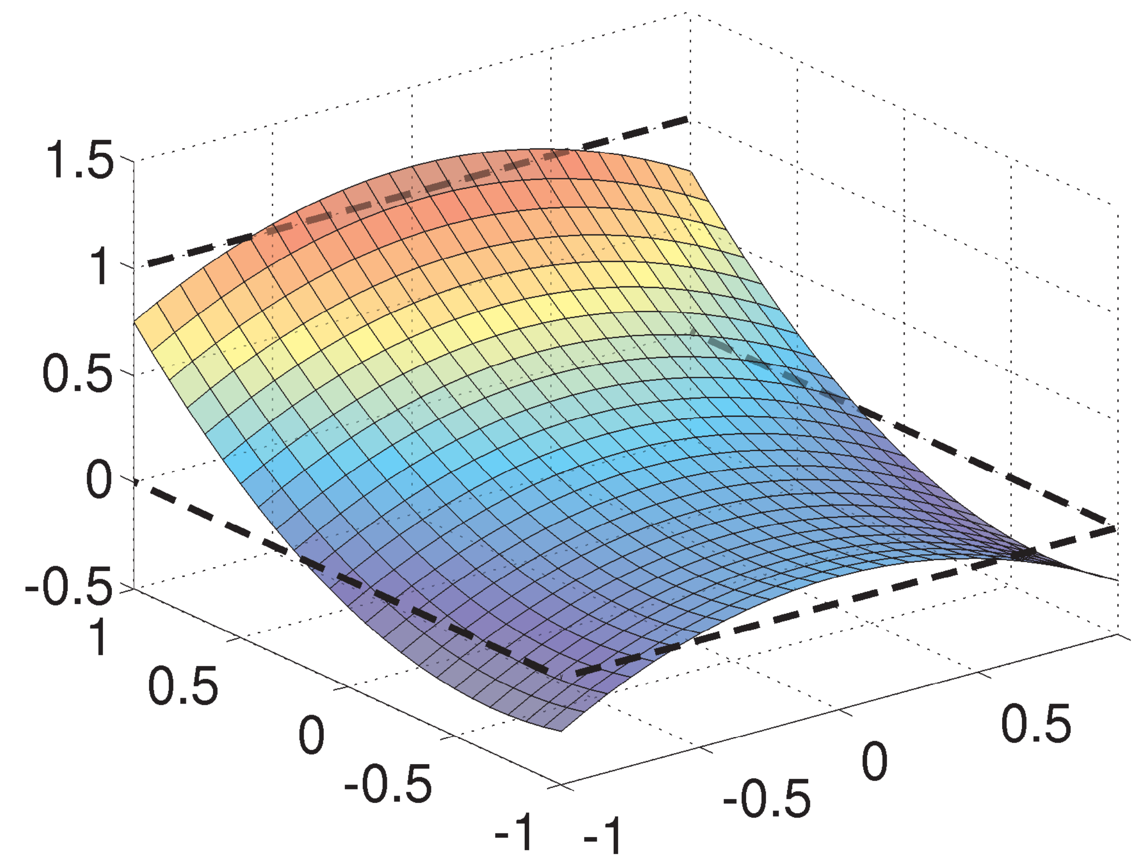


Bilinear Q_1 FE

- nodal values at cell vertices

Rotated bilinear $\tilde{Q}_1$ FE

- nodal values at cell midpoints
- integral mean-values at edges



Finite Element properties

- Uniform 128x128 grid of unit square with 5% stochastic disturbance

		$\triangle(A)$	M_C	M_L	NEQ	NA
	Q_1	8	✓	✓	16,641	82,680
mean value	$\sim Q_1^{par}$	6	✓	✓	33,024	140,483
	$\sim Q_1^{np}$	6	✓	✓	33,024	140,478
point value	$\sim Q_1^{par}$	6	-7.2E-07	(✓)	33,024	138,204
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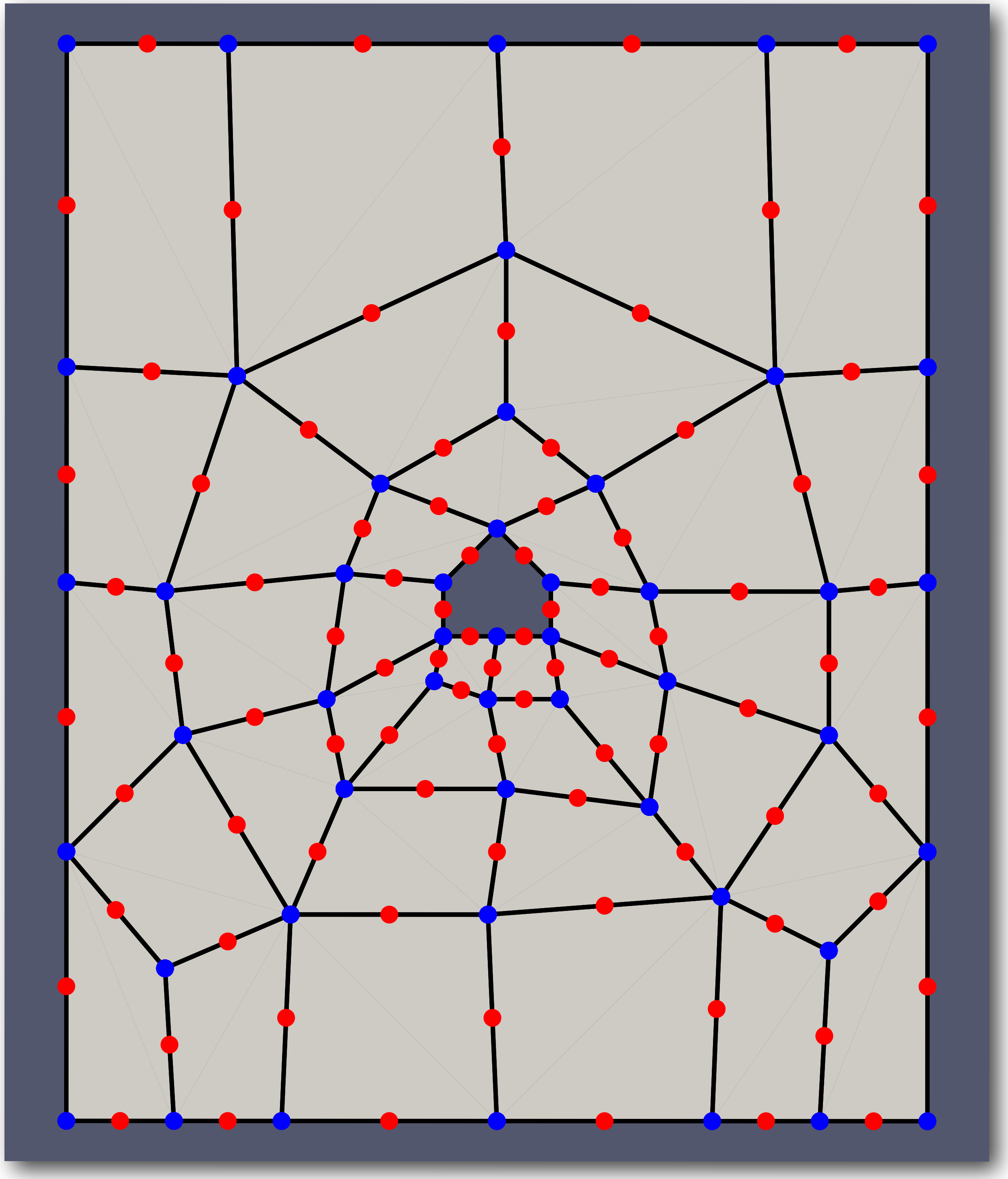
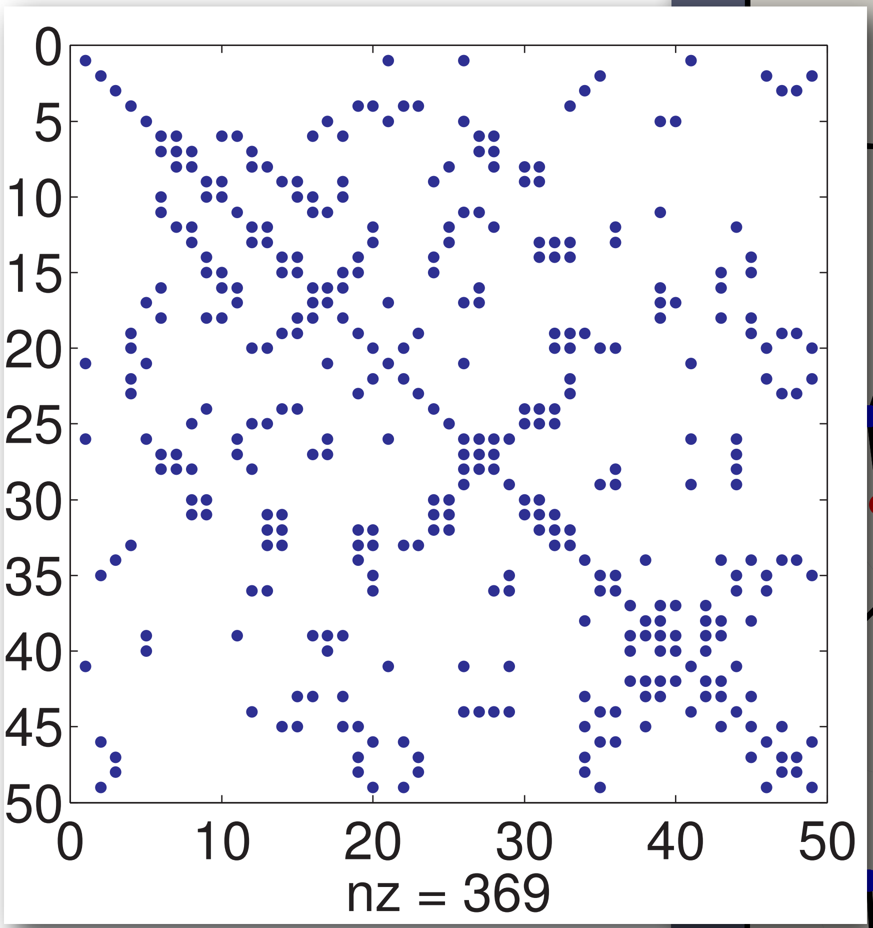
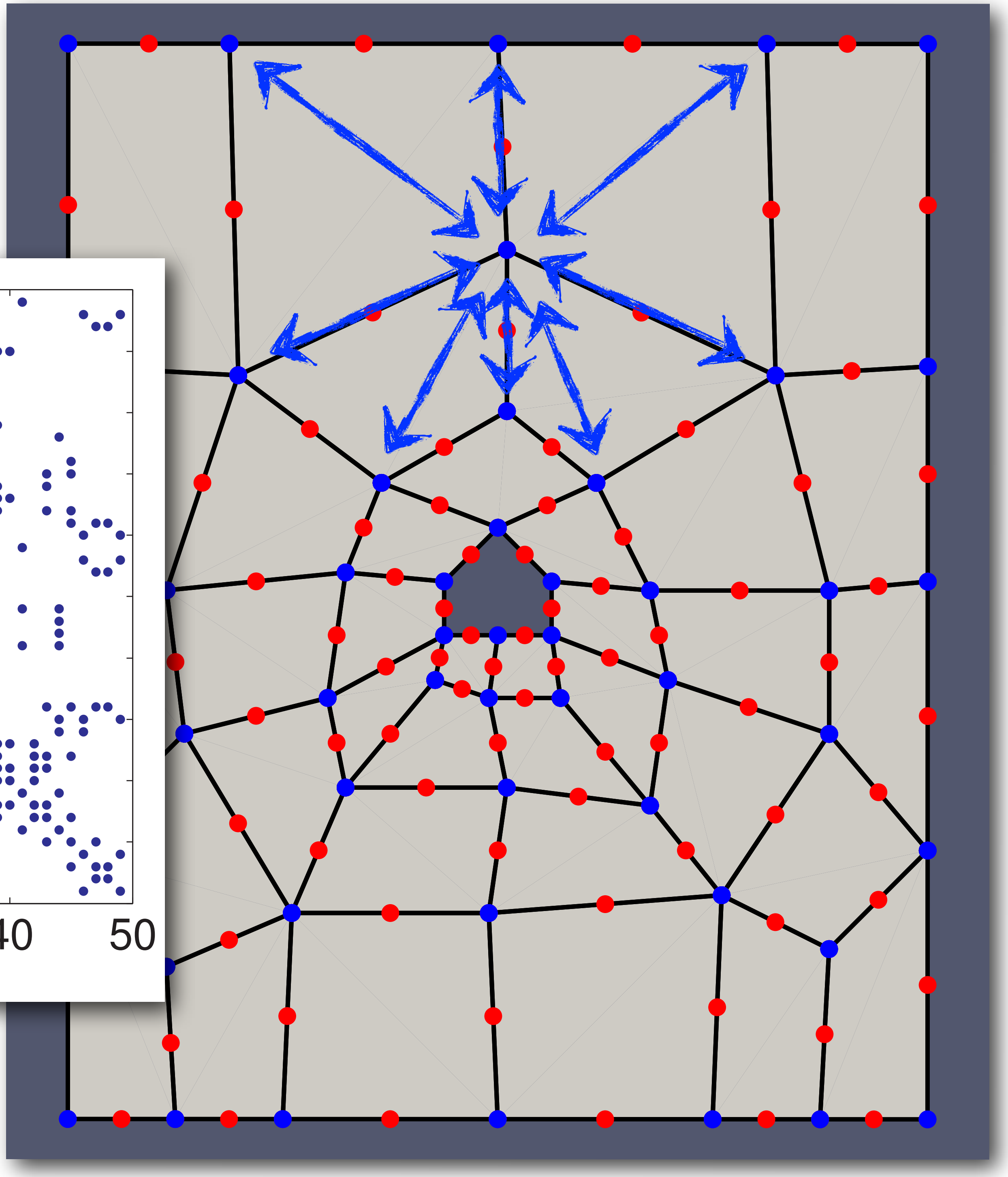


Finite Element properties

Q_1 FE





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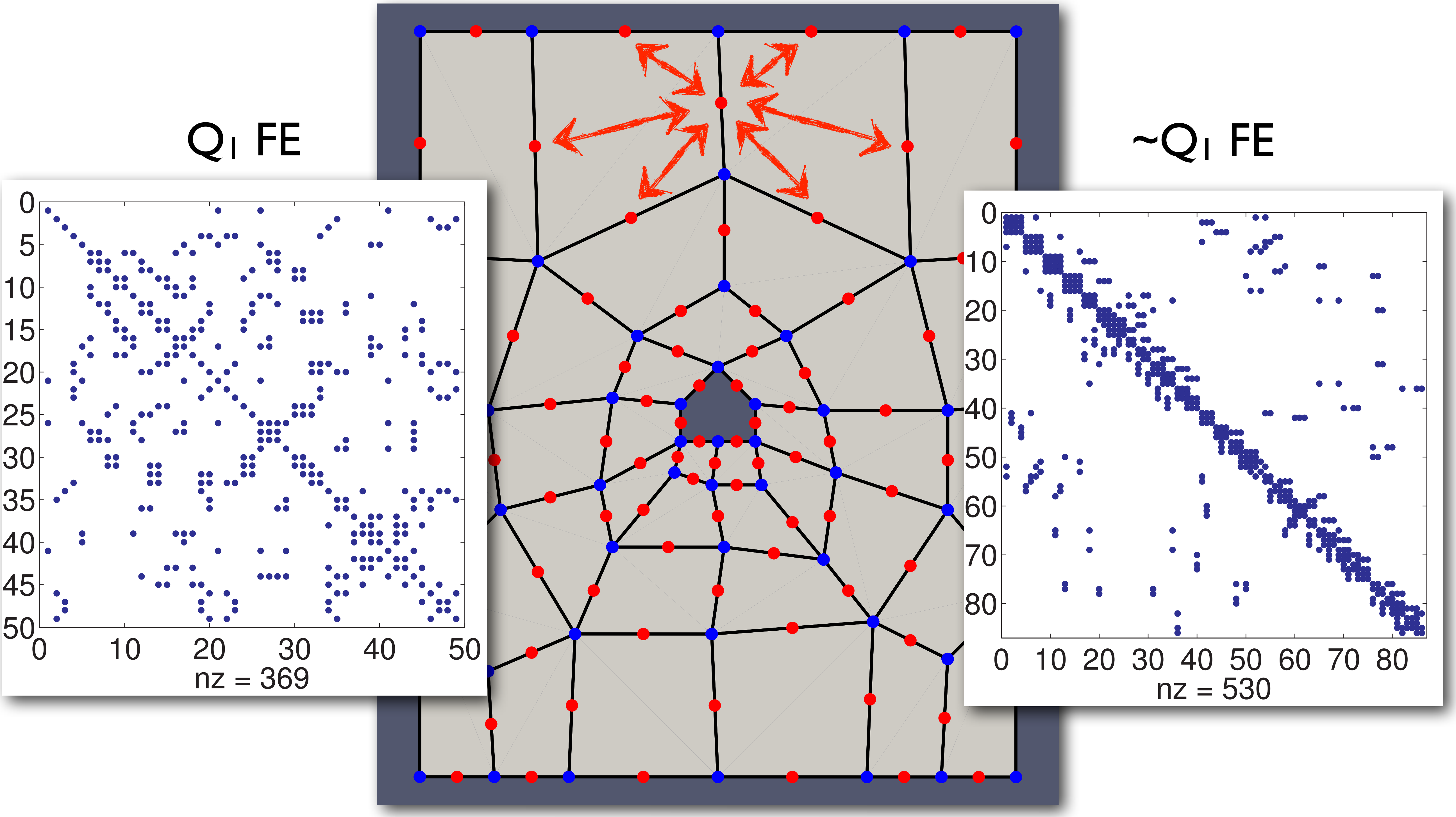


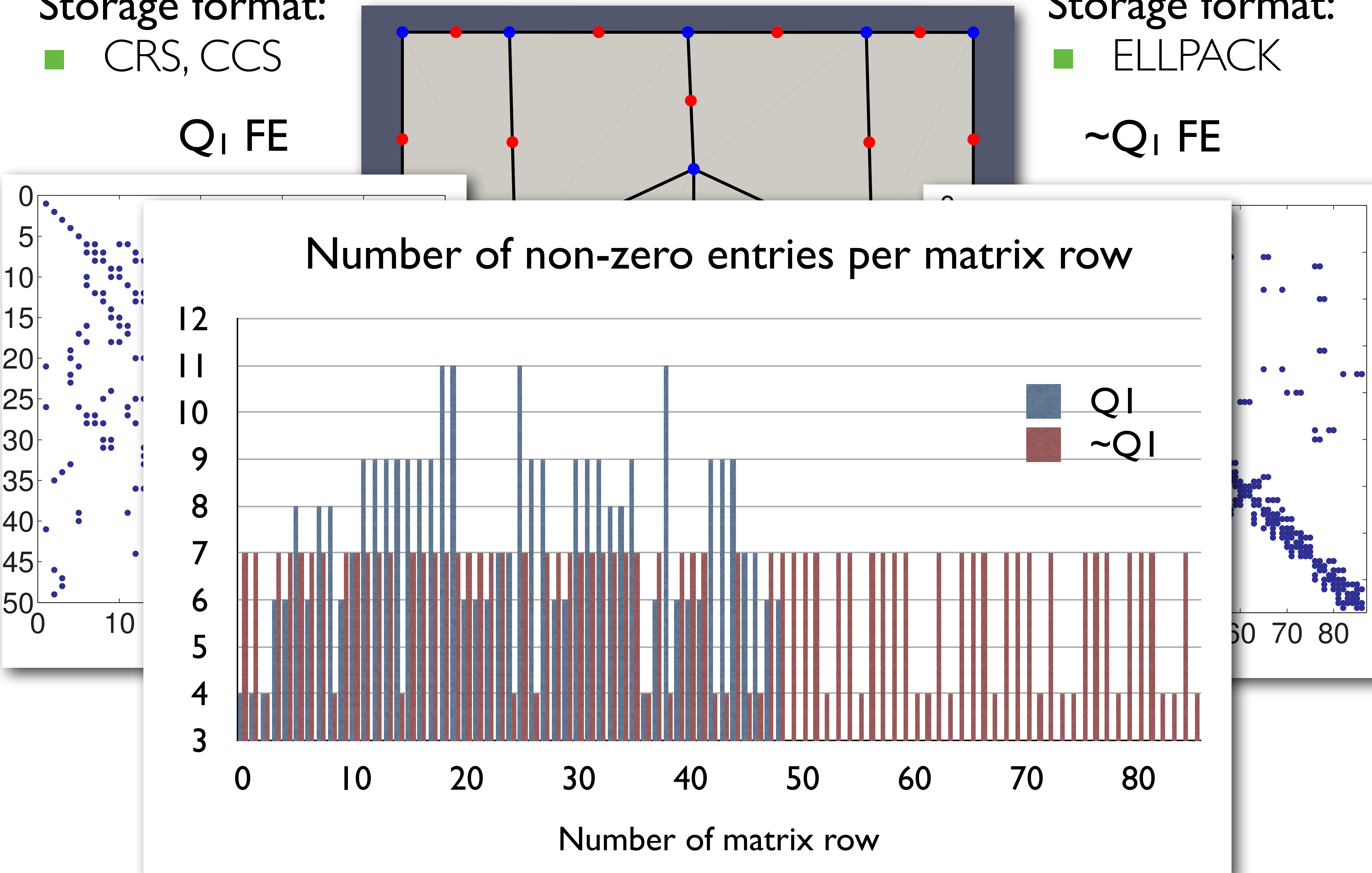
Finite Element properties

Storage format:
- CRS, CCS

Q_1 FE

Storage format:
- ELLPACK

$\sim Q_1$ FE

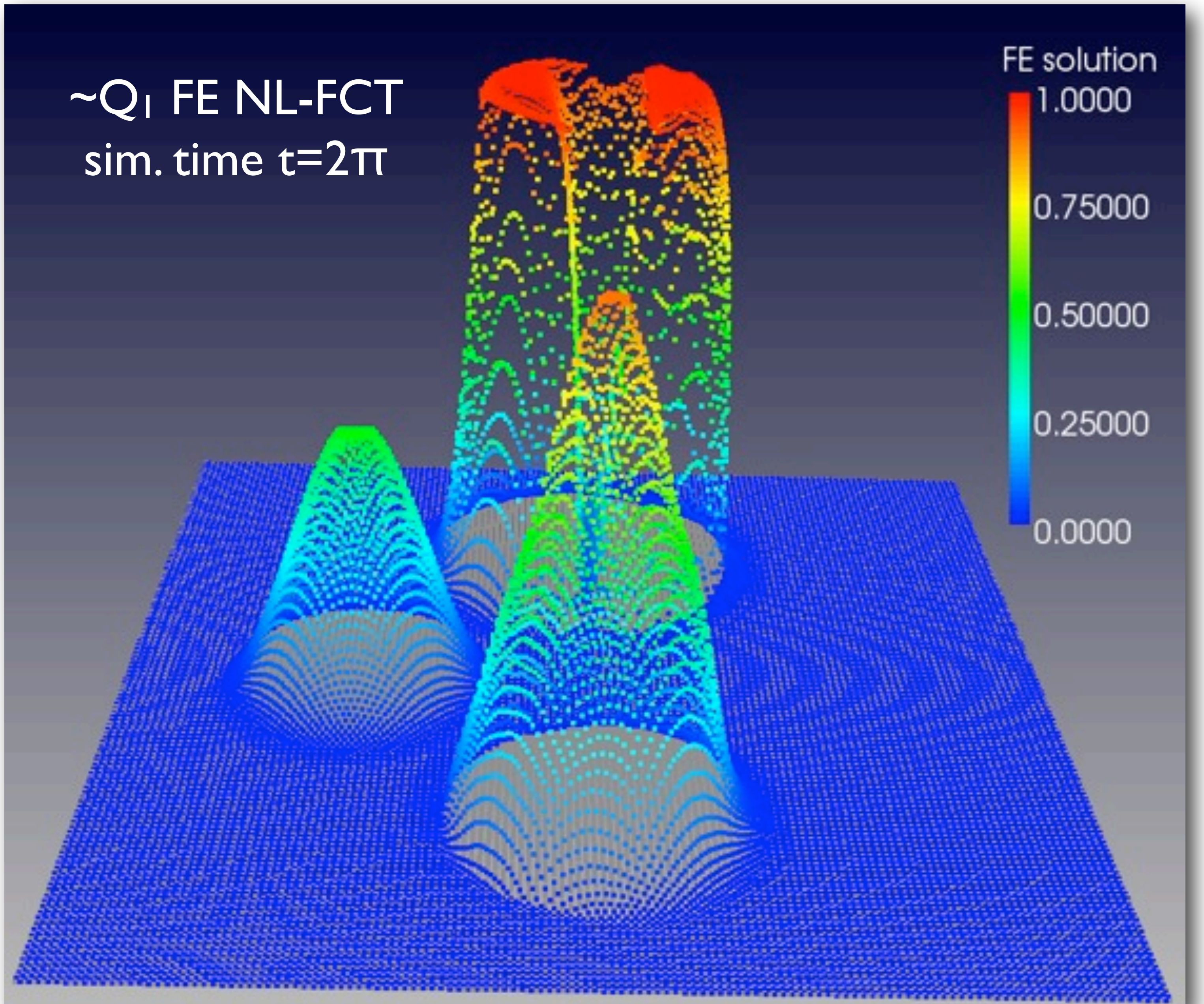


Solid body rotation

Pure convection problem

$$\dot{u} + \nabla \cdot (\mathbf{v}u) = 0 \ \text{ in } \Omega = (0,1)^2$$

$$u = 0 \ \text{ on } \Gamma_{\text{inflow}}$$



- Velocity field

  $\mathbf{v}(x,y) = (0.5 - y, x - 0.5)$

- Grid size

  $h = 1/2^l, l = 5, 6, \ldots$

- Stochastic grid disturbance

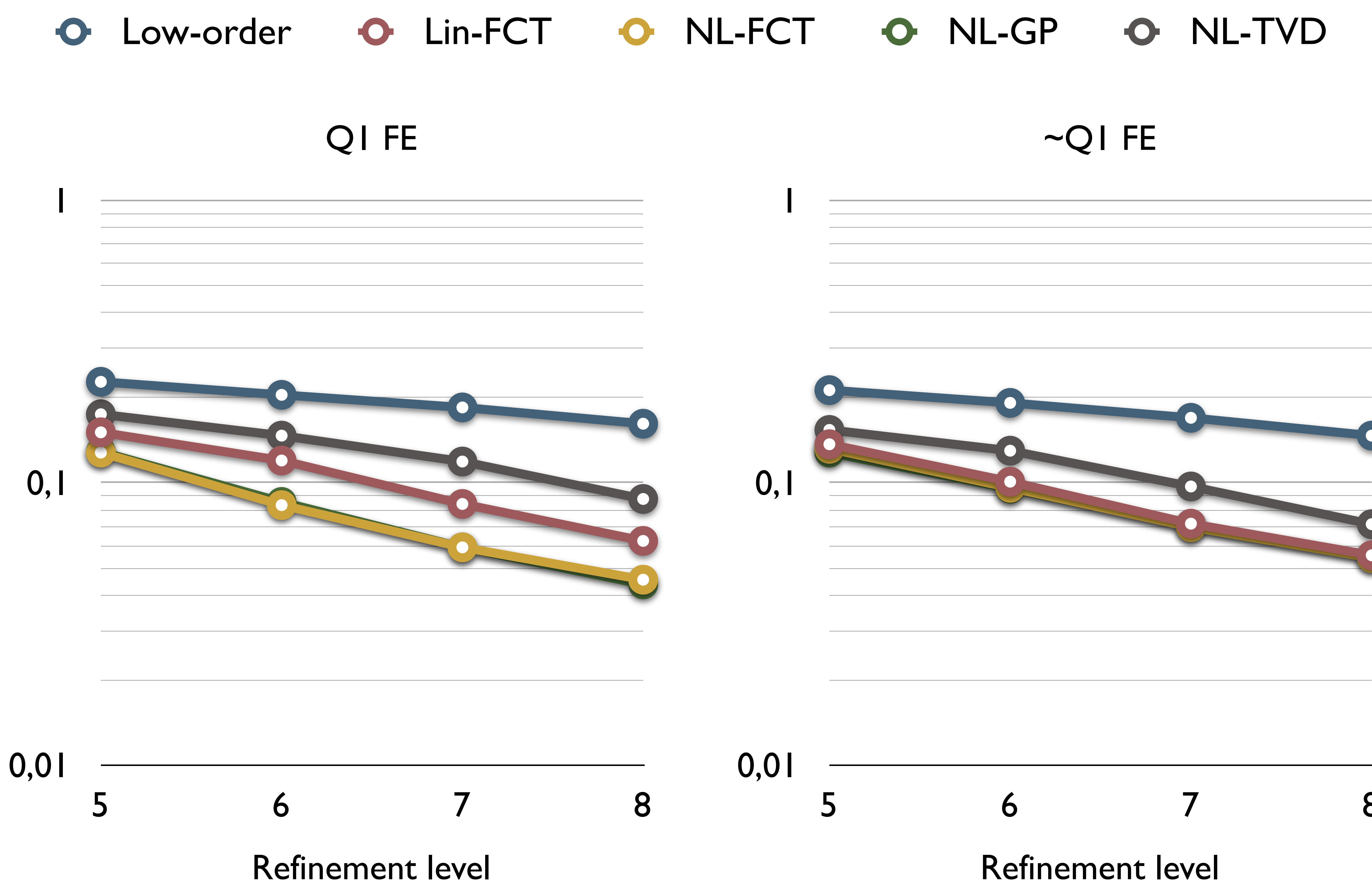
  $\delta \in \{0\%, 1\%, 5\%\}$

- Time step in Crank-Nicolson $\Delta t = 1.28 \cdot h$

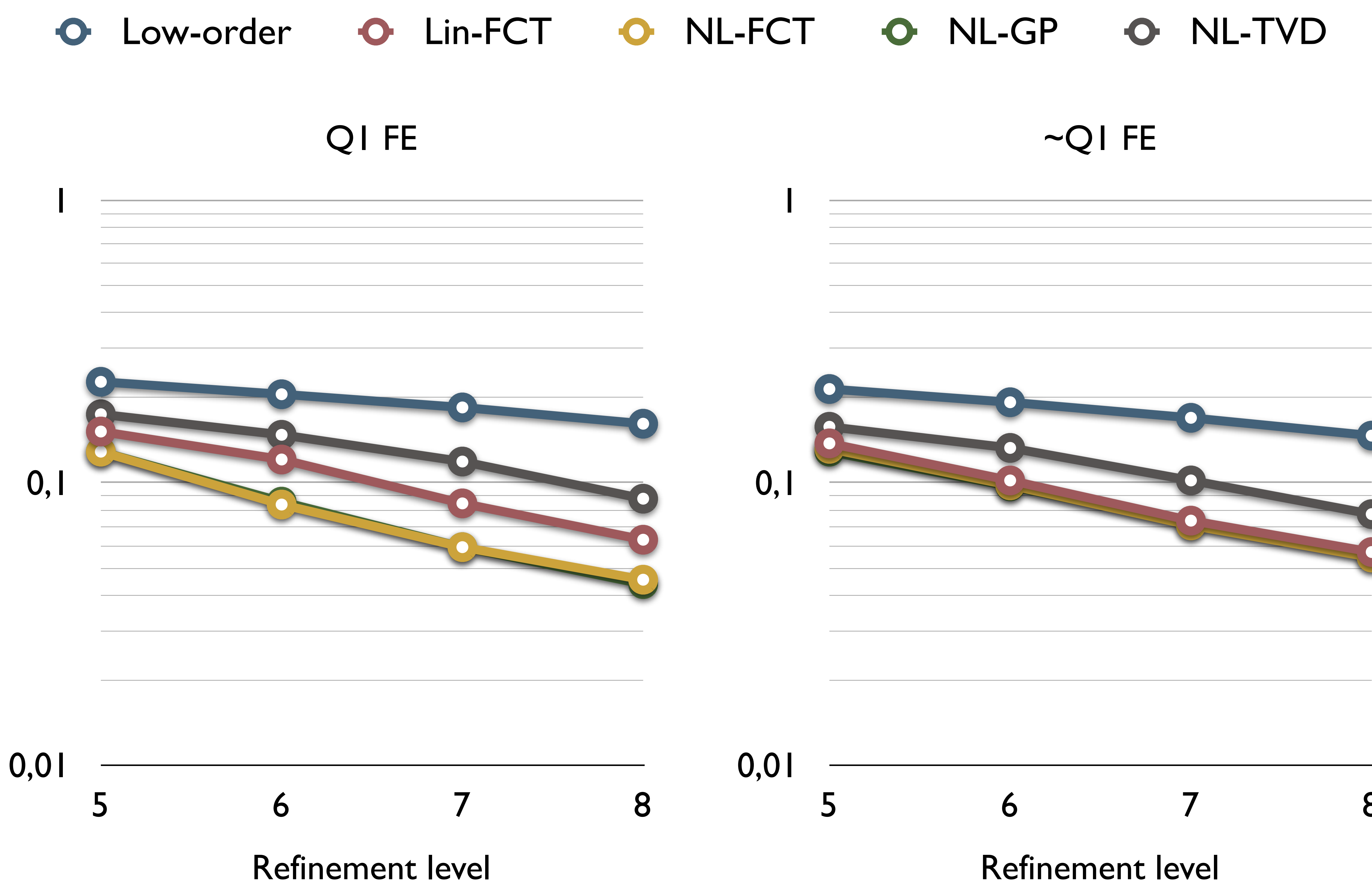
- Initial = exact solution at

  $t = 2\pi k,\ k \in \mathbb{N}$

SBR: *L_2-error (0% mesh disturbance)*



SBR: L_2-error (5% mesh disturbance)

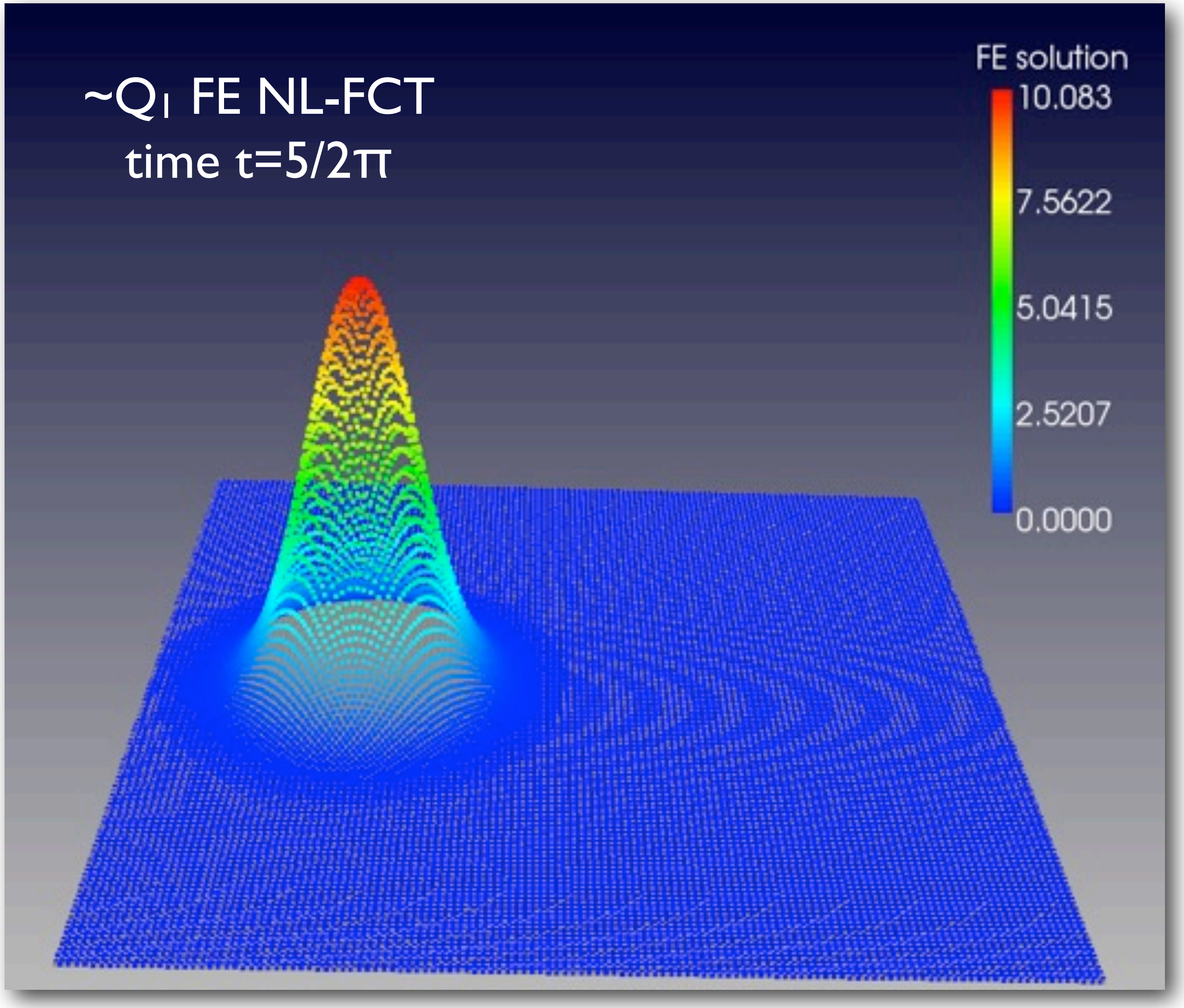


Rotation of a Gaussian hill

- Convection-diffusion equation

$$\dot{u} + \nabla \cdot (\mathbf{v}u - \epsilon \nabla u) = 0$$

$$\text{in } \Omega = (-1,1)^2$$



- Velocity field, diffusion coefficient

$$\mathbf{v}(x,y) = (-y, x), \quad \epsilon = 0.001$$

- Analytical solution

$$u(x,y,t) = \frac{1}{4\pi\epsilon t} e^{-\frac{r^2}{4\epsilon t}}$$

$$r^2 = (x - \hat{x})^2 + (y - \hat{y})^2$$

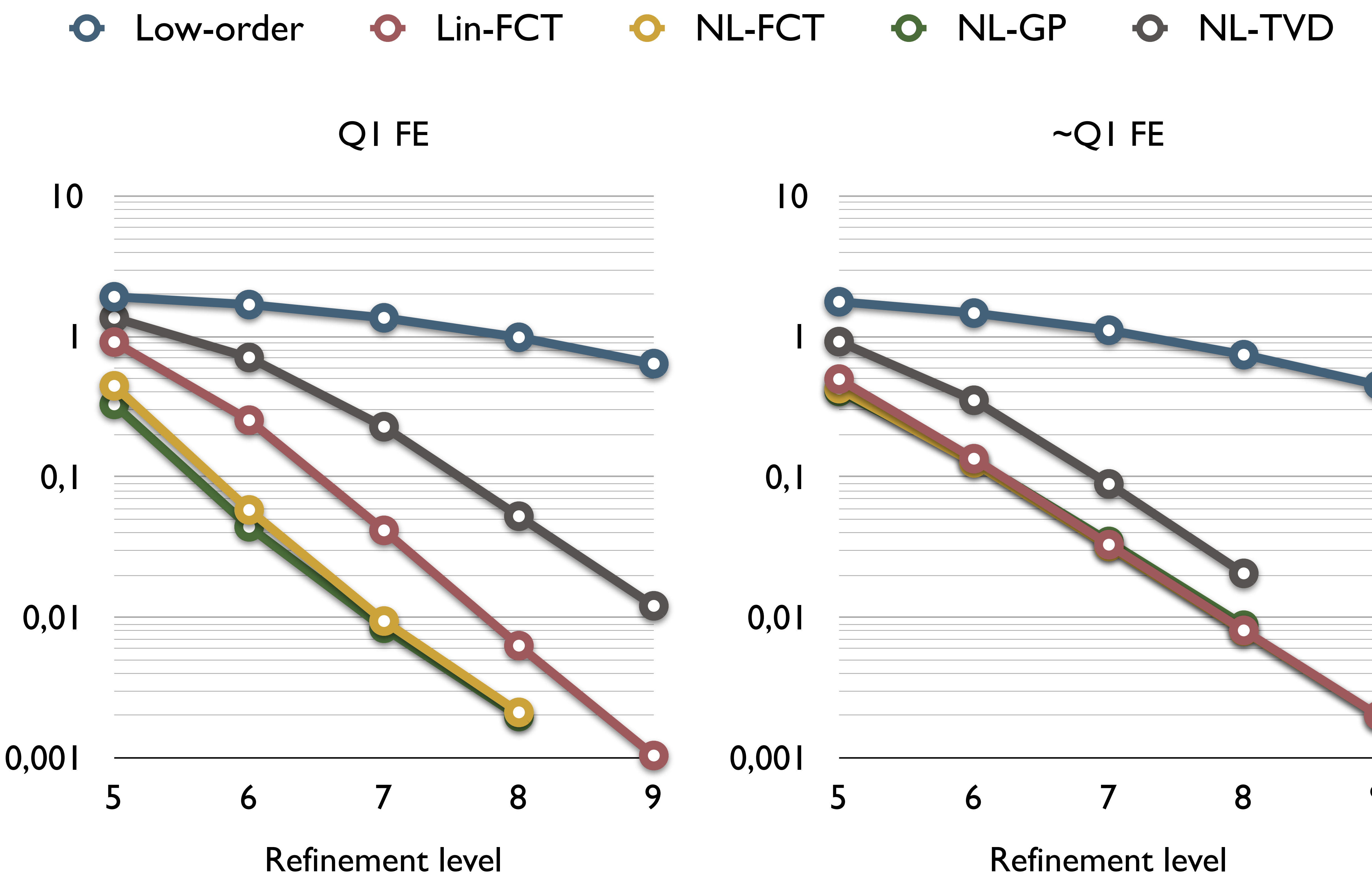
- Position of the peak

$$\hat{x}(t) = -0.5 \sin t$$

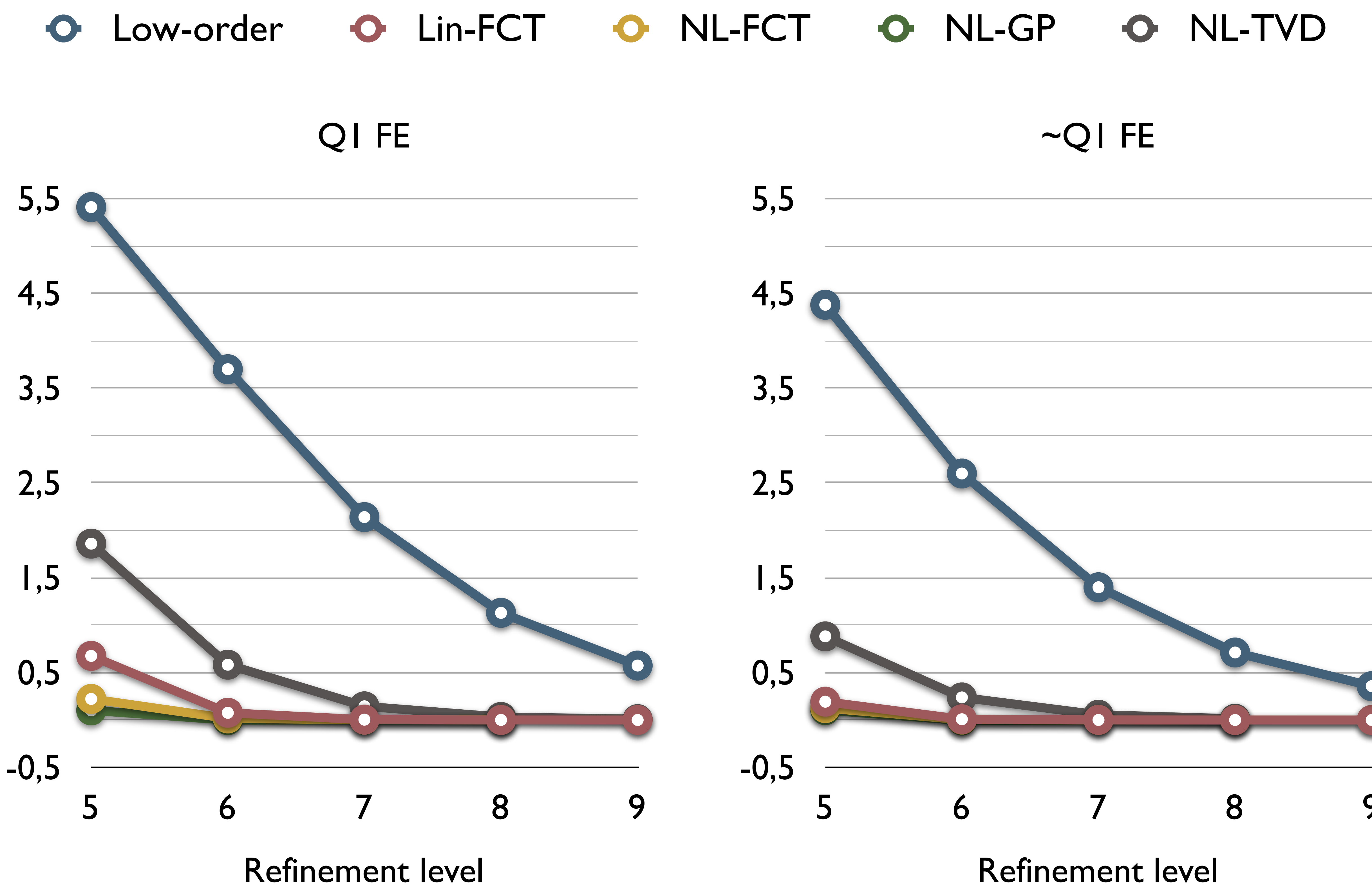
$$\hat{y}(t) = 0.5 \cos(t)$$

- Other parameters as before

RGH: L_2-error (0% mesh disturbance)



RGH: *dispersion-error (0% mesh disturbance)*



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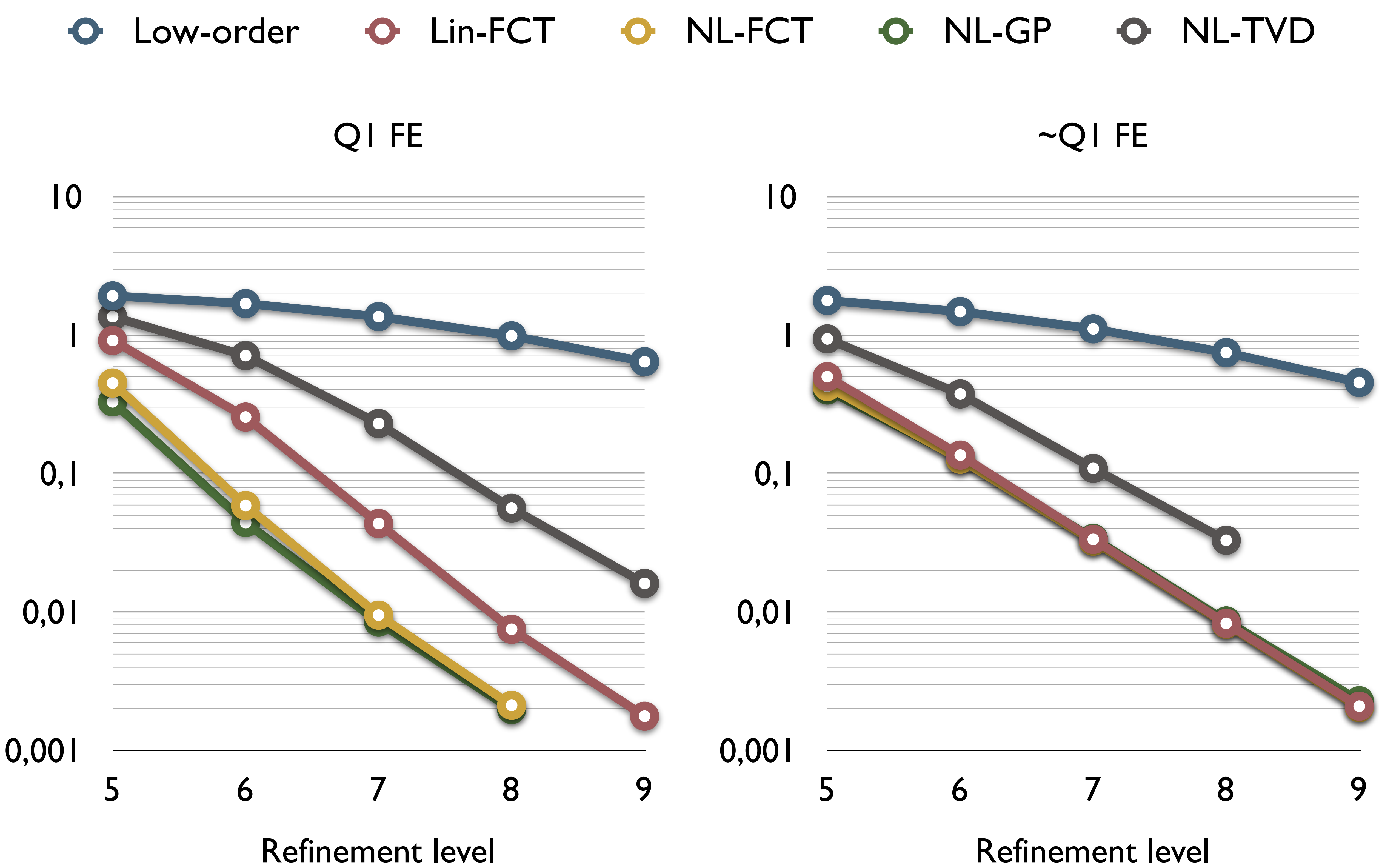

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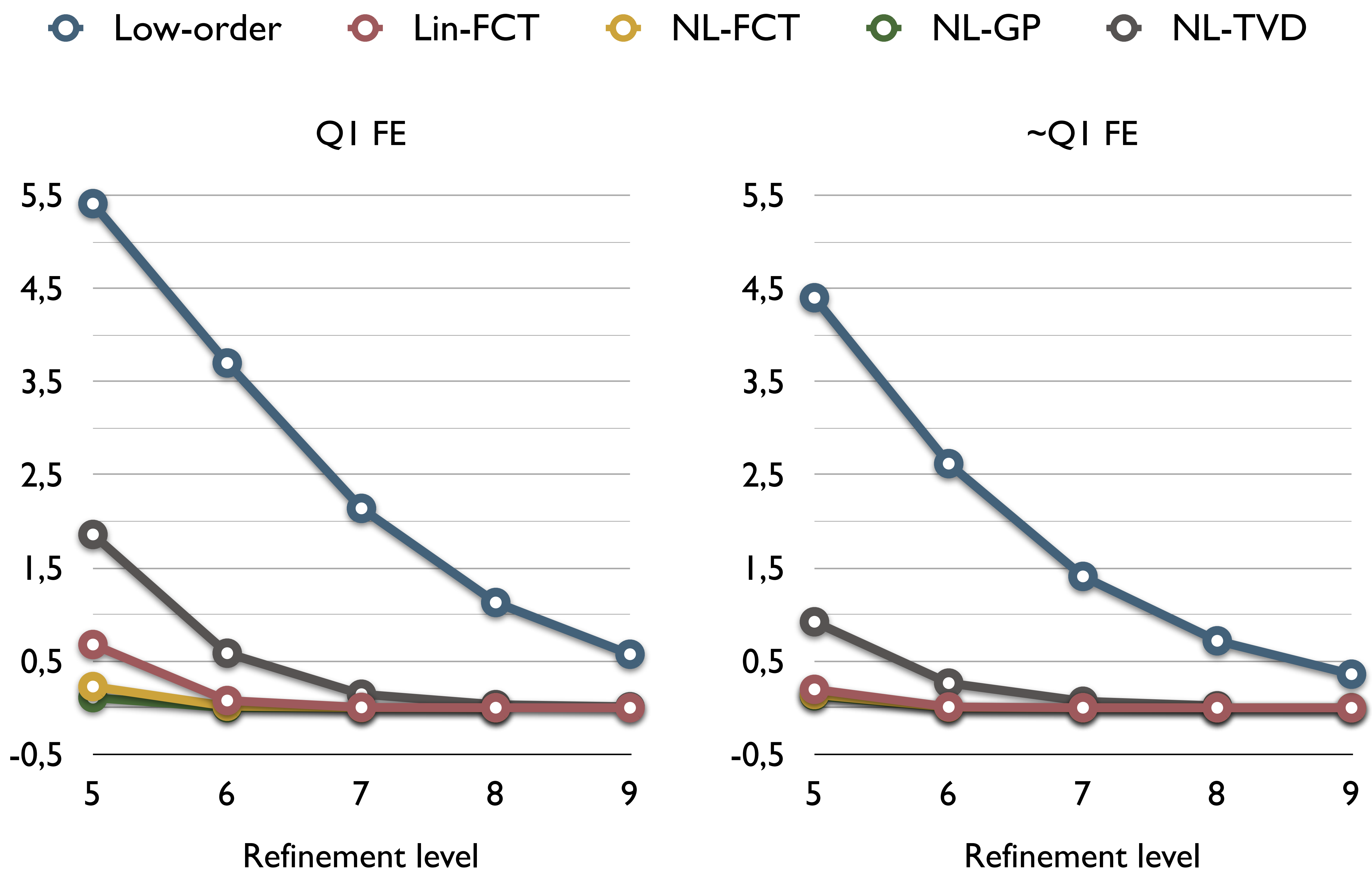
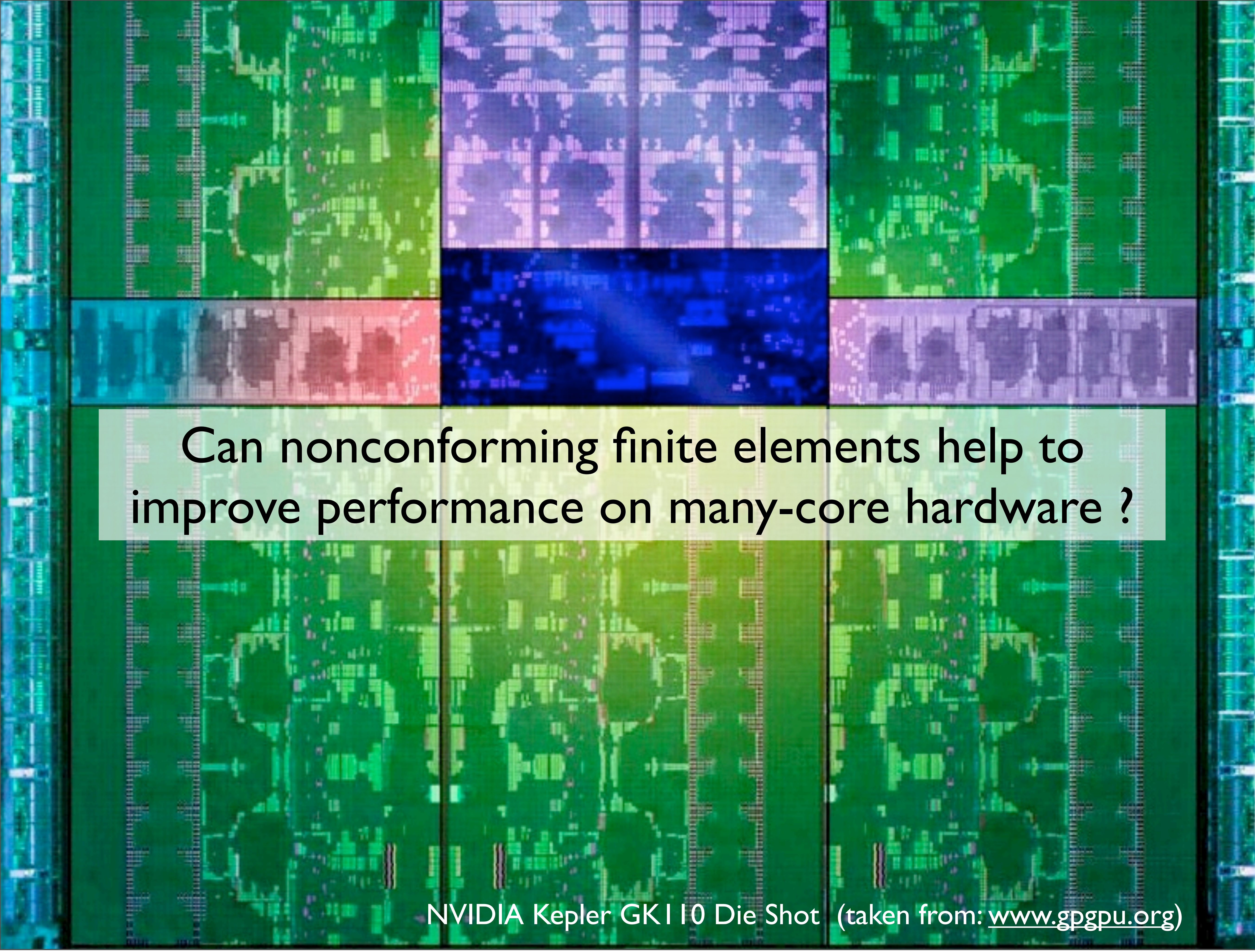


Taxonomy of finite elements

CONFORMING   NON CONFORMING

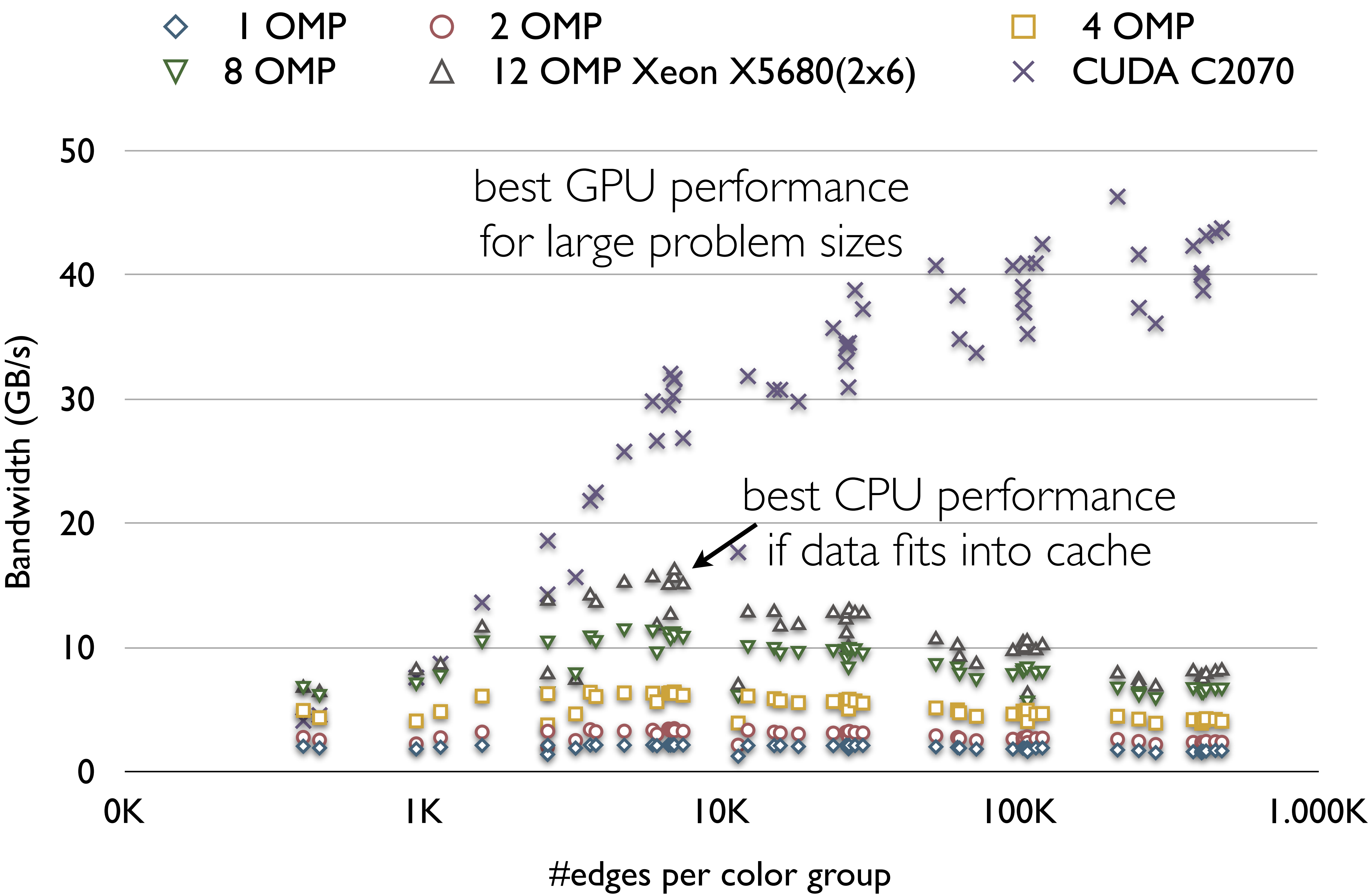
good	**accuracy**	good
small	**numerical diffusion**	small
smaller	**#DOFs, #edges**	larger
irregular	**sparsity pattern**	regular



Can nonconforming finite elements help to improve performance on many-core hardware ?

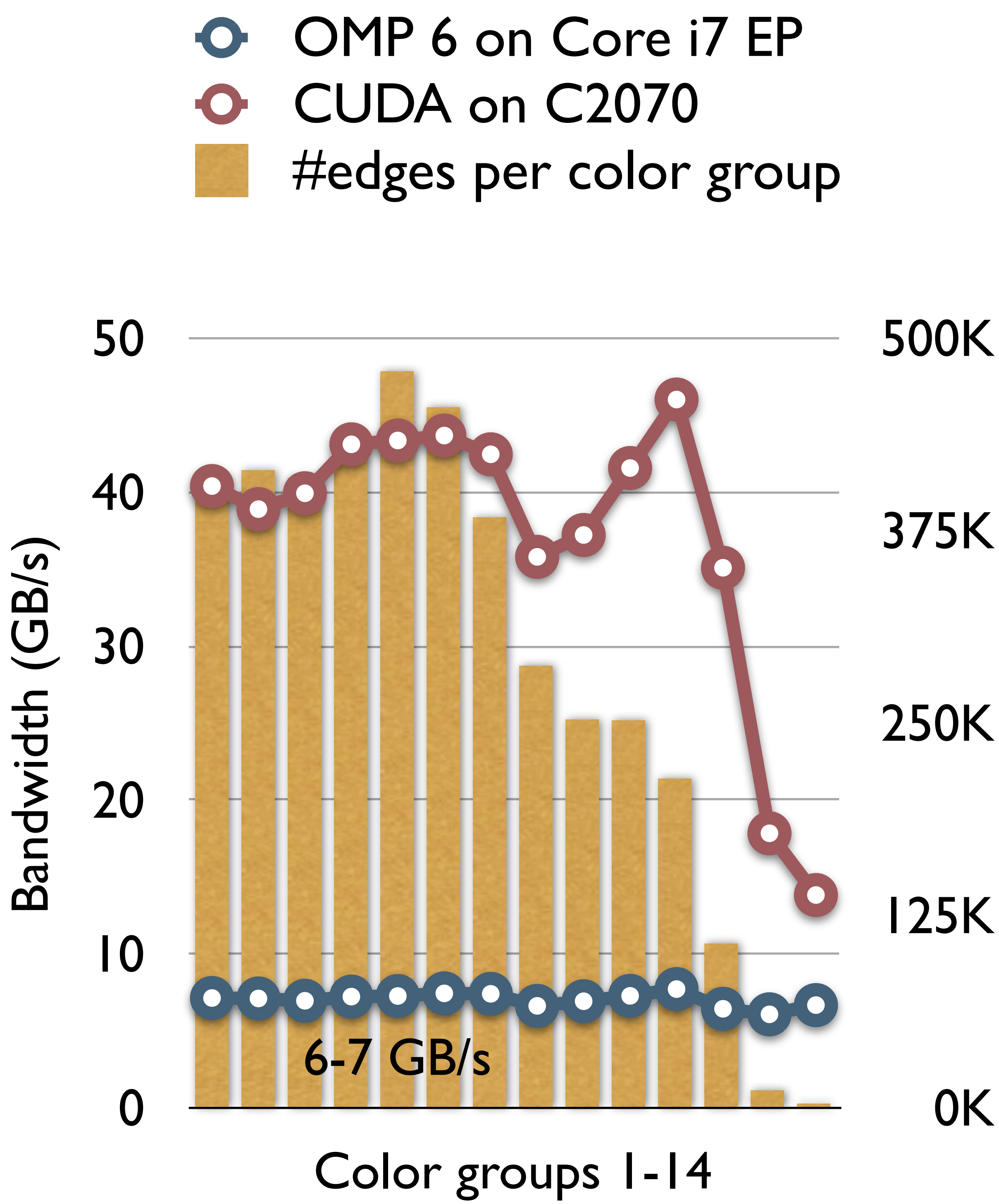
NVIDIA Kepler GK110 Die Shot (taken from: www.gpgpu.org)

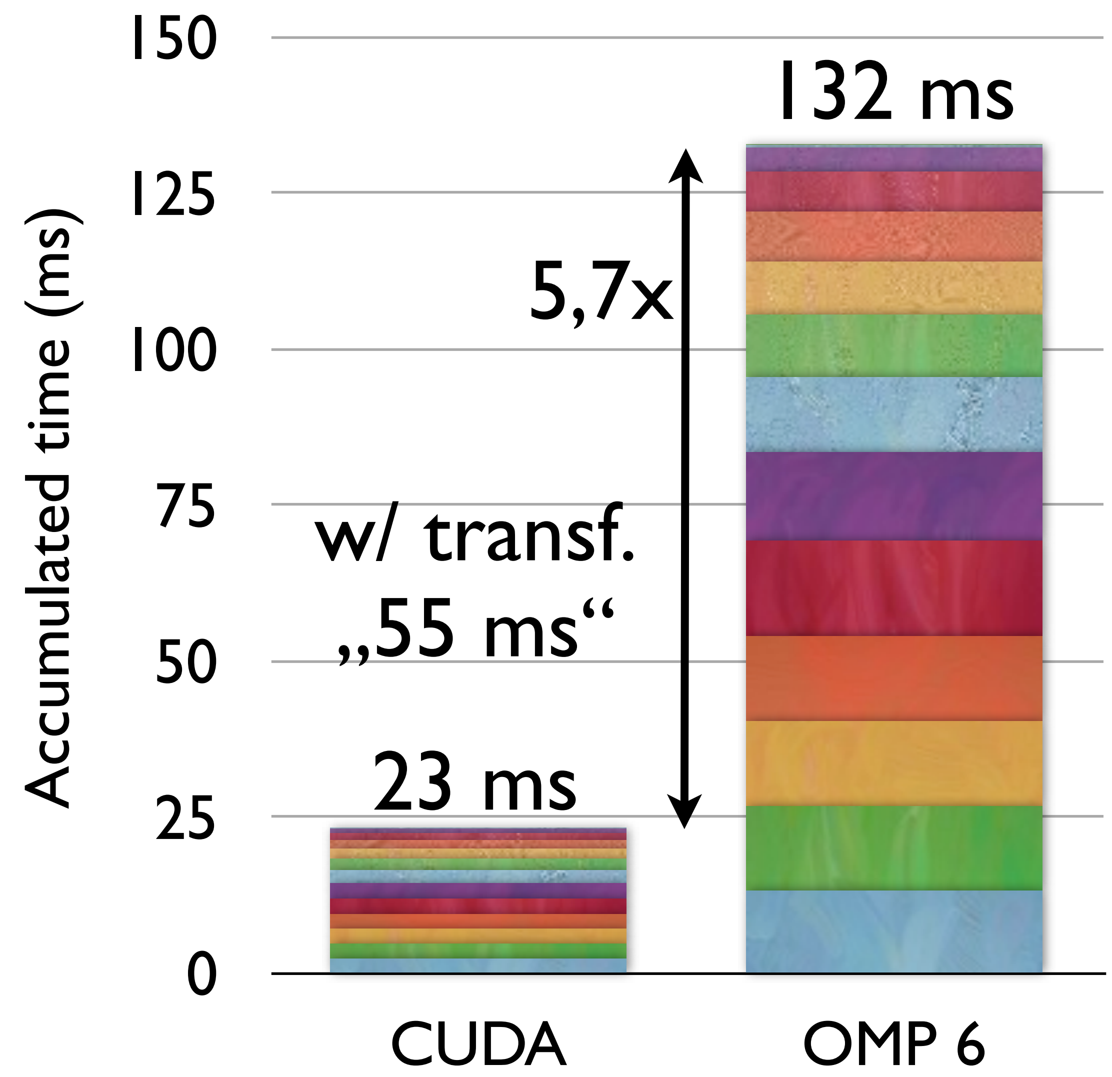
Example: *edge-based flux-assembly with Q_1 FE*



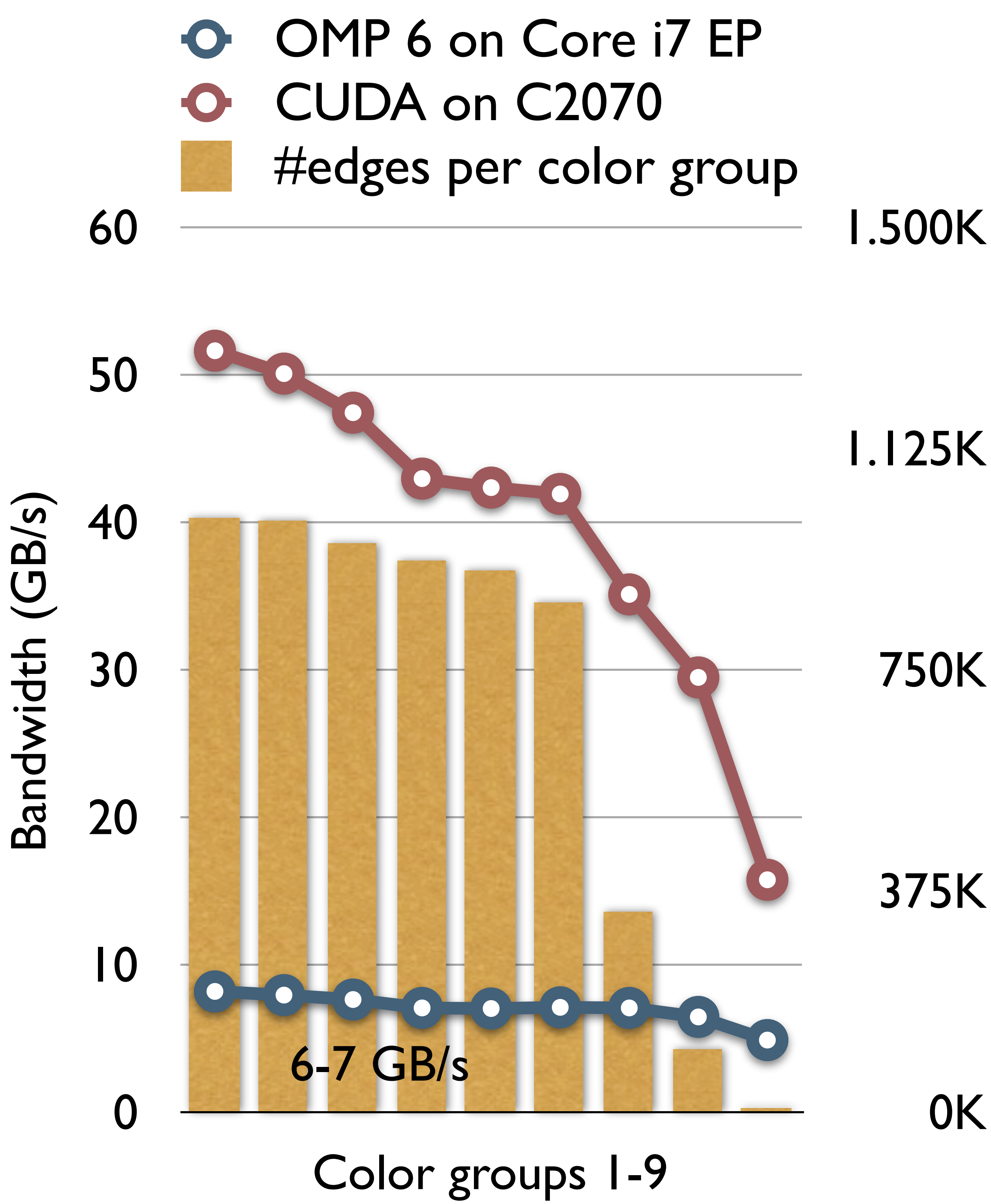
Example: *edge-based flux assembly with Q_1 FE*

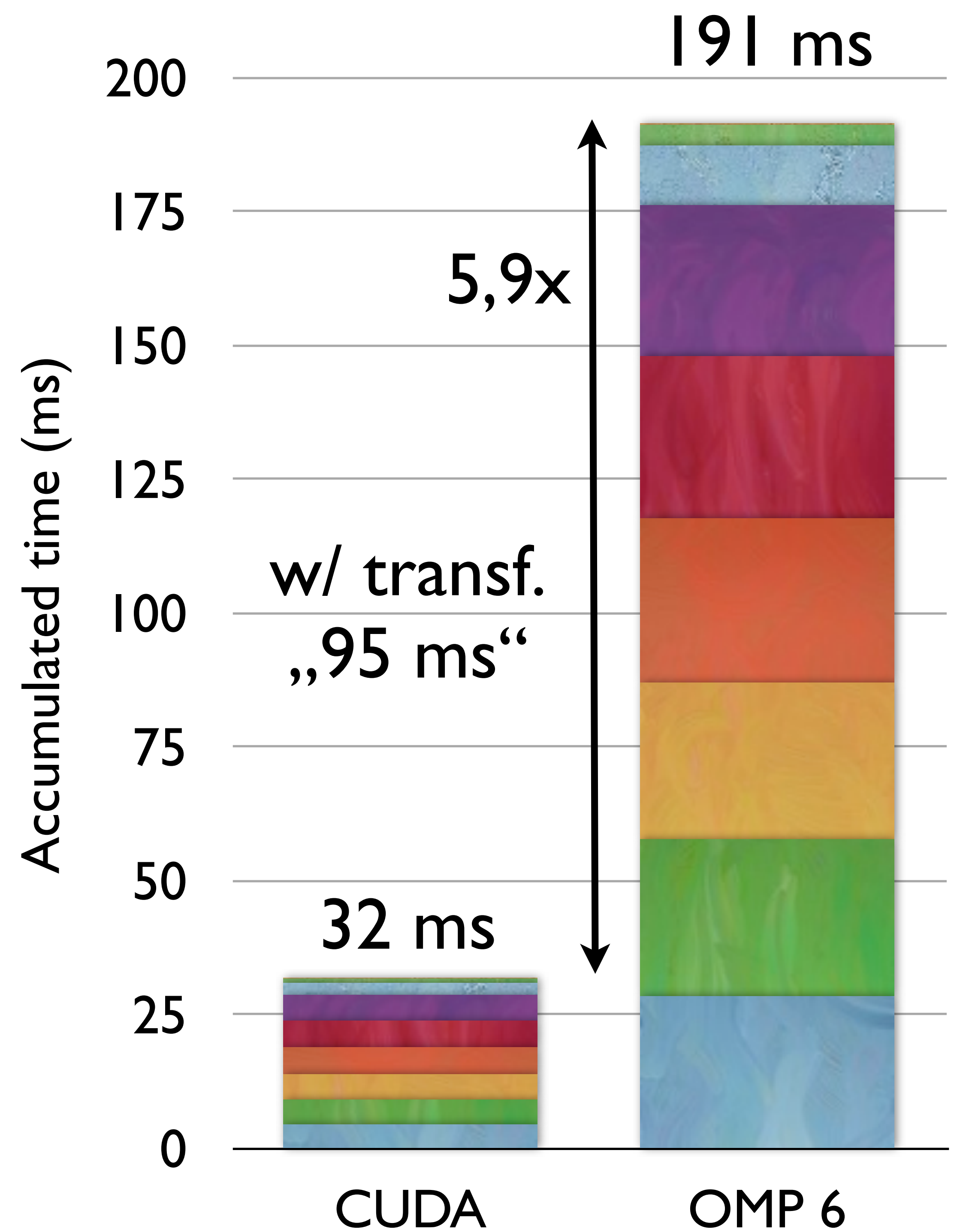
$$F_i = \sum_{k=1}^{N_{\text{colors}}} \sum_{ij \in \mathrm{CG}_k} \sigma_{ij}(U_j - U_i)$$





Example: *edge-based flux assembly with ~Q_1 FE*





Summary

Nonconforming $\tilde{Q}_1$ finite elements

- can be used within the algebraic flux correction framework
- are comparable to conforming FEs (accuracy/numerical diffusion)
- increase number of DOFs as compared to conforming Q_1 FEs
- lead to system matrices with regular structure favorable for HPC

Future plans

- Apply nonconforming AFC schemes to systems of equations
- Exploit benefits of nonconforming FEs for many-core architectures:
  speed up parallel (edge-based) assembly loops,
  implement more efficient matrix structures (ELLPACK),
  reduce communication costs in parallelized code

Acknowledgements

AFC schemes

- D. Kuzmin (University of Erlangen-Nuremberg)

CUDA/GPU programming

- D. Göddeke, D. Ribbrock, M. Geveler (TU Dortmund)

Featflow2

- M. Köster, P. Zajac (TU Dortmund)

Source code freely available at:

http://www.featflow.de/en/software/featflow2.html