On a failsafe flux limiting approach for the Euler equations

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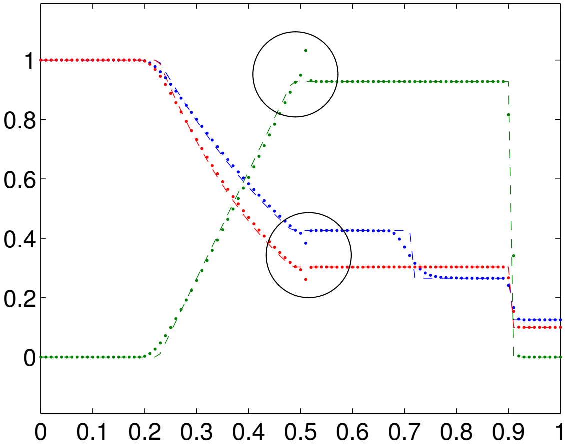
Motivation

- High-resolution schemes yield accurate results. But ...



Objective: to design flux corrected FEM with failsafe feature

Outline

1. High-resolution scheme
   - Finite element approximation
   - Flux-correction algorithm
   - Failsafe post-processing
2. Applications
   - Constrained initialization
   - Idealized Z-pinch implosion model
   - Ideal MHD equations
3. Conclusions

Finite element approximation

$$\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F}(U) = 0, \qquad U = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \rho E \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathcal{I} \\ (\rho E + p)\mathbf{v} \end{pmatrix}$$

- Weak formulation

$$\int_\Omega W \left[\frac{\partial U}{\partial t} + \nabla \cdot \mathbf{F}(U) \right] \mathrm{d}\mathbf{x} = 0, \quad \forall W \in \mathcal{W}$$

[1] C.A.J. Fletcher, CMAME 1983, 37(2), pp. 225–244

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- Weak formulation (using integration by parts)

$$\int_\Omega W \frac{\partial U}{\partial t}\,\mathrm{d}\mathbf{x} = \int_\Omega \nabla W \cdot \mathbf{F}(U)\,\mathrm{d}\mathbf{x} - \int_\Gamma W\,\mathbf{n} \cdot \mathbf{F}(U)\,\mathrm{d}s, \quad \forall W \in \mathcal{W}$$

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- Group representation[1] $U_j(t) = U(\mathbf{x}_j, t), \quad \mathbf{F}_j(t) = \mathbf{F}(U_j(t))$

$$U(\mathbf{x}, t) \approx \sum_j \varphi_j(\mathbf{x}) U_j(t), \qquad \mathbf{F}(U) \approx \sum_j \varphi_j(\mathbf{x}) \mathbf{F}_j(t)$$

[1] C.A.J. Fletcher, CMAME 1983, 37(2), pp. 225–244

Finite element approximation, cont'd

- Semi-discrete high-order scheme

$$\sum_j m_{ij} \frac{\mathrm{d}U_j}{\mathrm{d}t} = \sum_j (\mathbf{c}_{ji} - \mathbf{s}_{ij}) \cdot \mathbf{F}_j \qquad \forall i$$

$$m_{ij} = \int_\Omega \varphi_i \varphi_j \,\mathrm{d}\mathbf{x}, \qquad \mathbf{c}_{ji} = \int_\Omega \nabla \varphi_i \varphi_j \,\mathrm{d}\mathbf{x}, \qquad \mathbf{s}_{ij} = \int_\Gamma \varphi_i \varphi_j \mathbf{n} \,\mathrm{d}s$$

Finite element approximation, cont'd

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- Semi-discrete low-order scheme

$$m_i \frac{\mathrm{d}U_i}{\mathrm{d}t} = \sum_j (\mathbf{c}_{ji} - \mathbf{s}_{ij}) \cdot \mathbf{F}_j + \sum_{j \neq i} D_{ij}(U_j - U_i) \qquad \forall i$$

$m_i = \sum_j m_{ij} = \int_\Omega \varphi_i \,\mathrm{d}\mathbf{x}$, $\qquad D_{ij}$ artificial viscosity

Finite element approximation, cont'd

- Semi-discrete high-order scheme

$$\sum_j m_{ij} \frac{\mathrm{d}U_j}{\mathrm{d}t} = \sum_j (\mathbf{c}_{ji} - \mathbf{s}_{ij}) \cdot \mathbf{F}_j =: R_i^H \qquad \forall i$$

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$$m_i = \sum_j m_{ij} = \int_\Omega \varphi_i \,\mathrm{d}\mathbf{x}, \qquad D_{ij} \text{ artificial viscosity}$$

Finite element approximation, cont'd

- Relation between high- and low-order schemes

$$\sum_j m_{ij} \frac{\mathrm{d}U_j}{\mathrm{d}t} = R_i^H \quad \Leftrightarrow \quad m_i \frac{\mathrm{d}U_i}{\mathrm{d}t} = R_i^L + \sum_{j \neq i} F_{ij}$$

- Raw antidiffusive flux

$$F_{ij} = m_{ij} \left(\frac{\mathrm{d}U_i}{\mathrm{d}t} - \frac{\mathrm{d}U_j}{\mathrm{d}t} \right) + D_{ij}(U_i - U_j), \qquad F_{ji} = -F_{ij}$$

Objective: *linearize* the raw antidiffusive fluxes and *limit* them to prevent the generation of nonphysical under-/overshoots

Linearized FCT algorithm[2]

1. Compute the low-order solution at $t^{n+1} = t^n + \Delta t$

$$m_i \frac{U_i^L - U_i^n}{\Delta t} = \theta R_i^L(U^L) + (1-\theta) R_i^L(U^n), \qquad 0 < \theta \leq 1$$

[2] D. Kuzmin, JCP 2009, 228(7), pp. 2517–2534

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2. Approximate the time derivative

$$m_i \frac{\mathrm{d}U_i}{\mathrm{d}t} = R_i^L \qquad \Rightarrow \qquad \frac{\mathrm{d}U_i}{\mathrm{d}t} \approx \dot{U}_i^L = \frac{1}{m_i} R_i^L(U^L)$$

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3. Linearize the raw antidiffusive fluxes

$$F_{ij}^L = m_{ij}(\dot{U}_i^L - \dot{U}_j^L) + D_{ij}(U_i^L - U_j^L), \qquad F_{ji}^L = -F_{ij}^L$$

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4. Apply the *limited* antidiffusive fluxes

$$m_i U_i^{n+1} = m_i U_i^L + \Delta t \sum_{j \neq i} \alpha_{ij} F_{ij}^L, \qquad 0 \leq \alpha_{ij} = \alpha_{ji} \leq 1$$

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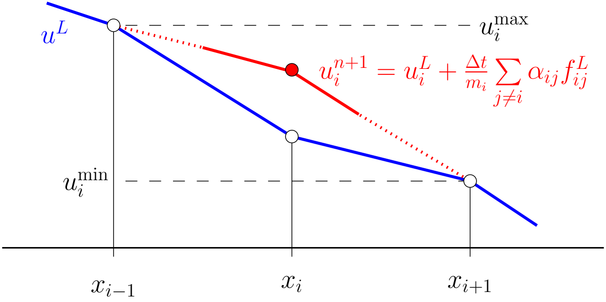
Flux limiting for *scalar equations*

$$m_i u_i^{n+1} = m_i u_i^L + \Delta t \sum_{j \neq i} \alpha_{ij} f_{ij}^L, \qquad f_{ji}^L = -f_{ij}^L, \qquad \alpha_{ij} = \alpha_{ji}$$

Zalesak's limiter[3] yields α_{ij}'s such that the nodal values of the corrected solution are bounded by the local extrema of the low-order solution



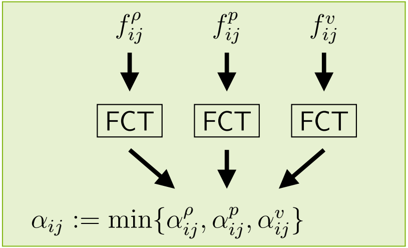
[3] S. Zalesak, JCP 1979, 31(3), pp. 335–362

Flux limiting for *systems*[4]

- Apply flux limiter to a set of control variables simultaneously

Nodal transformation

- $f_{ij}^{\rho} = \mathcal{T}_i^{\rho} F_{ij}^L$
- $f_{ij}^{p} = \mathcal{T}_i^{p} F_{ij}^L$
- $f_{ij}^{v} = \mathcal{T}_i^{v} F_{ij}^L$



- Apply correction factors to the *conservative* antidiffusive fluxes

$$m_i U_i^{n+1} = m_i U_i^L + \Delta t \sum_{j \neq i} \alpha_{ij} F_{ij}^L, \qquad F_{ji}^L = -F_{ij}^L, \qquad \alpha_{ij} = \alpha_{ji}$$

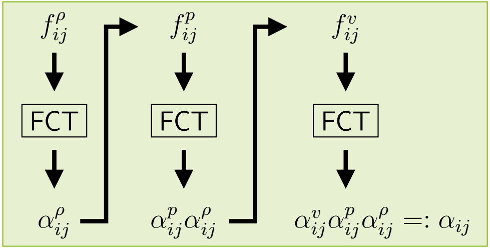
[4] D. Kuzmin, M. M, J.N. Shadid, M. Shashkov, JCP 2010, 229(23), pp. 8766–8779

Flux limiting for *systems*[4], cont'd

- Apply flux limiter to a set of control variables one after the other

Nodal transformation

- $f_{ij}^{\rho} = \mathcal{T}_i^{\rho} F_{ij}^L$
- $f_{ij}^{p} = \mathcal{T}_i^{p}(\alpha_{ij}^{\rho} F_{ij}^L)$
- $f_{ij}^{v} = \mathcal{T}_i^{v}(\alpha_{ij}^{p}\alpha_{ij}^{\rho} F_{ij}^L)$



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[4] D. Kuzmin, M. M, J.N. Shadid, M. Shashkov, JCP 2010, 229(23), pp. 8766–8779

Failsafe flux correction algorithm[4]

Flux correction machinery may fail to produce physically correct solutions:

$$\exists i: \quad u_i^{\min} \leq u_i^{FCT} \leq u_i^{\max} \quad \text{is violated for control variable } u$$

Remedy: enforce physically-motivated constraints by post-processing

$$m_i U_i^{FCT} = m_i U_i^L + \Delta t \sum_{j \neq i} \alpha_{ij} F_{ij}^L$$

$$\Leftrightarrow \quad m_i {U_i}^L = m_i U_i^{FCT} - \Delta t \sum_{j \neq i} \alpha_{ij} F_{ij}^L$$

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$$m_i U_i^{(k)} = m_i U_i^{FCT} - \Delta t \sum_{j \neq i} \beta_{ij}^{(k)} \alpha_{ij} F_{ij}^L, \quad k = 1, ..., K$$

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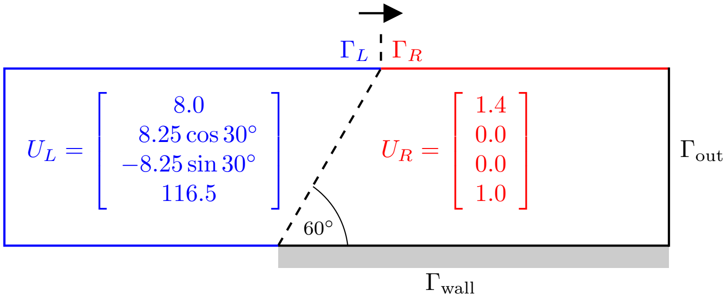
Example: $\beta_{ij}^{(0)} \equiv 0, \quad \beta_{ij}^{(k)} := \begin{cases} k/K & \text{if failure is detected at } i, j \\ \beta_{ij}^{(k-1)} & \text{otherwise} \end{cases}$

[4] D. Kuzmin, M. M, J.N. Shadid, M. Shashkov, JCP 2010, 229(23), pp. 8766–8779

Double Mach reflection[5]

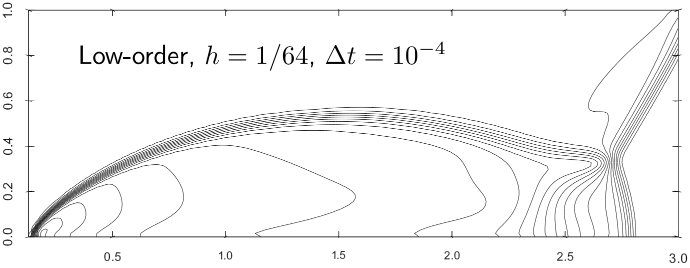
Test: Roe-linearization + FCT, structured grid, Q_1 finite elements

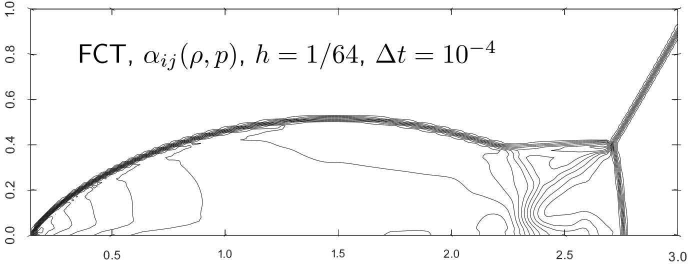
$T = 0.2$, Crank Nicolson time stepping ($\theta = 0.5$), $\Delta t = 64h \cdot 10^{-4}$



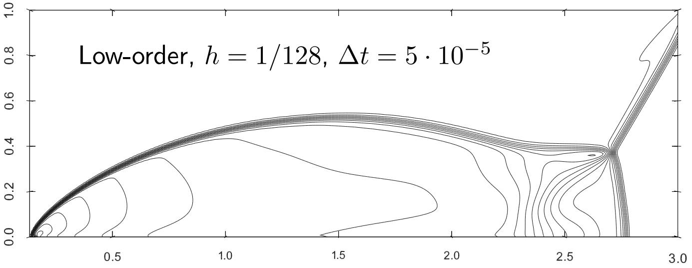
[5] P.R. Woodward, P. Colella, JCP 54, 115 (1984), pp. 115–173

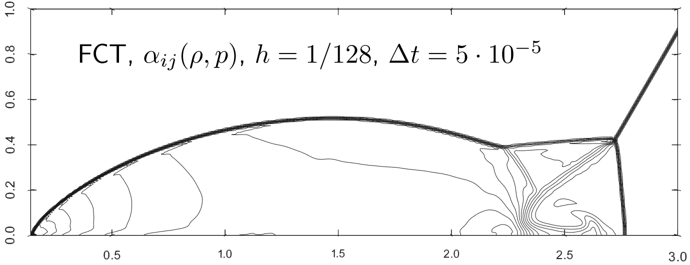
Double Mach reflection



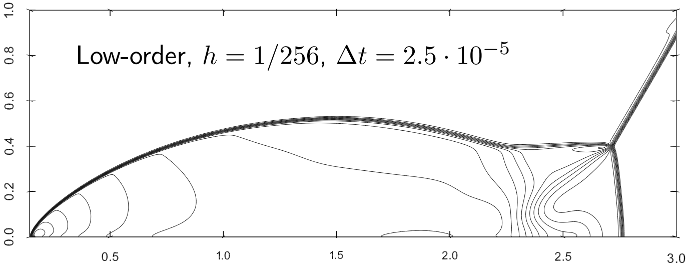


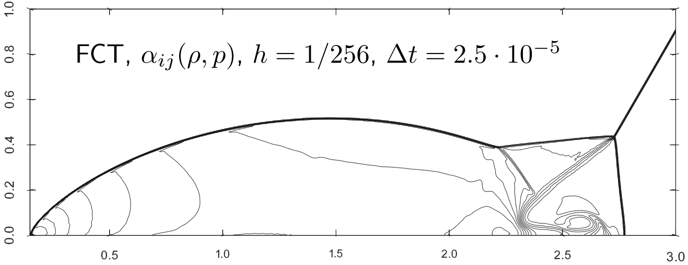
Double Mach reflection





Double Mach reflection





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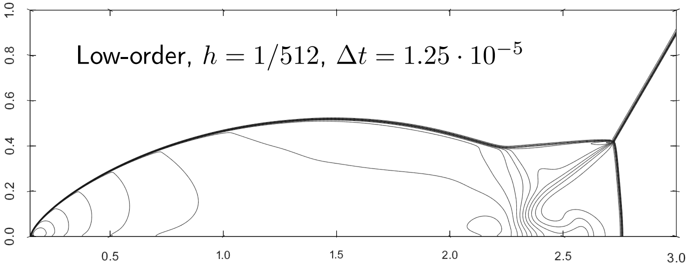

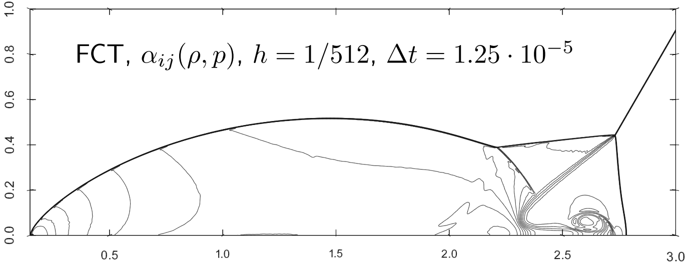

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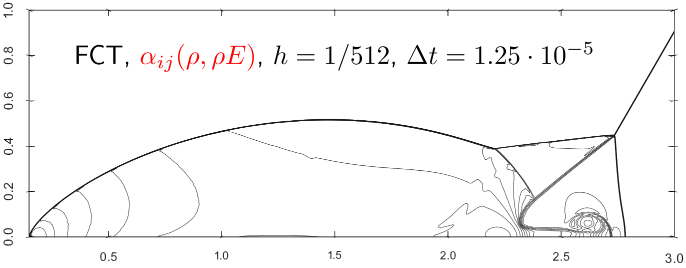
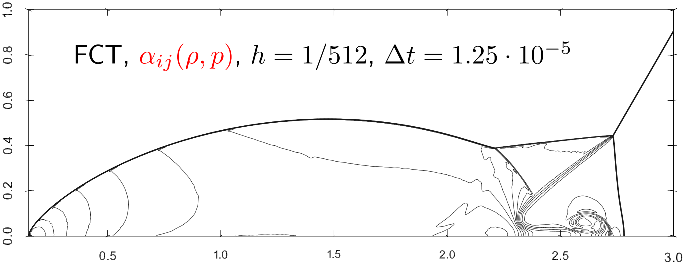
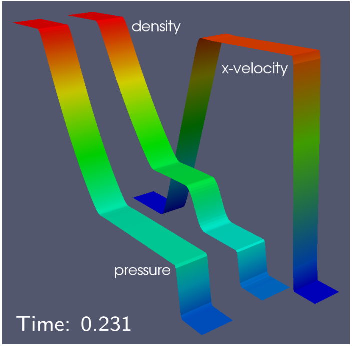




Sod's shock tube problem

Test: Rusanov-type dissipation + FCT, $\alpha_{ij}(\rho, p)$, $10n \times n$ grid, Q_1 FEs

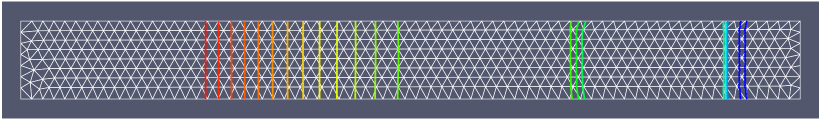
$$\kappa_u = \log \frac{\|u_{2h} - u_{4h}\|_1}{\|u_h - u_{2h}\|_1} / \log 2$$



	Crank Nicolson time stepping			
	FCT		Low-order	
n_{fine}	κ_ρ	κ_p	κ_ρ	κ_p
20	0.624	1.027	0.193	0.623
40	0.970	1.003	0.421	0.671
80	1.079	1.005	0.575	0.701
160	1.073	1.005	0.624	0.730

	Backward Euler time stepping			
	FCT		Low-order	
n_{fine}	κ_ρ	κ_p	κ_ρ	κ_p
20	0.671	0.982	0.190	0.619
40	0.980	0.950	0.416	0.669
80	0.977	0.947	0.575	0.701
160	0.981	0.945	0.624	0.730

Sod's shock tube problem



Coarse mesh with contour plot of density variable at time $T = 0.231$

	Crank Nicolson time stepping				Backward Euler time stepping			
	FCT		Low-order		FCT		Low-order	
#trias	κ_ρ	κ_p	κ_ρ	κ_p	κ_ρ	κ_p	κ_ρ	κ_p
18,176	0.925	0.876	0.364	0.665	0.955	0.841	0.357	0.662
72,704	0.874	0.800	0.539	0.679	0.820	0.732	0.536	0.679
290,816	0.806	0.934	0.614	0.718	0.765	0.875	0.616	0.719
1,163,264	0.948	0.966	0.641	0.739	0.889	0.905	0.642	0.740

Linearized FCT algorithm yields accurate and non-oscillatory solutions using P_1 and Q_1 finite elements on structured and unstructured meshes, respectively.

Outline

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   - Flux-correction algorithm
   - Failsafe post-processing

2. Applications
   - Constrained initialization
   - Idealized Z-pinch implosion model
   - Ideal MHD equations

3. Conclusions

Constrained initialization

Test: discontinuous initial data with P_1 FEs on unstructured mesh

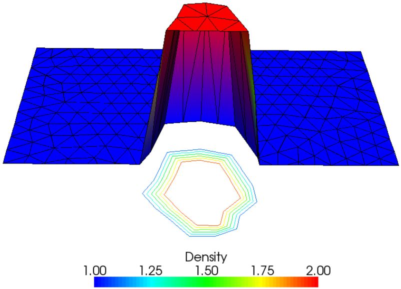
$$\Omega = (-0.5, 0.5)^2, \quad \Omega_{\text{in}} = \{(x, y) \in \Omega | r = \sqrt{x^2 + y^2} < 0.13\}$$

	Ω_{in}	$\Omega \setminus \Omega_{\text{in}}$
ρ_0	2.0	1.0
$\mathbf{v}_0$	0.0	0.0
p_0	15.0	1.0

- Pointwise initialization

$$U_h(x_i, y_i) := U_0(x_i, y_i)$$

is not conservative!



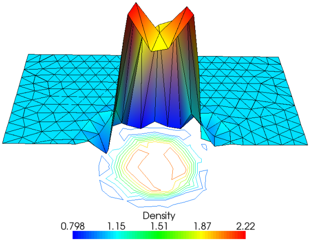
$\int_\Omega \rho_0 \, \mathrm{d}\mathbf{x}$	$\int_\Omega (\rho E)_0 \, \mathrm{d}\mathbf{x}$	$\int_\Omega \rho_h \, \mathrm{d}\mathbf{x}$	$\int_\Omega (\rho E)_h \, \mathrm{d}\mathbf{x}$
1.05309	4.35825	1.04799	4.17949

Constrained initialization, cont'd

- Conservative initialization by L_2 projection

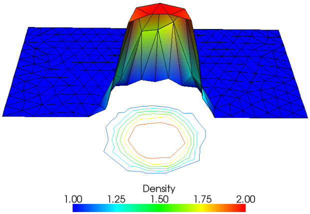
$$U_h = \sum_j \varphi_j U_j \ : \quad \int_\Omega W_h U_h \mathrm{d}\mathbf{x} = \int_\Omega W_h U_0 \mathrm{d}\mathbf{x} \qquad \forall W_h \in \mathcal{W}_h$$

Consistent mass matrix



$$\sum_j m_{ij} U_j^C = \int_\Omega \varphi_i U_0 \mathrm{d}\mathbf{x}$$

Lumped mass matrix



$$m_i U_i^L = \int_\Omega \varphi_i U_0 \mathrm{d}\mathbf{x}$$

Constrained initialization, cont'd

- Relation between consistent and lumped L_2 projection

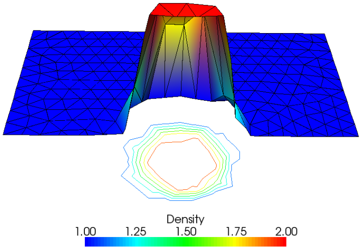
$$m_i U_i^C = m_i U_i^L + \sum_{j \neq i} F_{ij}, \qquad F_{ij} = m_{ij}(U_i^C - U_j^C)$$

- Constrained L_2 projection

$$m_i U_i = m_i U_i^L + \sum_{j \neq i} \alpha_{ij} F_{ij}$$

Apply failsafe flux correction machinery to compute

$$0 \leq \alpha_{ij} = \alpha_{ji} \leq 1$$



Outline

Idealized Z-pinch implosion model[6]

- Generalized Euler system coupled with scalar tracer equation

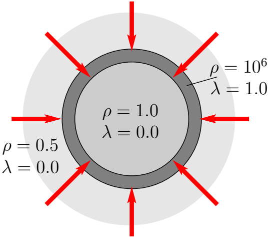
$$\frac{\partial}{\partial t}\begin{bmatrix}\rho \\ \rho\mathbf{v} \\ \rho E \\ \rho\lambda\end{bmatrix} + \nabla\cdot\begin{bmatrix}\rho\mathbf{v} \\ \rho\mathbf{v}\otimes\mathbf{v} + p\mathcal{I} \\ \rho E\mathbf{v} + p\mathbf{v} \\ \rho\lambda\mathbf{v}\end{bmatrix} = \begin{bmatrix}0 \\ \mathbf{f} \\ \mathbf{f}\cdot\mathbf{v} \\ 0\end{bmatrix}$$

- Equation of state

$$p = (\gamma - 1)\rho(E - 0.5|\mathbf{v}|^2)$$

- Non-dimensional Lorentz force

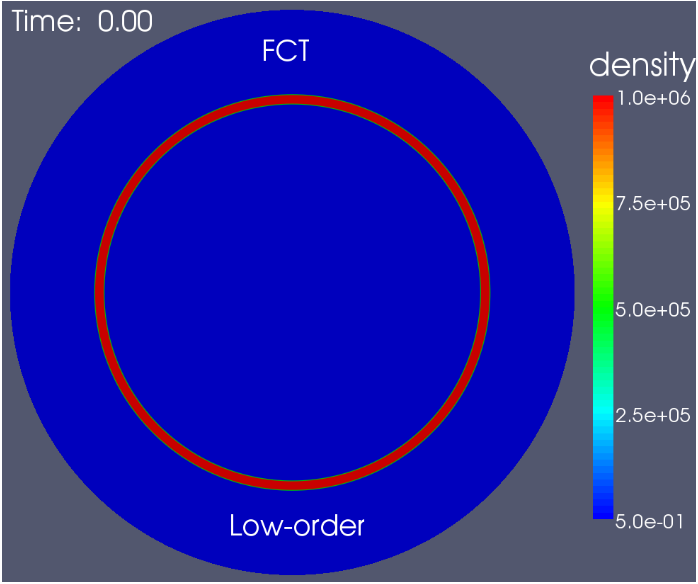
$$\mathbf{f} = (\rho\lambda)\left(\frac{I(t)}{I_{\max}}\right)^2 \frac{\hat{\mathbf{e}}_r}{r_{\text{eff}}}, \quad 0 \le \lambda \le 1$$



$\mathbf{v} = 0.0$, $p = 1.0$ everywhere

[6] J.W. Banks, J.N. Shadid, IJNMF 2009, 61(7), pp. 725–751

Idealized Z-pinch implosion



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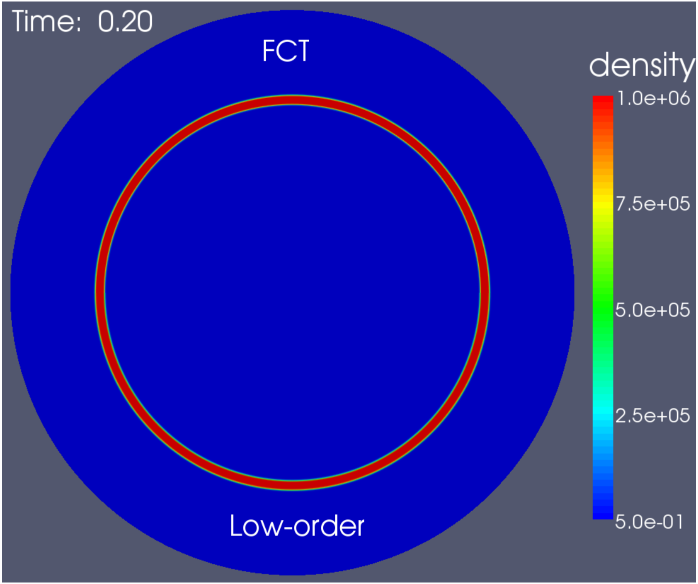

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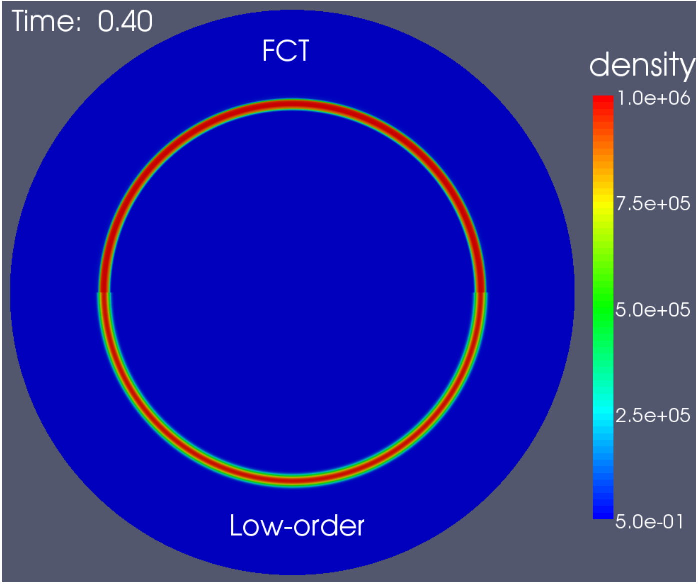

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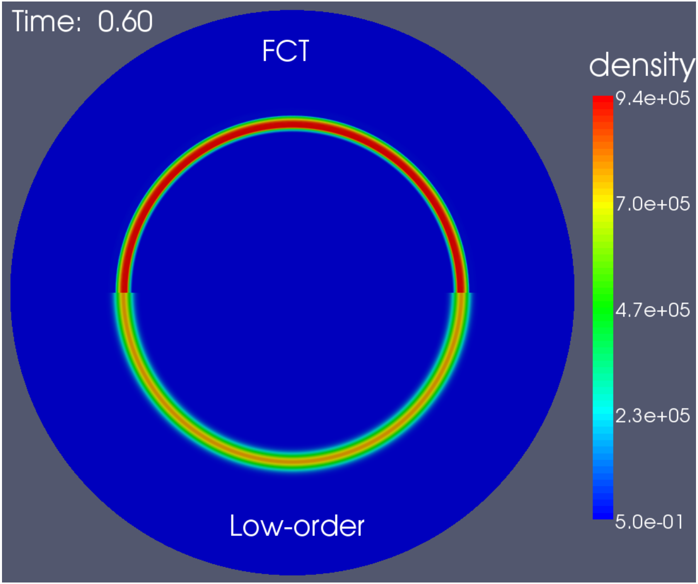

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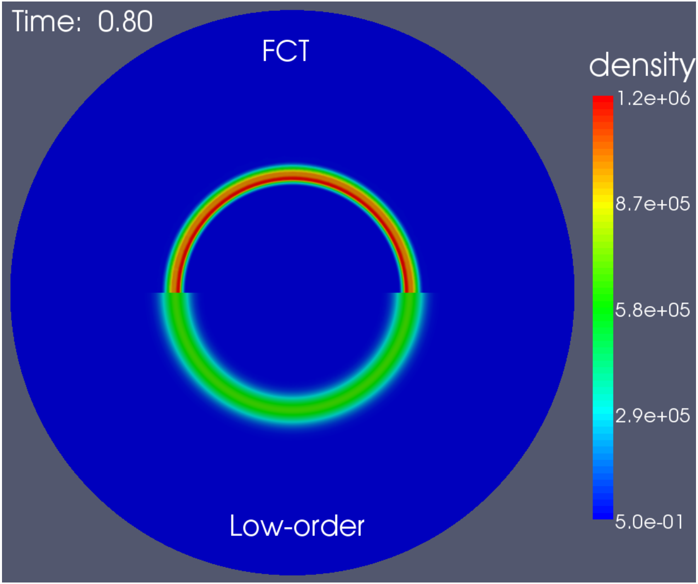

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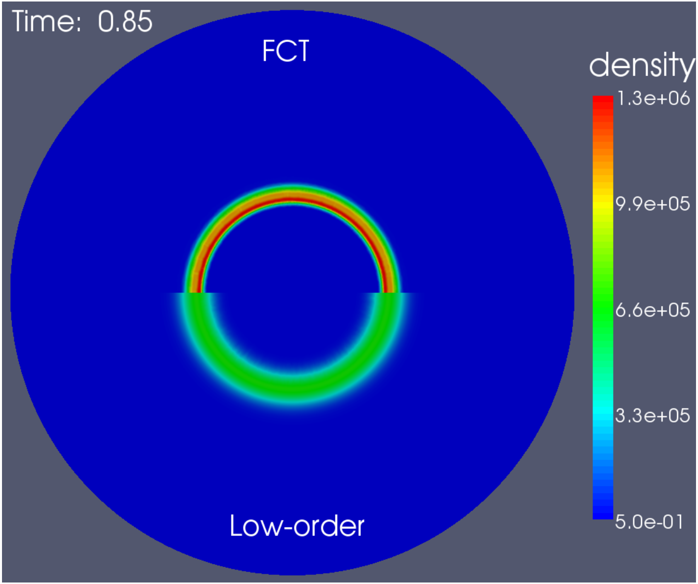

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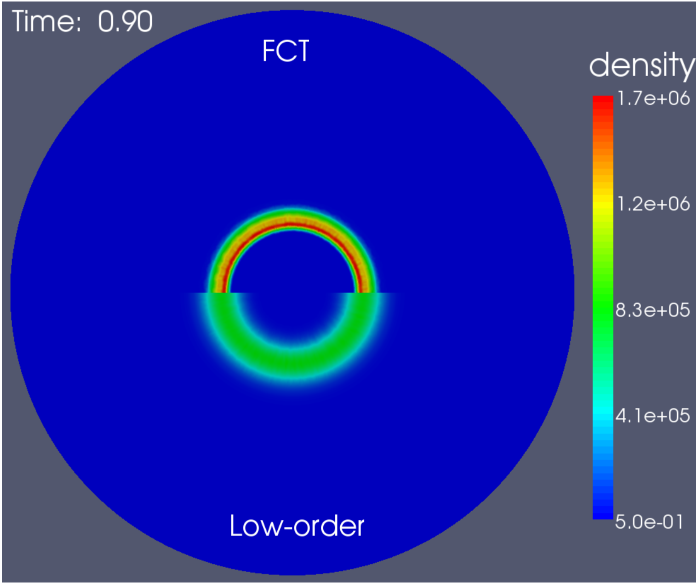

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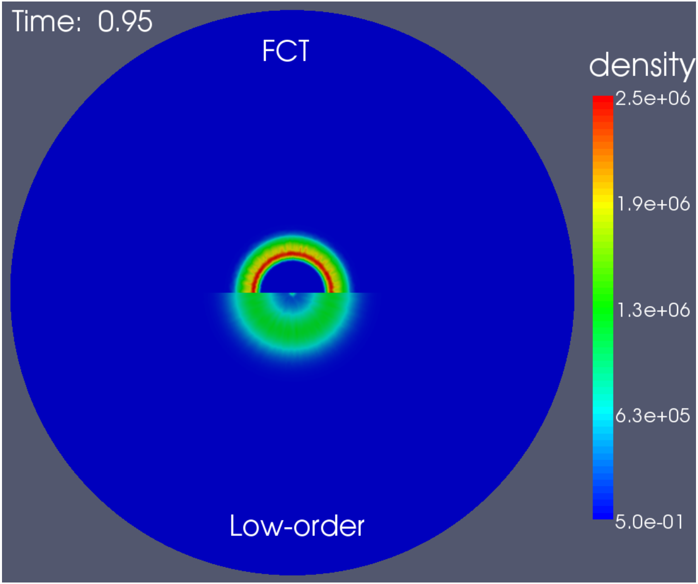

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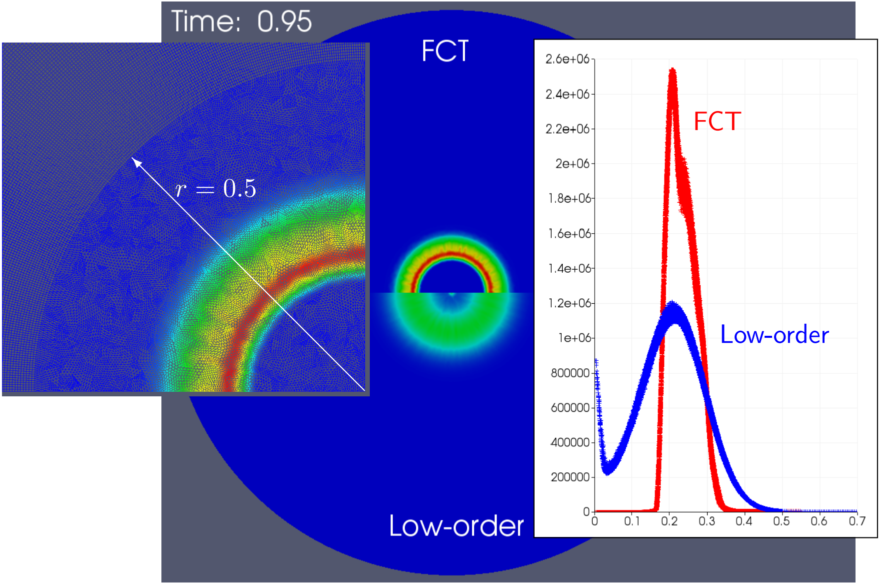


Outline

Ideal MHD equations

- Idealized MHD equations

$$\frac{\partial}{\partial t}\begin{bmatrix}\rho\\ \rho\mathbf{v}\\ \mathbf{B}\\ \rho E\end{bmatrix}+\nabla\cdot\begin{bmatrix}\rho\mathbf{v}\\ \rho\mathbf{v}\otimes\mathbf{v}+p\mathcal{I}-\mathbf{B}\otimes\mathbf{B}\\ \mathbf{v}\otimes\mathbf{B}-\mathbf{B}\otimes\mathbf{v}\\ \rho E\mathbf{v}+p\mathbf{v}-(\mathbf{B}\cdot\mathbf{v})\mathbf{B}\end{bmatrix}=0$$

  subject to $\nabla\cdot\mathbf{B}=0$

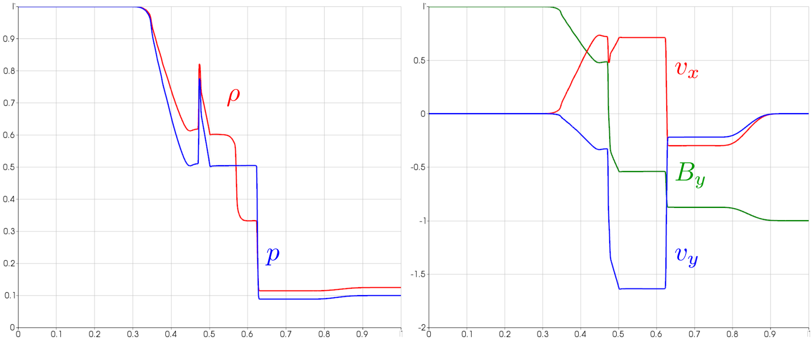
- Divergence involution in 1D: $\partial_x B_x = 0 \;\Rightarrow\; B_x = const$
- Hyperbolic conservation laws for 7 variables: ρ, $\mathbf{v}$, B_y, B_z, ρE
- Roe matrix (for arbitrary γ) by Cargo and Gallice[7]
- FCT limiter is applied to control variables ρ, p, B_y and B_z

[7] P. Cargo, G. Gallice, JCP 1997, 136(2), pp.446–466

Shock tube problem

- $\gamma = 1.4, \quad B_x = 0.75, \quad t_{\text{fin}} = 0.1,$ 800 grid points

$$(\rho, \mathbf{v}, B_y, B_z, p)^T = \begin{cases} (1.0 \quad, 0.0, \quad 1.0, 0.0, 1.0)^T & \text{if } x \leq 0.5 \\ (0.125, 0.0, -1.0, 0.0, 0.1)^T & \text{if } x > 0.5 \end{cases}$$



Outline

1. High-resolution scheme
   - Finite element approximation
   - Flux-correction algorithm
   - Failsafe post-processing
2. Applications
   - Constrained initialization
   - Idealized Z-pinch implosion model
   - Ideal MHD equations
3. Conclusions

Conclusions

- Failsafe flux correction algorithm
  - ensures boundedness of physical quantities
  - preserves symmetry on unstructured grids
  - is applicable to ‘challenging’ applications
  - can be turned into a constrained L_2 projection
- Future research
  - extension to multidimensional MHD equations
  - treatment of the $\nabla \cdot \mathbf{B} = 0$ involution
  - consider non-conforming FEs

Conclusions

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Thank you for your attention!

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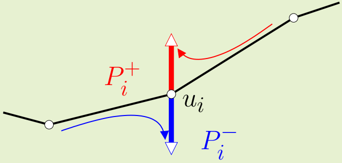
Zalesak's flux limiter[5]



- Consider sums of positive/negative antidiffusive fluxes into each node

[5] S. Zalesak, JCP 1979, 31(3), pp. 335–362

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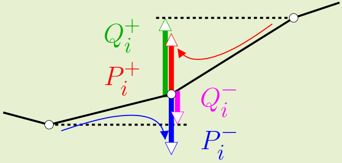

- Consider sums of positive/negative antidiffusive fluxes into each node
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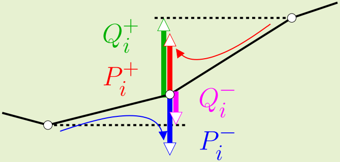

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- Compute nodal correction factors

$$R_i^+ = \min\{1, Q_i^+/P_i^+\} \quad \text{and} \quad R_i^- = \min\{1, Q_i^-/P_i^-\}$$

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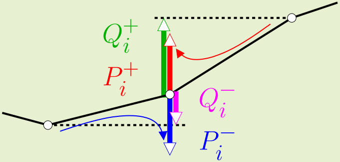

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- Limit antidiffusive flux for edge ij by

$$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\} & \text{for positive fluxes} \\ \min\{R_i^-, R_j^+\} & \text{for negative fluxes} \end{cases}$$

[5] S. Zalesak, JCP 1979, 31(3), pp. 335–362

Extended version of Zalesak's FCT limiter

Input: auxiliary solution u^L and antidiffusive fluxes f_{ij}^u, where $f_{ji}^u \neq f_{ij}^u$

1. Sums of positive/negative antidiffusive fluxes into node i

$$P_i^+ = \textstyle\sum_{j \neq i} \max\{0, f_{ij}^u\}, \qquad P_i^- = \textstyle\sum_{j \neq i} \min\{0, f_{ij}^u\}$$

2. Upper/lower bounds based on the local extrema of u^L

$$Q_i^+ = m_i(u_i^{\max} - u_i^L), \qquad Q_i^- = m_i(u_i^{\min} - u_i^L)$$

3. Correction factors $\alpha_{ij}^u = \alpha_{ji}^u$ to satisfy the FCT constraints

$$\alpha_{ij}^u = \min\{R_{ij}, R_{ji}\}, \quad R_{ij} = \begin{cases} \min\{1, Q_i^+/P_i^+\} & \text{if } f_{ij}^u \geq 0 \\ \min\{1, Q_i^-/P_i^-\} & \text{if } f_{ij}^u < 0 \end{cases}$$

Node-based transformation of control variables

- Conservative variables: density, momentum, total energy

$$U_i = [\rho_i, (\rho\mathbf{v})_i, (\rho E)_i], \qquad F_{ij} = \left[f_{ij}^{\rho}, \mathbf{f}_{ij}^{\rho v}, f_{ij}^{\rho E}\right], \qquad F_{ji} = -F_{ij}$$

- Primitive variables $V = TU$: density, velocity, pressure

$$V_i = [\rho_i, \mathbf{v}_i, p_i], \qquad \mathbf{v}_i = \frac{(\rho\mathbf{v})_i}{\rho_i}, \qquad p_i = (\gamma - 1)\left[(\rho E)_i - \frac{|(\rho\mathbf{v})_i|^2}{2\rho_i}\right]$$

$$G_{ij} = \left[f_{ij}^{\rho}, \mathbf{f}_{ij}^{v}, f_{ij}^{p}\right] = T(U_i)F_{ij}, \qquad T(U_j)F_{ji} = G_{ji} \neq -G_{ij}$$

- Raw antidiffusive fluxes for the velocity and pressure

$$\mathbf{f}_{ij}^{v} = \frac{\mathbf{f}_{ij}^{\rho v} - \mathbf{v}_i f_{ij}^{\rho}}{\rho_i}, \qquad f_{ij}^{p} = (\gamma - 1)\left[\frac{|\mathbf{v}_i|^2}{2} f_{ij}^{\rho} - \mathbf{v}_i \cdot \mathbf{f}_{ij}^{\rho v} + f_{ij}^{\rho E}\right]$$