Algebraic Flux Correction

Basic concepts, recent results for nonconforming finite elements, and aspects of parallelization

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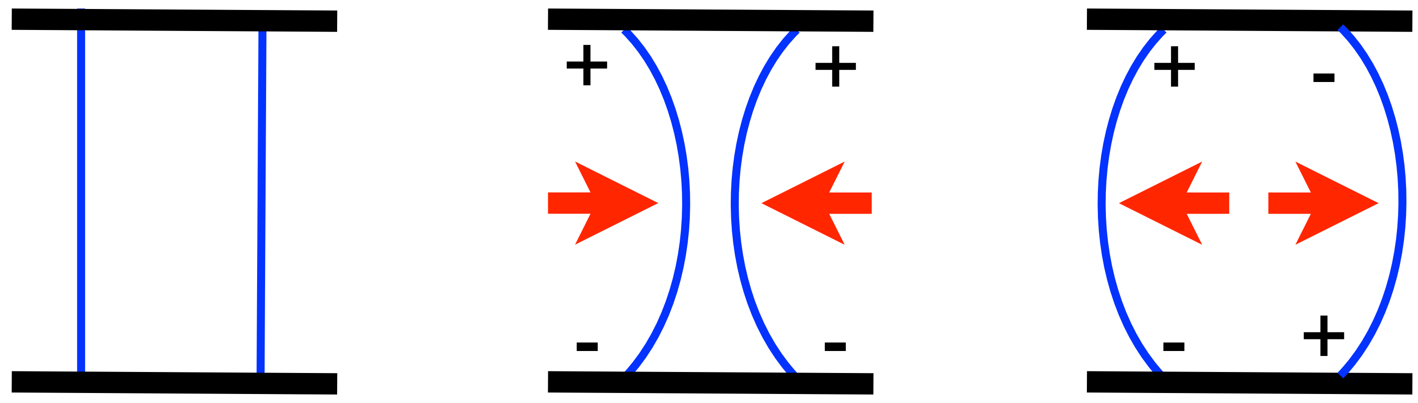
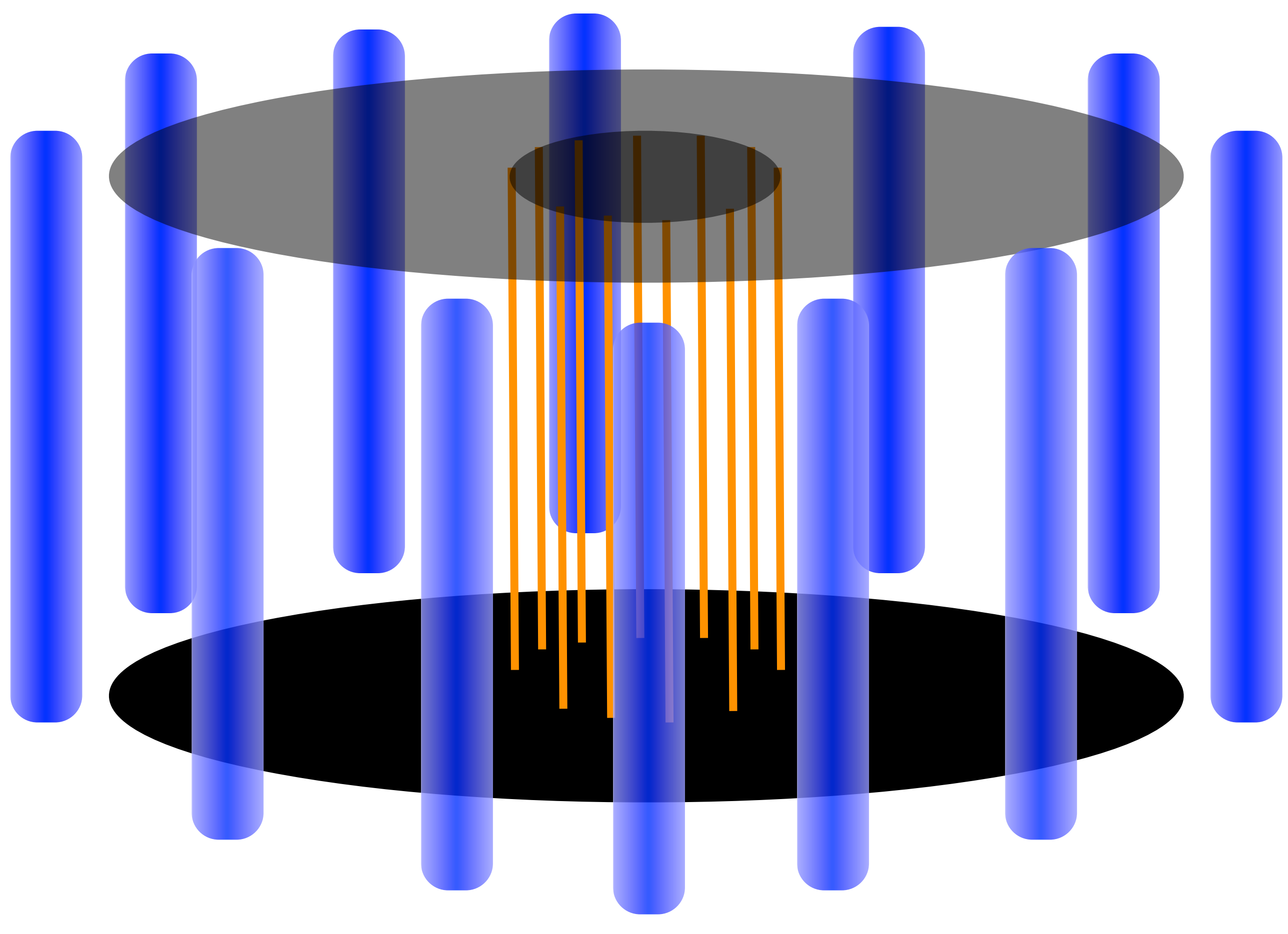
Sample application

- Magnetic force between two parallel wires



- Magnetic forces in the Z-machine (Sandia National Labs)



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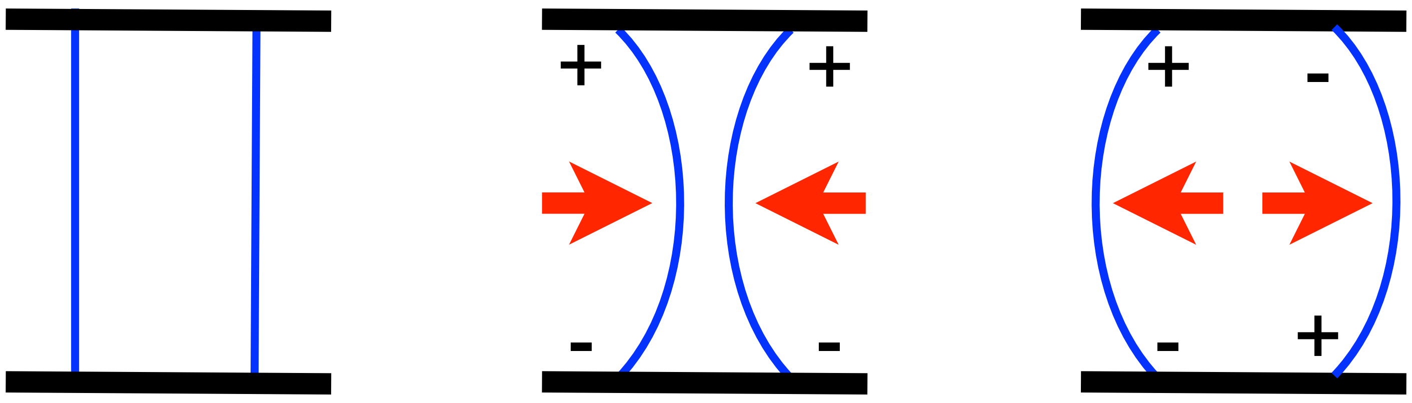

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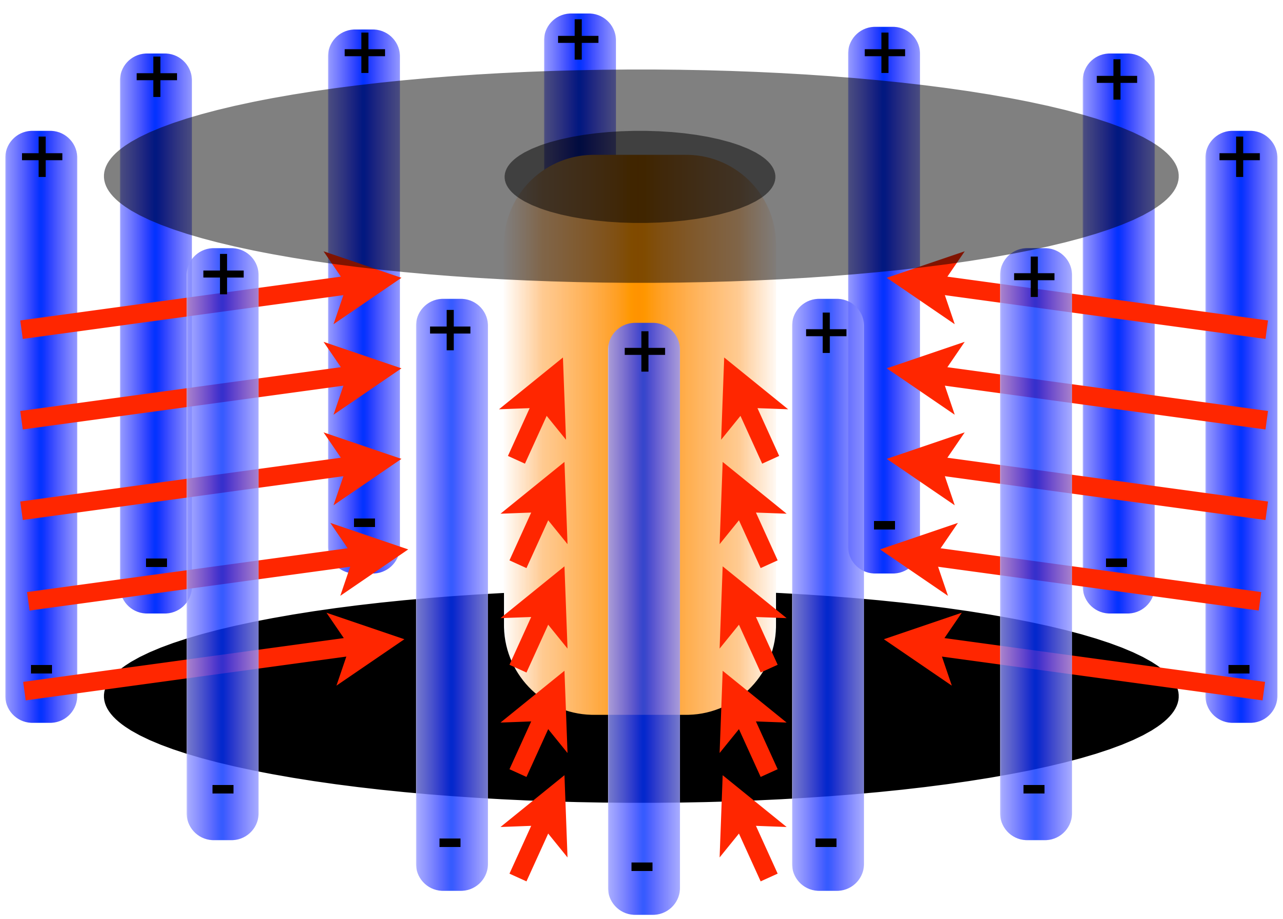

Idealized Z-Pinch implosions

- Phenomenological model by Banks and Shadid[(a)]

$$\partial_t \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \rho E \end{pmatrix} + \nabla \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} + p\mathcal{I} \\ \rho E \mathbf{v} + p \mathbf{v} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{f} \\ \mathbf{f} \cdot \mathbf{v} \end{pmatrix}$$

$$0 \leq \lambda \leq 1 \qquad \partial_t(\lambda \rho) + \nabla \cdot (\lambda \rho \mathbf{v}) = 0$$

- Lorentz force term and equation of state

$$\mathbf{f} = (\rho \lambda) \frac{12(1 - t^4)t}{\max\{r, 10^{-4}\}} \hat{\mathbf{e}}_r$$

$$p = (\gamma - 1)\rho(E - 0.5\|\mathbf{v}\|^2)$$

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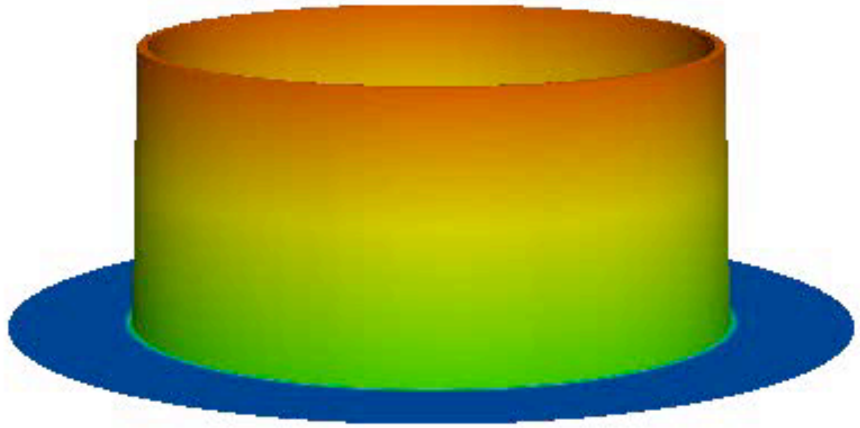

Numerical challenges

- transient transport processes with strongly varying time scales
- accurate resolution of moving fronts both in time and space
- fluid quantities like mass density must not become negative
- large nonlinear coupled systems of (hyperbolic) conservation laws

Algebraic Flux Correction, abbr. **AFC**, family of high-resolution schemes for convection-dominated transport and anisotropic diffusion problems

- universal stabilization approach based on algebraic design criteria
- approved for conforming (multi-)linear finite element schemes
- found complicated to extend to higher-order finite elements[(e)]

Outline

- Review of algebraic flux correction schemes in the context of conforming finite elements
- Extension of the algebraic flux correction paradigm to nonconforming finite elements
- Parallelization of edge-based AFC schemes
- Summary and Outlook

Algebraic Flux Correction

Part I: Basic concepts

Design criteria
Discrete upwinding
AFC-type schemes

Model problem: $\partial_t u + \nabla \cdot \mathbf{f} = 0, \quad \mathbf{f} = \mathbf{v}u$

- FEM approximation $\quad (w_h, \partial_t u_h + \nabla \cdot \mathbf{f}_h)_\Omega = 0 \quad \forall w_h \in W_h$

$$u_h = \sum_j \varphi_j(\mathbf{x}) u_j(t), \qquad \mathbf{f}_h = \sum_j \varphi_j(\mathbf{x}) \mathbf{f}(u_j(t))$$

- Galerkin method

$$(\varphi_i, \varphi_j)_\Omega \qquad (\varphi_i, -\nabla \varphi_j)_\Omega \cdot \mathbf{v}_j$$

$$\forall i : \quad \sum_j m_{ij} \dot{u}_j = \sum_j k_{ij} u_j$$

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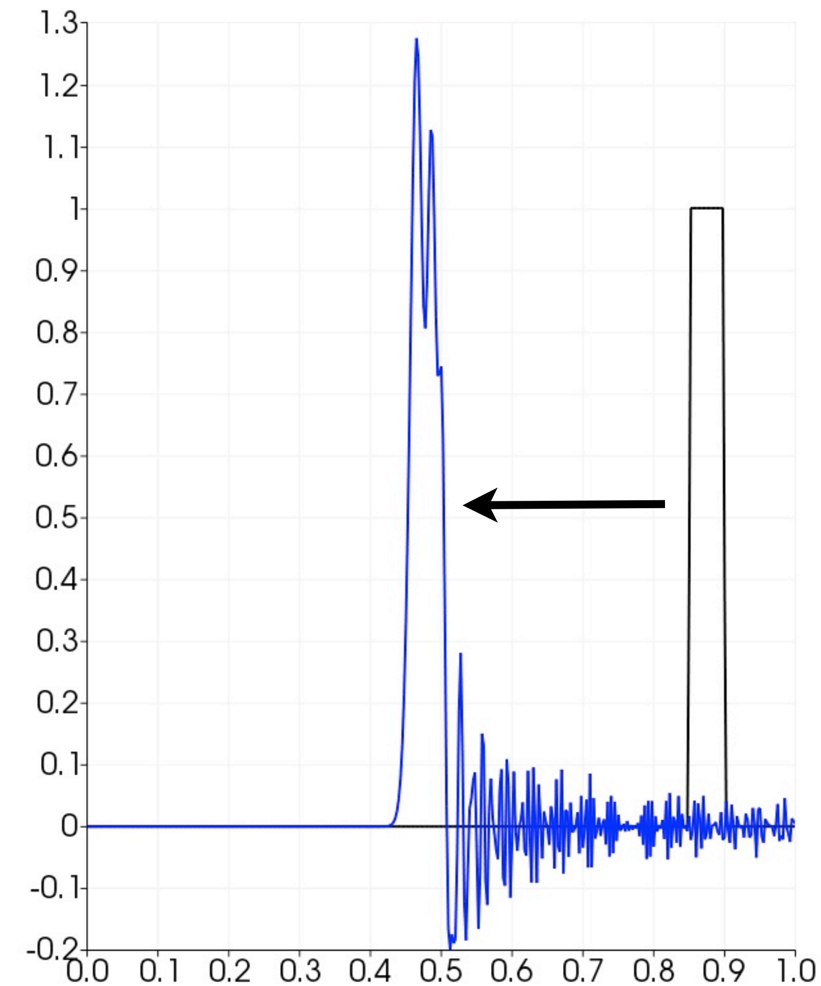
- **Problem**
  generation of spurious oscillations near steep gradients occur unless stabilization of the convective term is performed
  - algebraic flux correction (AFC)



Local Extremum Diminishing[(b)]

- Let the semi-discrete system be given by

$$m_i \dot{u}_i = \sum_{j \neq i} \sigma_{ij}(u_j - u_i) \qquad \begin{array}{ll} m_i > 0, & \forall i \\ \sigma_{ij} \geq 0 & \forall j \neq i \end{array}$$

Then local extrema are not enhanced

$$u_i = \left\{ \begin{array}{l} u^{\max} \\ u^{\min} \end{array} \right. \quad \Rightarrow \quad \dot{u}_i = \frac{1}{m_i} \sum_{j \neq i} \sigma_{ij}(u_j - u^{\substack{\max \\ \min}}) \left\{ \begin{array}{l} \leq 0 \\ \geq 0 \end{array} \right.$$

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- Galerkin method violates LED constraint

$$\sum_j m_{ij} \dot{u}_j = \sum_j k_{ij} u_j = \sum_{j \neq i} k_{ij}(u_j - u_i) + u_i \sum_j k_{ij}$$

intrinsic coupling

$\exists k_{ij} < 0,\ j \neq i$

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$$\underbrace{\sum_j m_{ij} \dot{u}_j}_{\text{intrinsic coupling}} = \sum_j k_{ij} u_j = \sum_{j \neq i} k_{ij}(u_j - u_i) + u_i \underbrace{\sum_j k_{ij}}_{=:\, g_i}$$

intrinsic coupling

$\exists k_{ij} < 0,\ j \neq i$

$=: g_i$

Discrete Upwinding[(c)]

- Prerequisite: $m_{ij} = (\varphi_i, \varphi_j)_\Omega \geq 0 \ \forall i,j$

$$\wedge \ m_i = \sum_j m_{ij} > 0 \ \forall i$$

Galerkin method
+ mass lumping
+ artificial diffusion

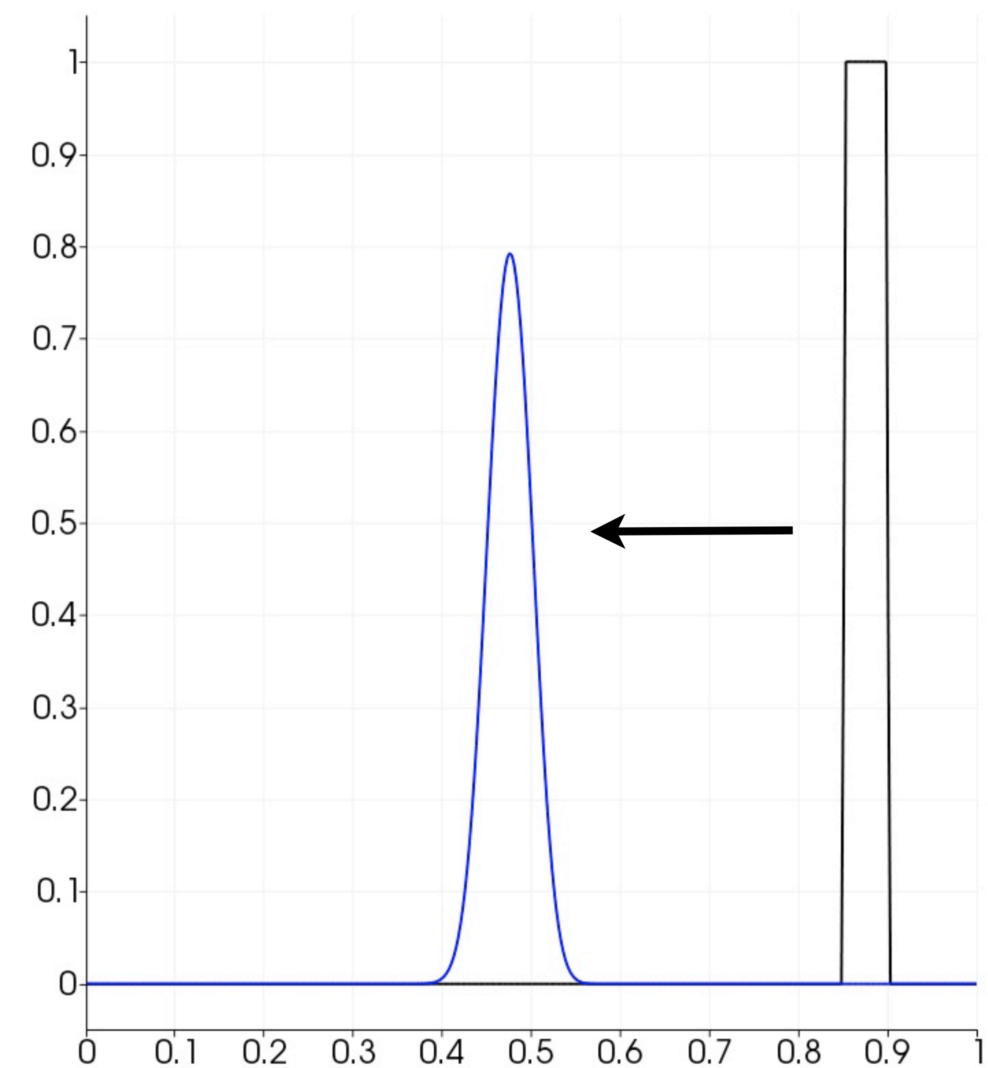
- **Low-order method**

$$m_i \dot{u}_i = \sum_{j \neq i} [k_{ij} + d_{ij}](u_j - u_i) + g_i$$

- artificial diffusion coefficient

$$d_{ij} = \max\{-k_{ij}, 0, -k_{ji}\} = d_{ji}$$

$$\Rightarrow \begin{cases} l_{ij} := k_{ij} + d_{ij} \geq 0 \\ l_{ji} := k_{ji} + d_{ji} \geq 0 \end{cases}$$



- satisfies the LED constraint
  but it is overly diffusive

Example in 1D

$$\partial_t u + v\partial_x u = 0, \quad v = \text{const}$$

- Galerkin method $M\dot{u} = Ku$

$$\frac{\Delta x}{6}\begin{bmatrix} 2 & 1 & & \\ 1 & 4 & 1 & \\ & 1 & 4 & 1 \\ & & 1 & 2 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{bmatrix} = \frac{v}{2}\begin{bmatrix} 1 & -1 & & \\ 1 & 0 & -1 & \\ & 1 & 0 & -1 \\ & & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

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mass lumping

- Low-order method

$$\frac{\Delta x}{2} \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{bmatrix} = \frac{v}{2} \begin{bmatrix} 1 & -1 & & \\ 1 & 0 & -1 & \\ & 1 & 0 & -1 \\ & & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

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artificial diffusion

- Low-order method

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- Low-order method $M_L\dot{u} = [K + D]u$

$$\frac{\Delta x}{2}\begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{bmatrix}\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \end{bmatrix} = v\begin{bmatrix} 0 & 0 & & \\ 1 & -1 & 0 & \\ & 1 & -1 & 0 \\ & & 1 & -1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

Algebraic Flux Correction(c-l)

- **AFC-type method**

$$m_i \dot{u}_i = \sum_{j \neq i} l_{ij}(u_j - u_i) + g_i + \sum_{j \neq i} \alpha_{ij} f_{ij}$$

Galerkin method
+ mass lumping
+ artificial diffusion
+ limited antidiffusion

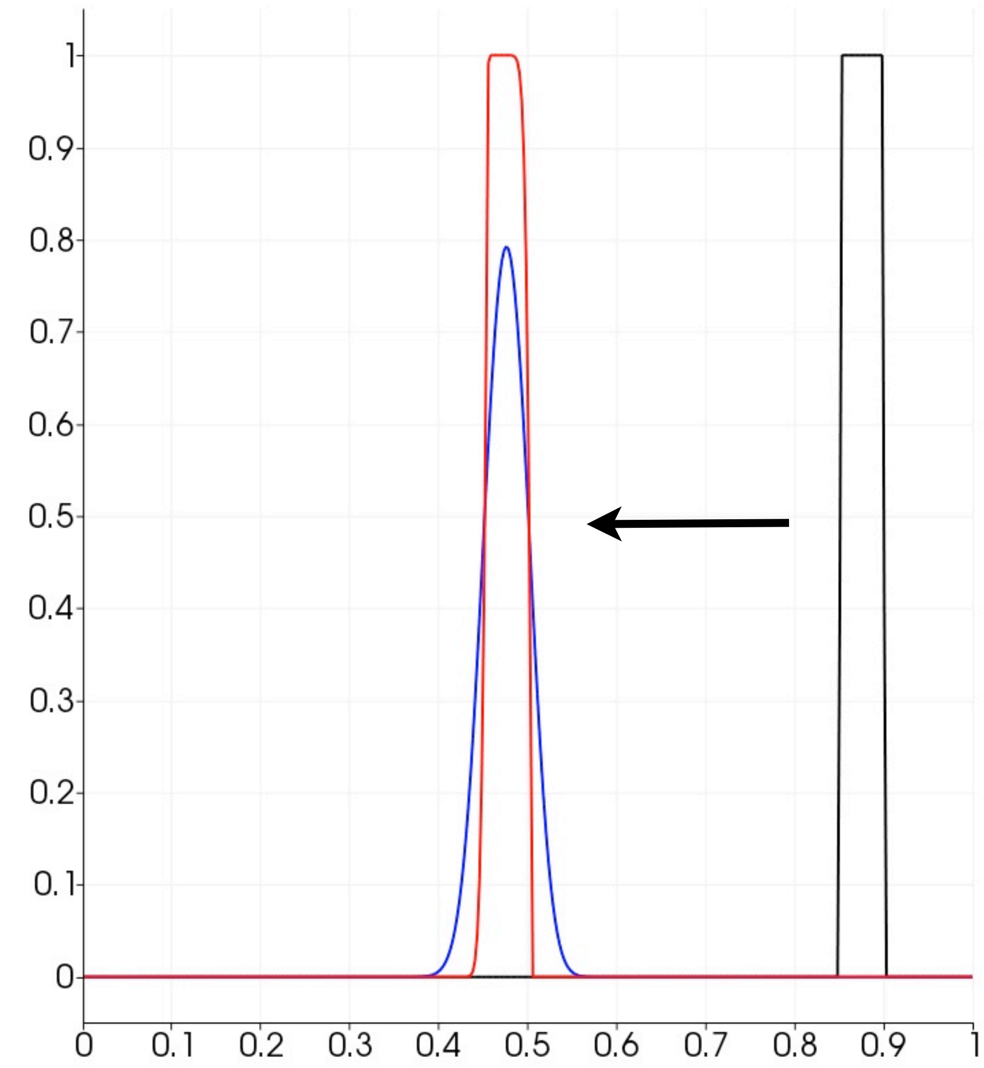
- raw antidiffusive fluxes

$$f_{ij} = m_{ij}(\dot{u}_i - \dot{u}_j) + d_{ij}(u_i - u_j)$$
$$f_{ji} = -f_{ij}$$

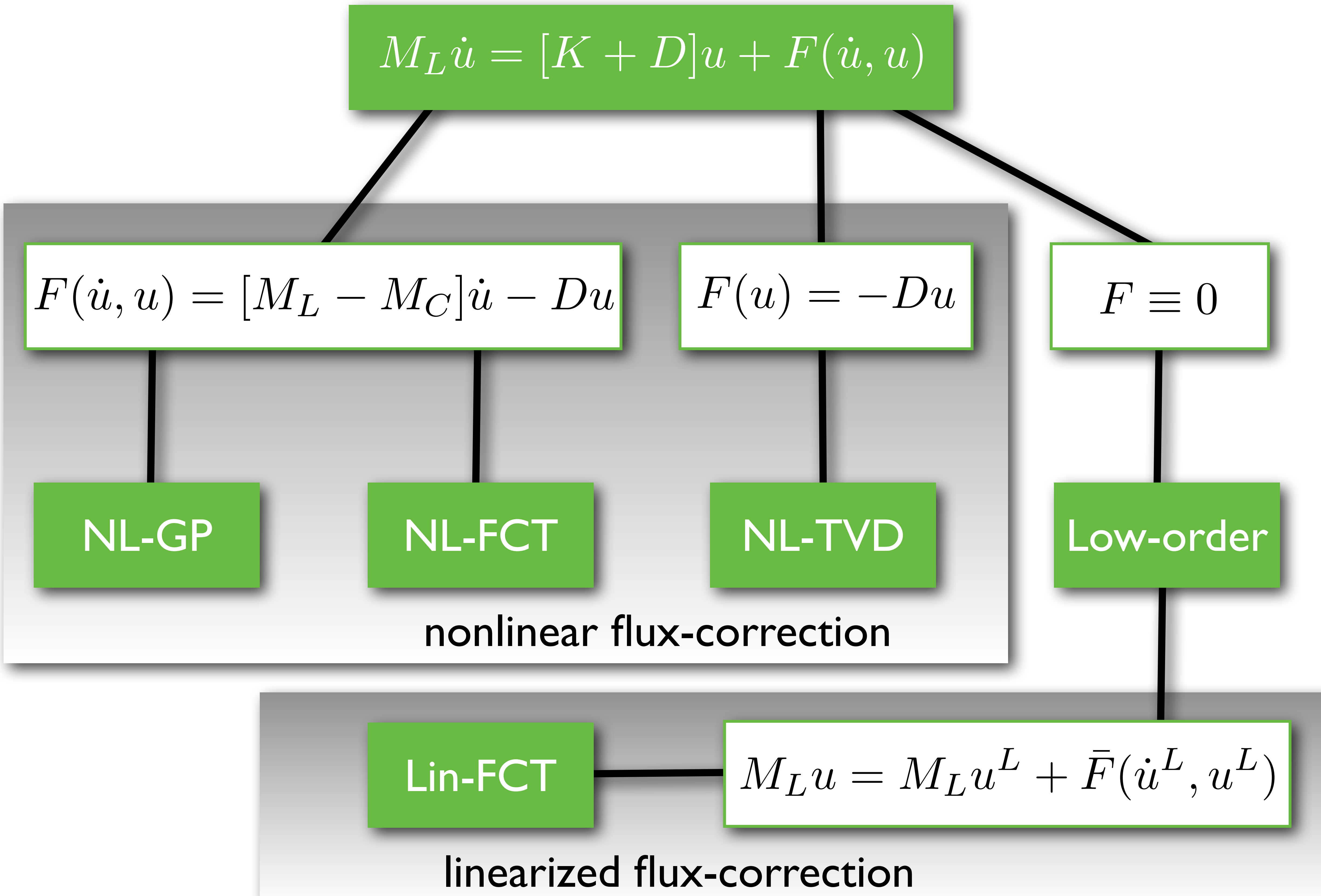
- edge-wise correction factors

$$0 \leq \alpha_{ij} = \alpha_{ji} \leq 1$$

are computed by some variant of Zalesak's multidimensional limiter



Family of AFC-type method



Linearized FCT algorithm[f]

- Compute low-order predictor by Crank-Nicolson time stepping scheme

$$\left(\frac{1}{\Delta t}M_L - \frac{1}{2}[K+D]\right)u^L = \frac{1}{\Delta t}M_L u^n + \frac{1}{2}[K+D]u^n$$

- Approximate the time derivative

$$\dot{u}^L = M_L^{-1}[K+D]u^L$$

- Linearize the raw antidiffusive fluxes and perform prelimiting

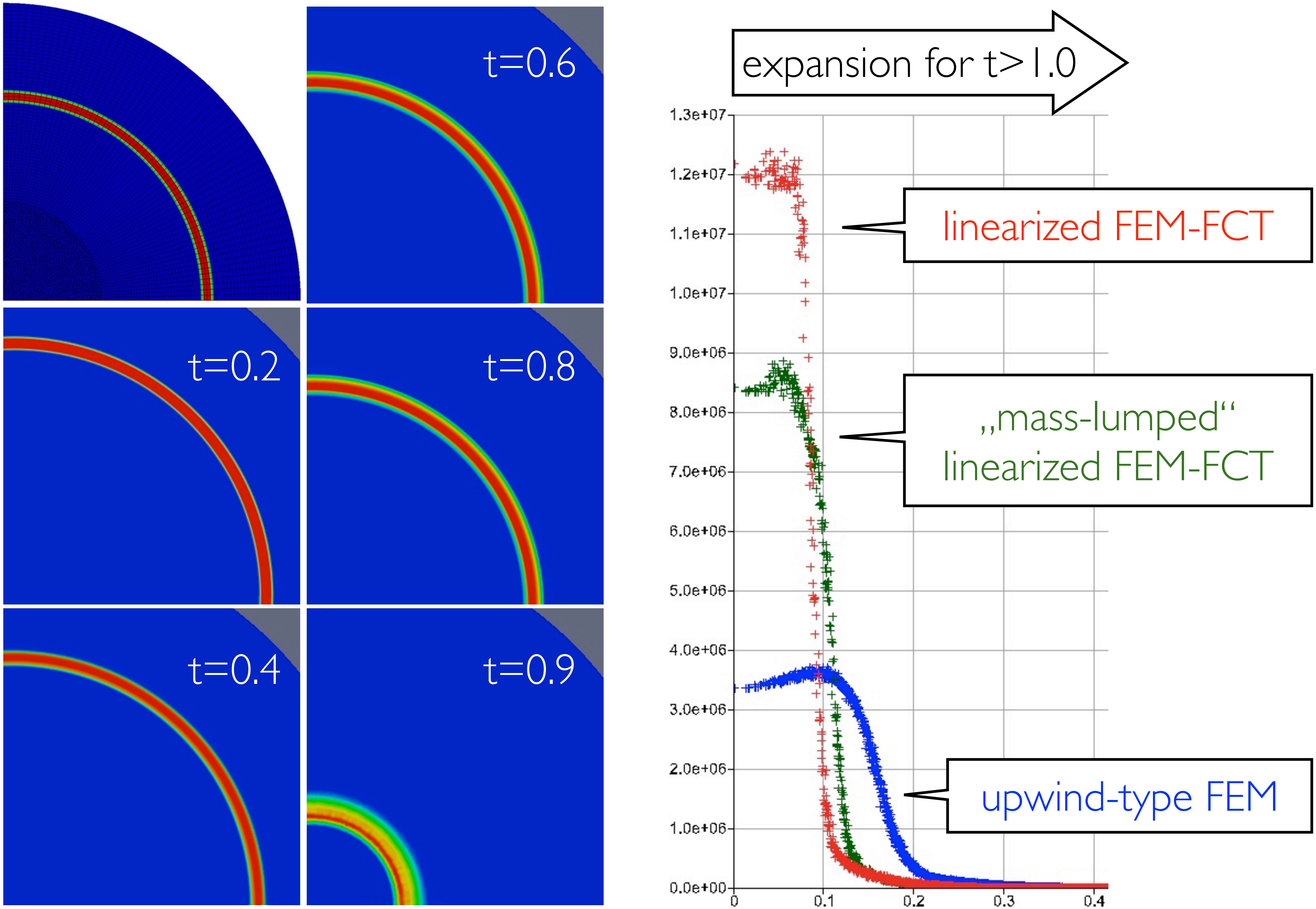
$$f_{ij} = m_{ij}(\dot{u}_i^L - \dot{u}_j^L) + d_{ij}(u_i^L - u_j^L)$$

$$f'_{ij} := 0 \quad \text{if} \quad f_{ij}(u_j^L - u_i^L) > 0$$

- Compute correction factors α_{ij} and update end-of-step solution

$$u_i^{n+1} = u_i^L + \frac{\Delta t}{m_i}\sum_{j\neq i}\alpha_{ij} f'_{ij}$$

Linearized FCT for the Z-pinch problem[g,h]



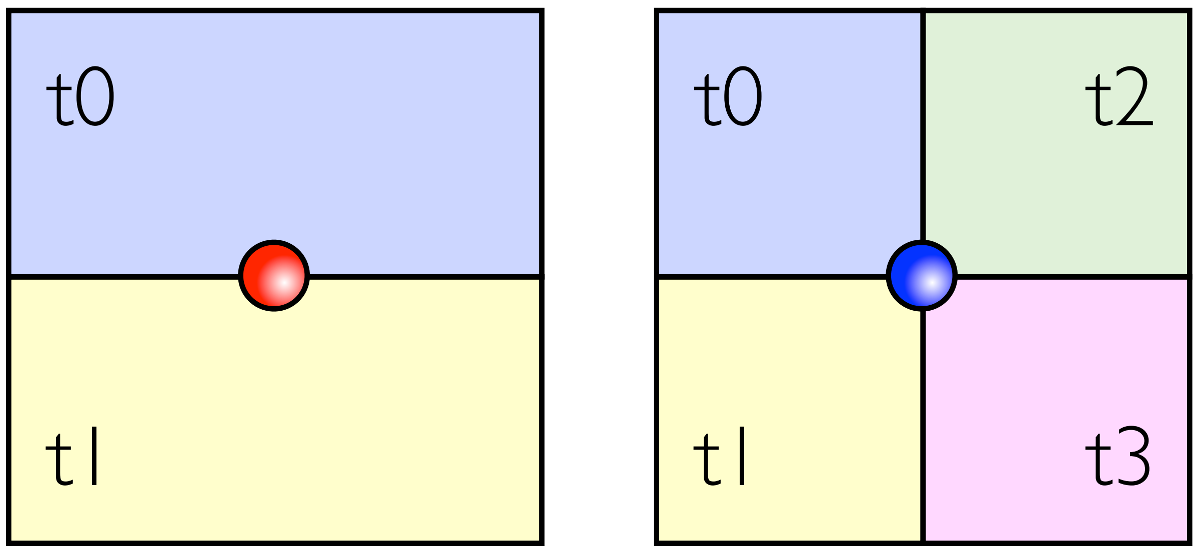
Algebraic Flux Correction

Part II: Extension to nonconforming finite elements

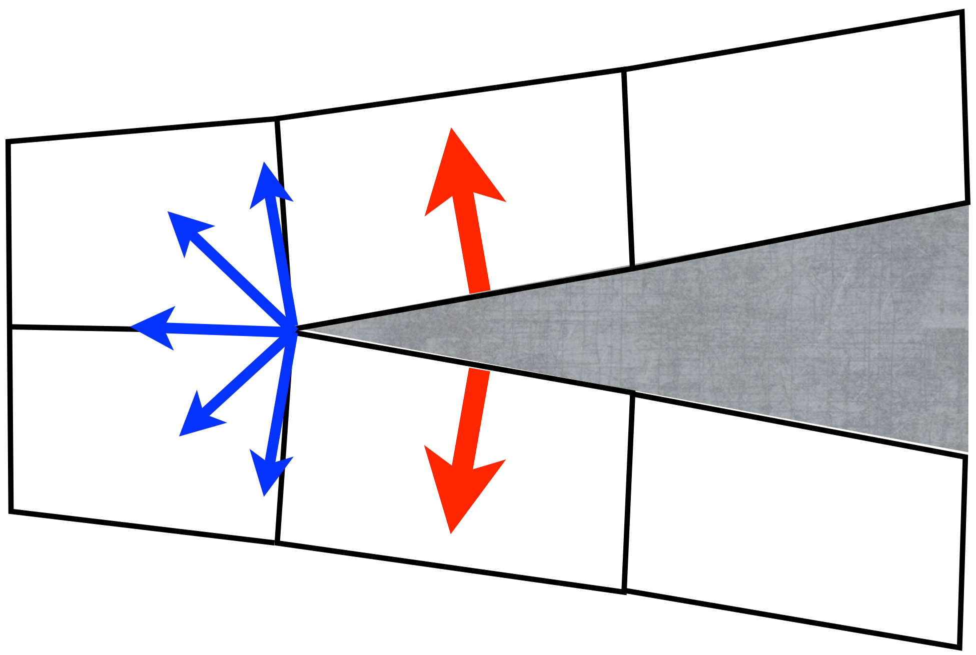
Finite element spaces
Review of design criteria
Numerical examples

Reasons for nonconforming finite elements

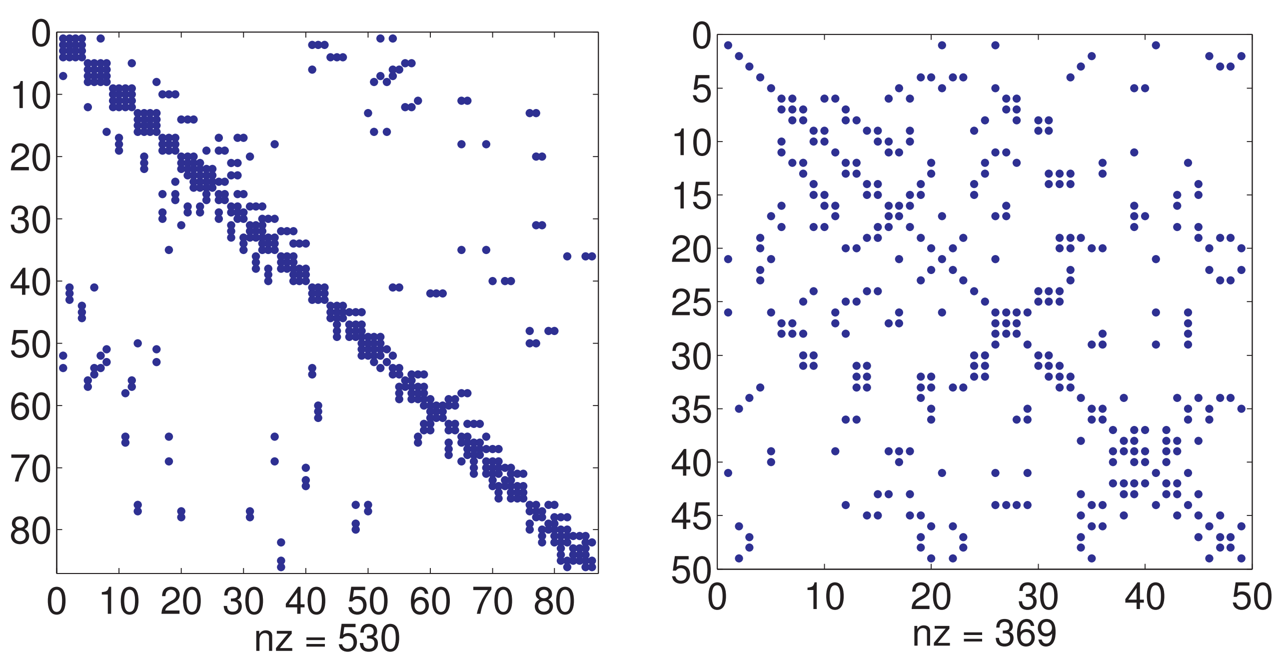
Reduction of communication costs in parallel computations



Unique definition of normal vectors



‚Regular' sparsity pattern



Scientific curiosity

Do nonconforming approximations work in the AFC machinery?

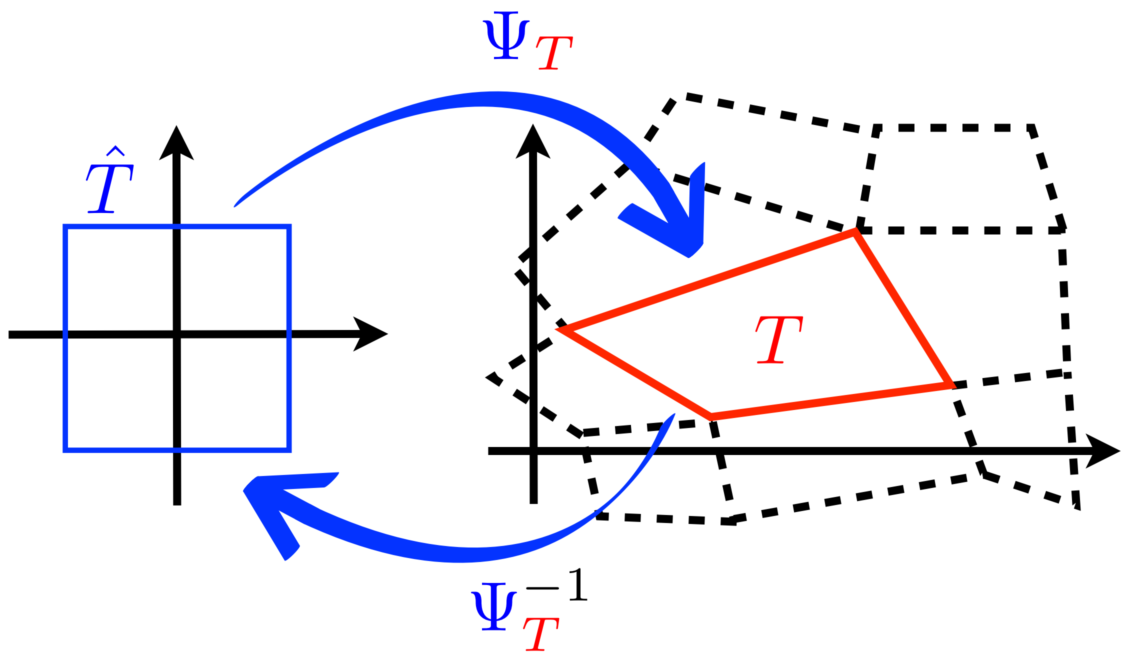
Parametric finite elements

- Reference map

  $$\Psi_T \; : \; \hat{T} := [-1,1]^2 \mapsto T \in \mathcal{T}_h$$

- Polynomial space

  $$\mathcal{Q}(T) = \{q = \hat{q} \circ \Psi_T^{-1},\, \hat{q} \in \hat{\mathcal{Q}}(\hat{T})\}$$



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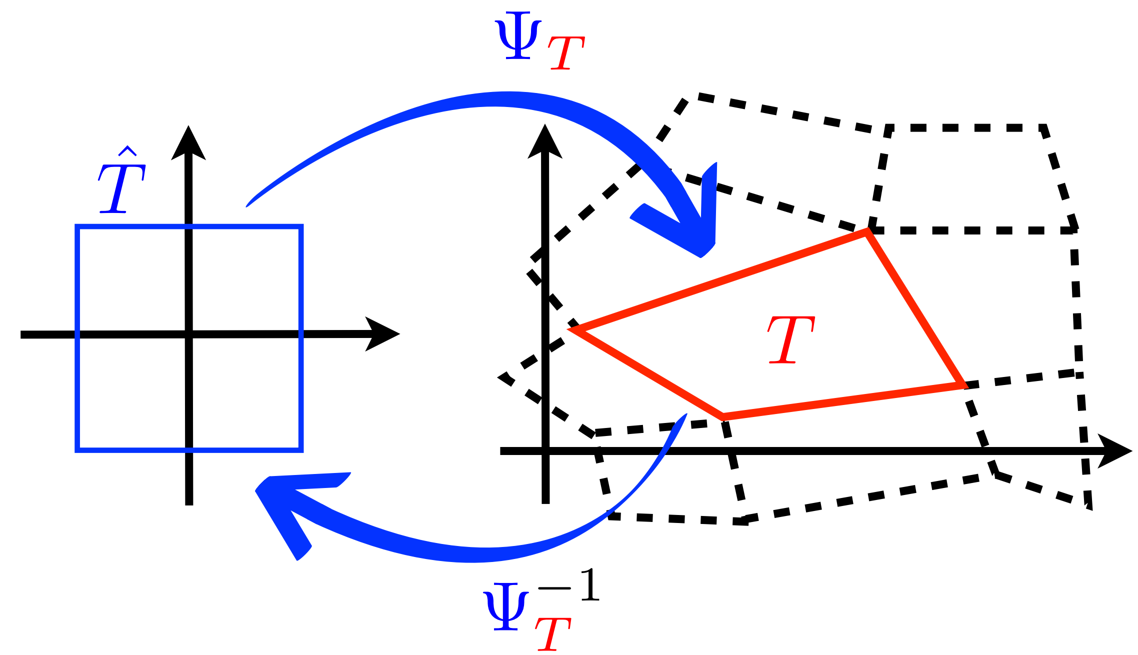


$\hat{Q}_1(\hat{T}) = \mathrm{span}\langle 1, \hat{x}, \hat{y}, \hat{x}\hat{y}\rangle$	$\hat{Q}_1^{\mathrm{nc}}(\hat{T}) = \mathrm{span}\langle 1, \hat{x}, \hat{y}, \hat{x}^2 - \hat{y}^2\rangle$
■ AFC is known to work $\mathrm{supp}(\varphi_i) \cap \mathrm{supp}(\varphi_j) \neq \emptyset$ $\Leftrightarrow\ m_{ij} = (\varphi_i, \varphi_j)_\Omega > 0$	■ need to verify prerequisites $m_{ij} = (\varphi_i, \varphi_j)_\Omega \geq 0,\ \forall i, j$ $m_i = \sum_j m_{ij} > 0,\ \forall i$

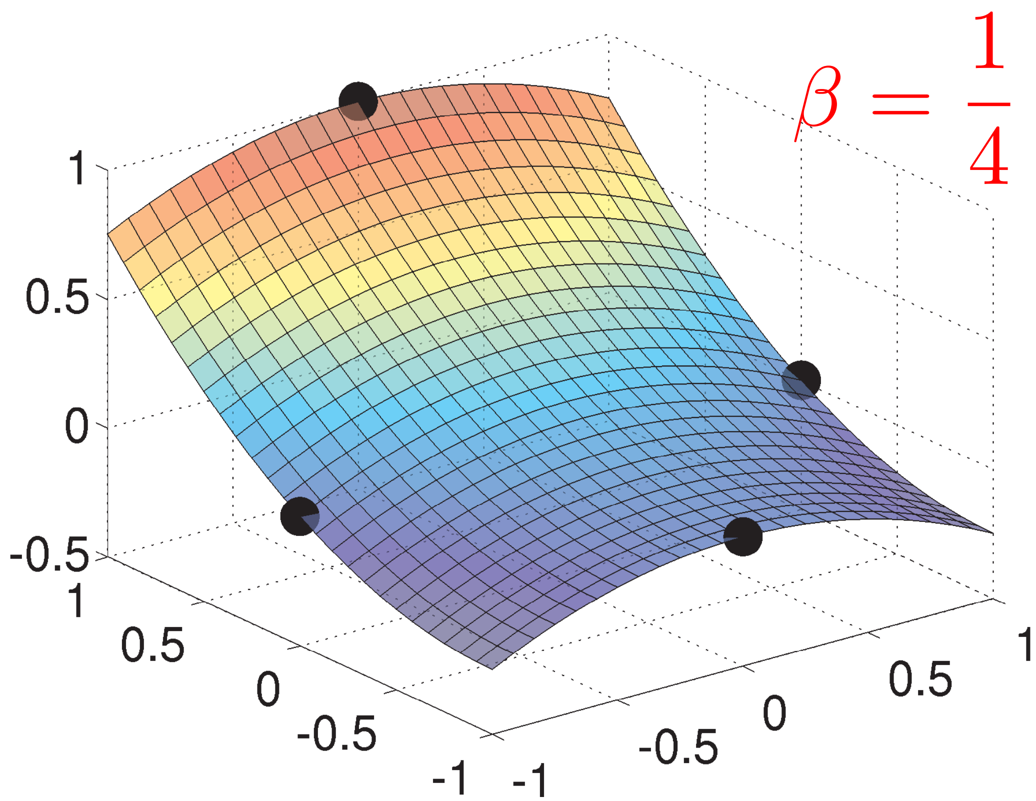
Rannacher-Turek element(m)

- Rotated bilinear shape functions

$$\hat{\varphi}^{(k)}(\hat{x},\hat{y}) = \begin{cases} \frac{1}{4} \pm \frac{1}{2}\hat{x} + \beta(\hat{x}^2 - \hat{y}^2), & k = 1,3 \\ \frac{1}{4} \pm \frac{1}{2}\hat{y} - \beta(\hat{x}^2 - \hat{y}^2), & k = 2,4 \end{cases}$$

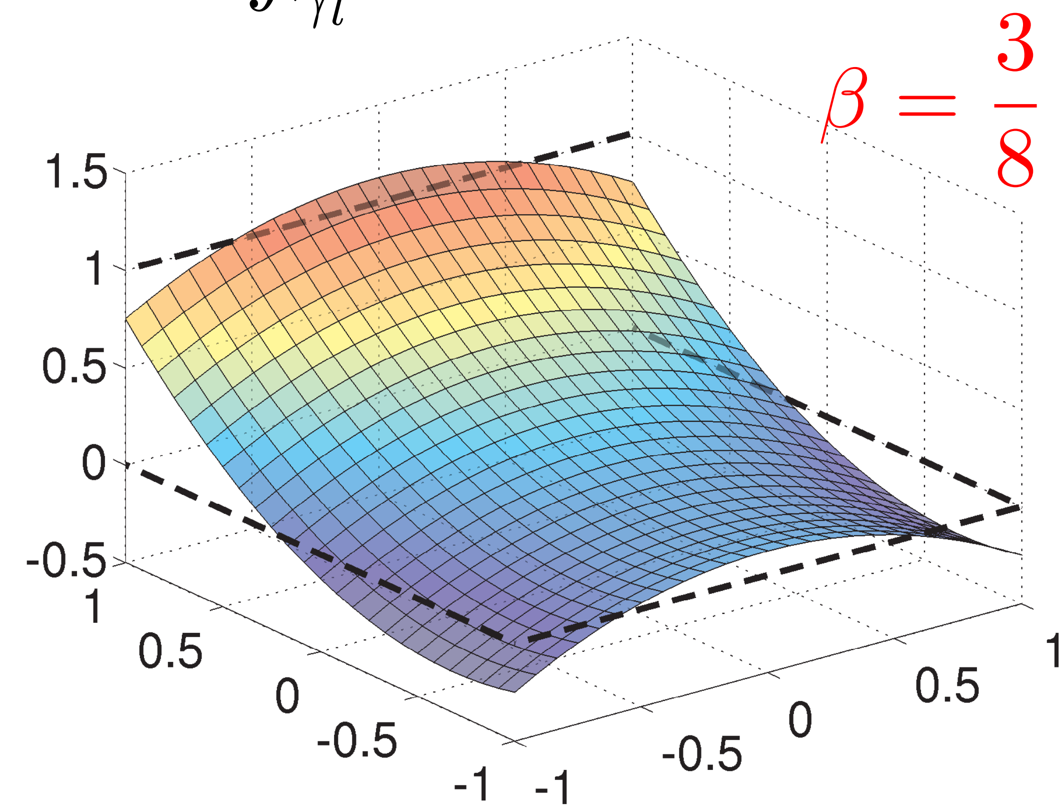
- Based on midpoint values

$$\hat{\varphi}^{(k)}(\hat{\mathbf{m}}_l) = \delta_{kl}$$



- Based on integral mean values

$$|\hat{\gamma}_l|^{-1} \int_{\hat{\gamma}_l} \hat{\varphi}^{(k)}(\hat{x},\hat{y})\,\mathrm{d}\gamma = \delta_{kl}$$



Matrix analysis

- Local mass matrix evaluated on the reference element
  - Midpoint based variant does not satisfy the prerequisites

$$\hat{M}_{\text{"}1/4\text{"}} = \frac{1}{180} \begin{pmatrix} 113 & 37 & -7 & 37 \\ 37 & 113 & 37 & -7 \\ -7 & 37 & 113 & 37 \\ 37 & -7 & 37 & 113 \end{pmatrix}$$

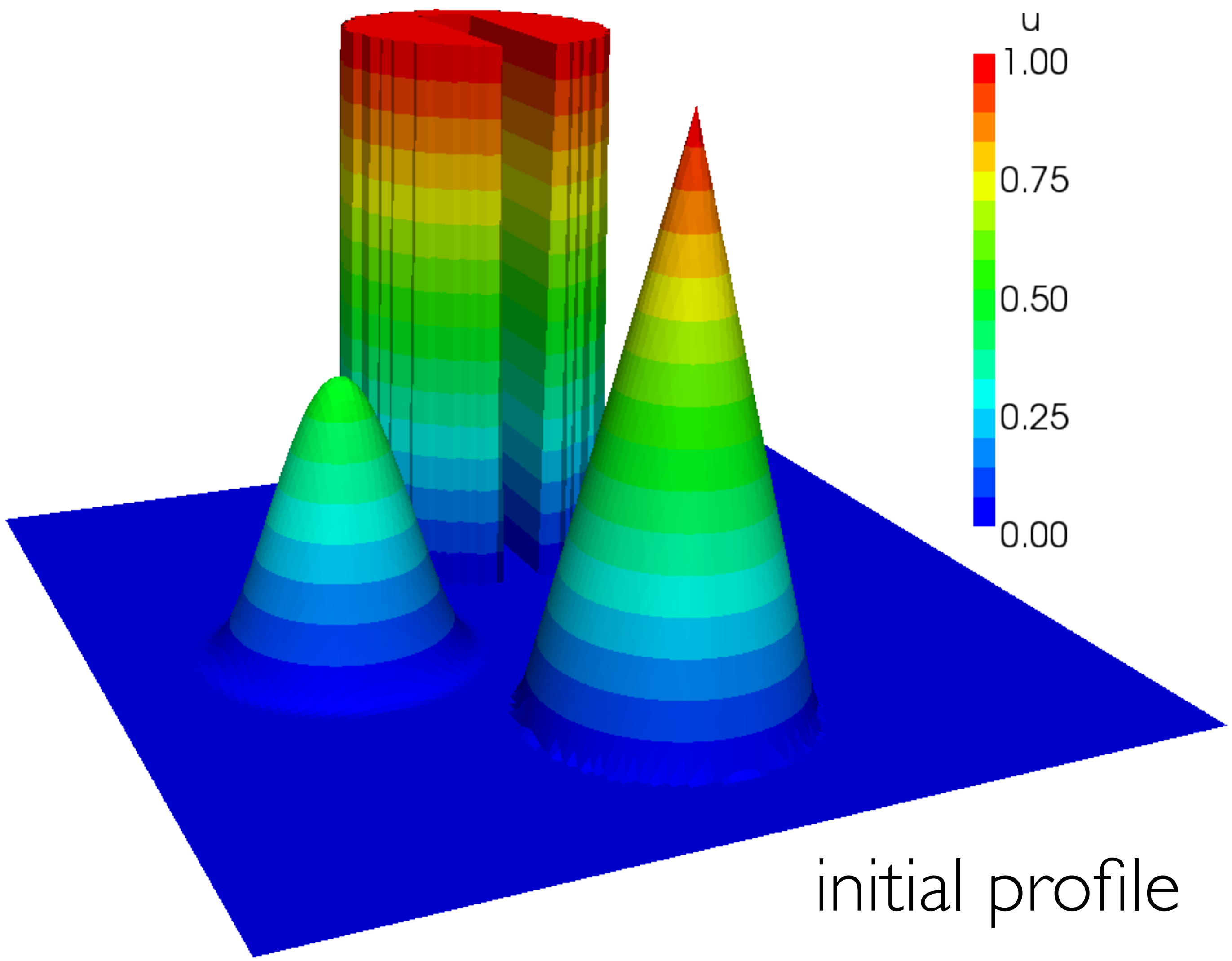
  - Integral mean value based variant satisfies prerequisites

$$\hat{M}_{\text{"}3/8\text{"}} = \frac{1}{60} \begin{pmatrix} 41 & 9 & 1 & 9 \\ 9 & 41 & 9 & 1 \\ 1 & 9 & 41 & 9 \\ 9 & 1 & 9 & 41 \end{pmatrix}$$

Solid body rotation

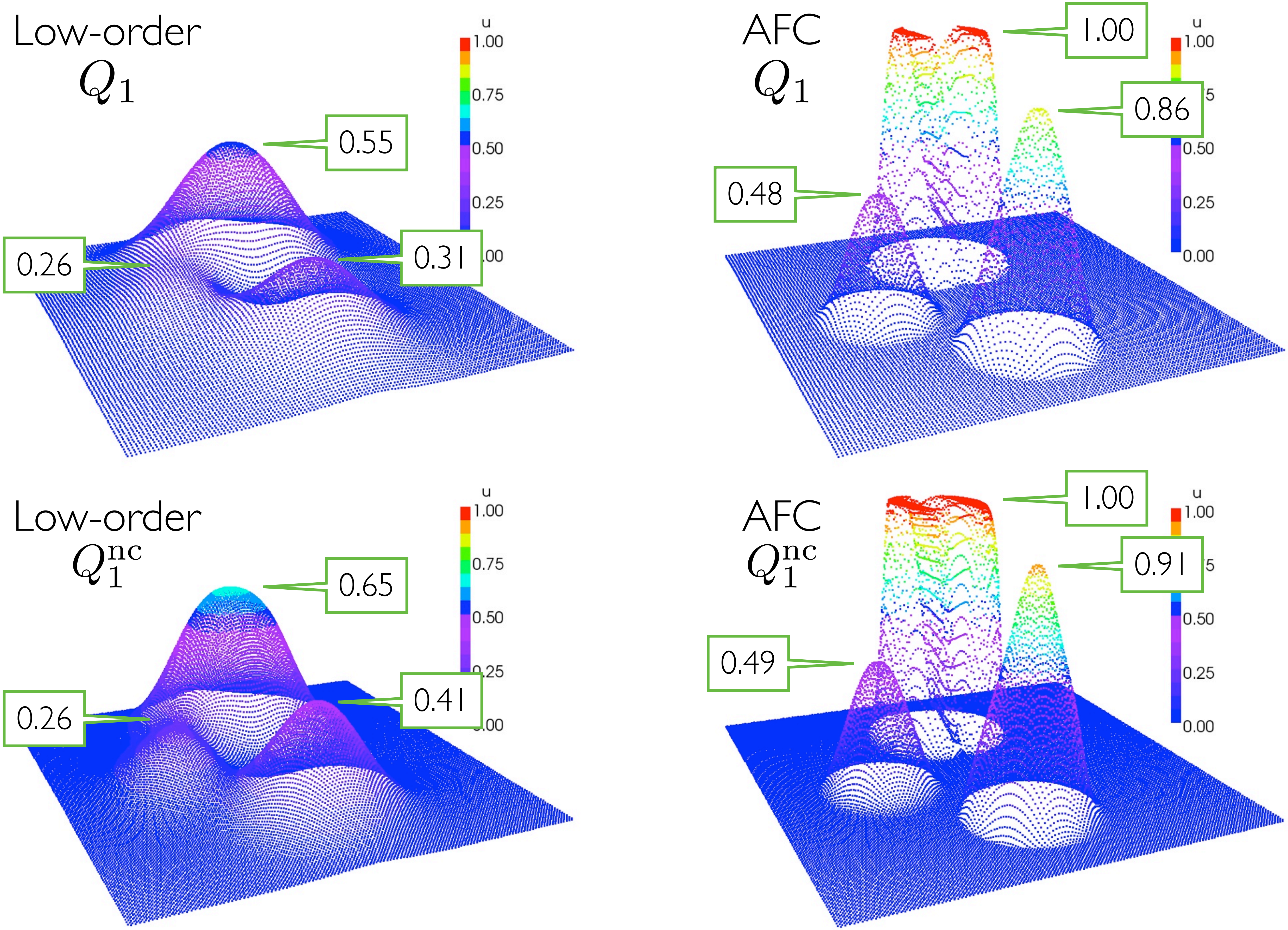
$$\dot{u} + \nabla \cdot (\mathbf{v}u) = 0 \;\text{ in } (0,1)^2$$
$$u = 0 \;\text{ on } \Gamma_{\text{inflow}}$$

- Velocity field
  $\mathbf{v} = (0.5 - y, x - 0.5)$
- Grid size
  $h = 1/2^l, l = 5, 6, \ldots$
- Crank-Nicolson scheme
  $\Delta t = 1.28 \cdot h$
- Initial = exact solution at
  $t = 2\pi k,\; k \in \mathbb{N}$



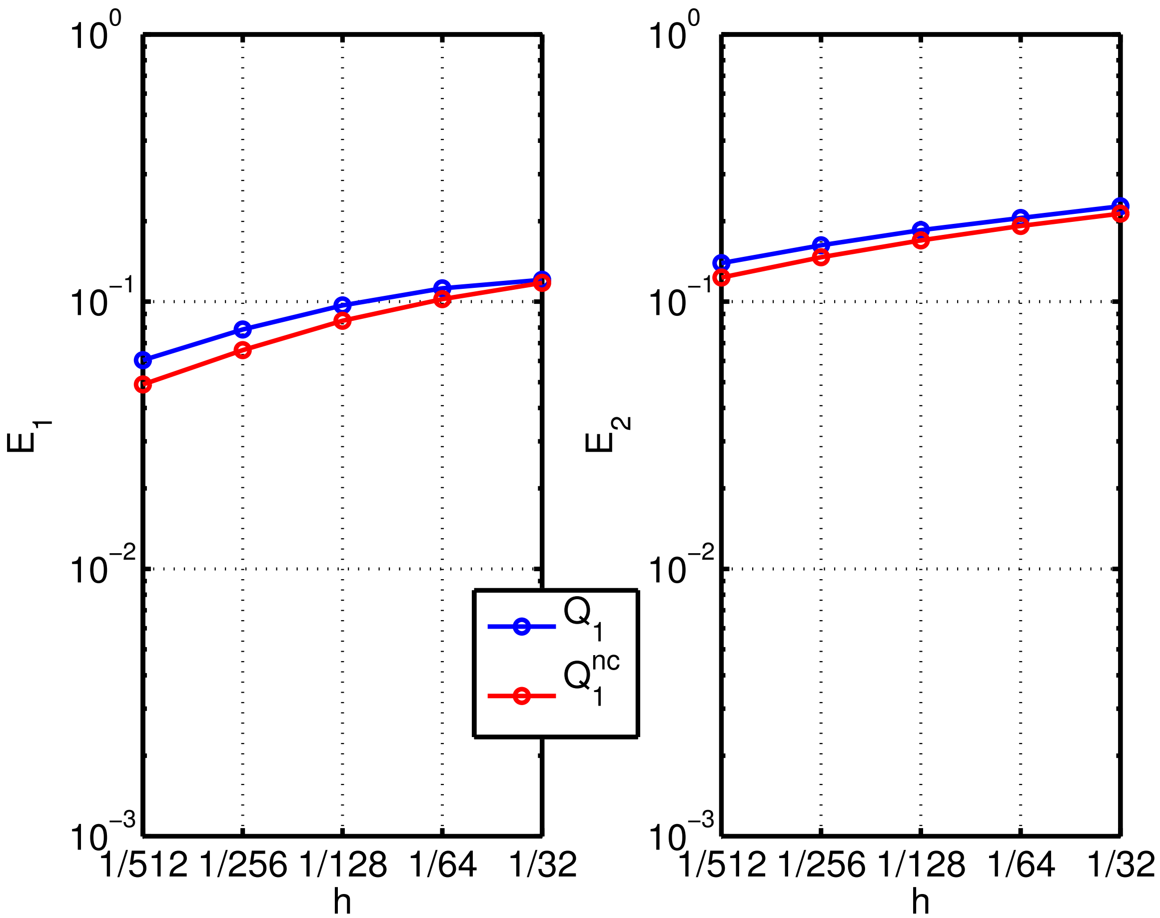
initial profile

SBR: bilinear *vs.* Rannacher-Turek elements

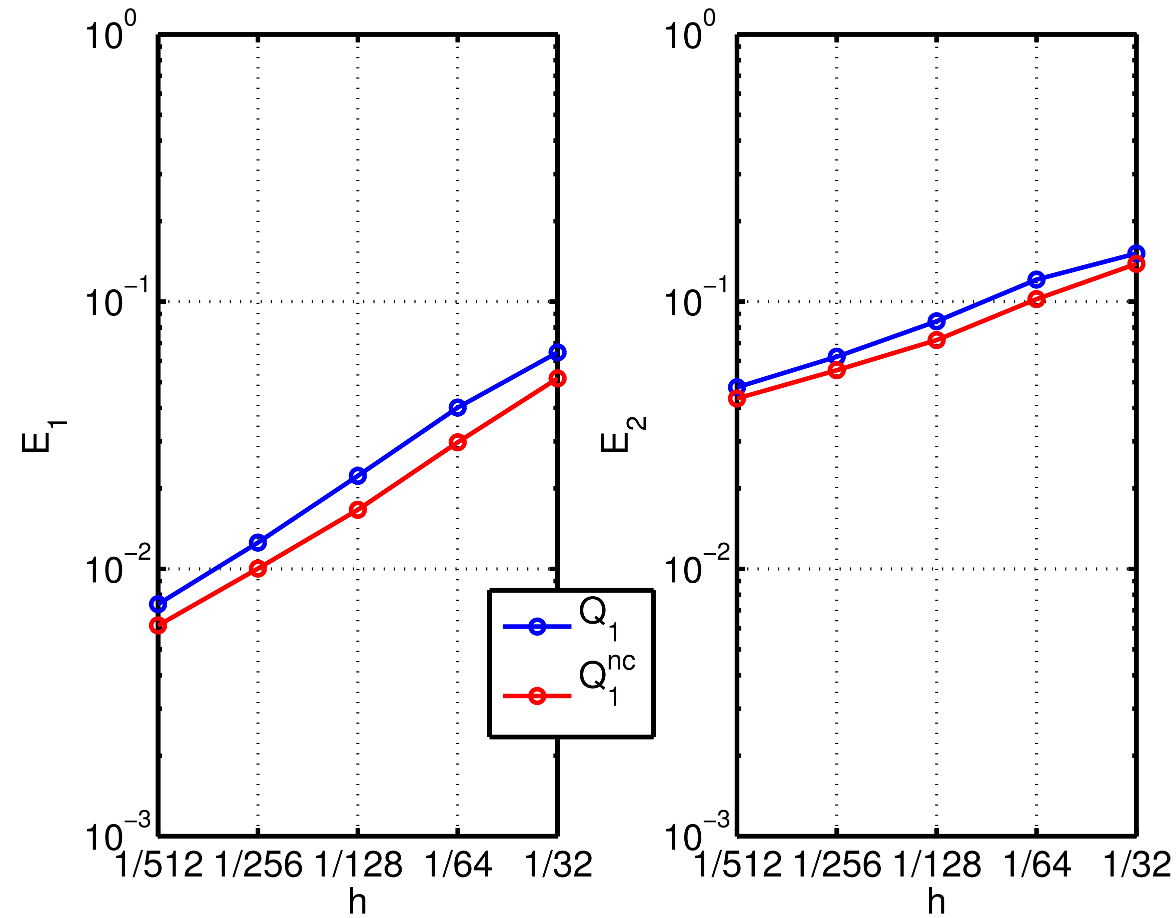


SBR: convergence history

Low-order method



AFC-type method



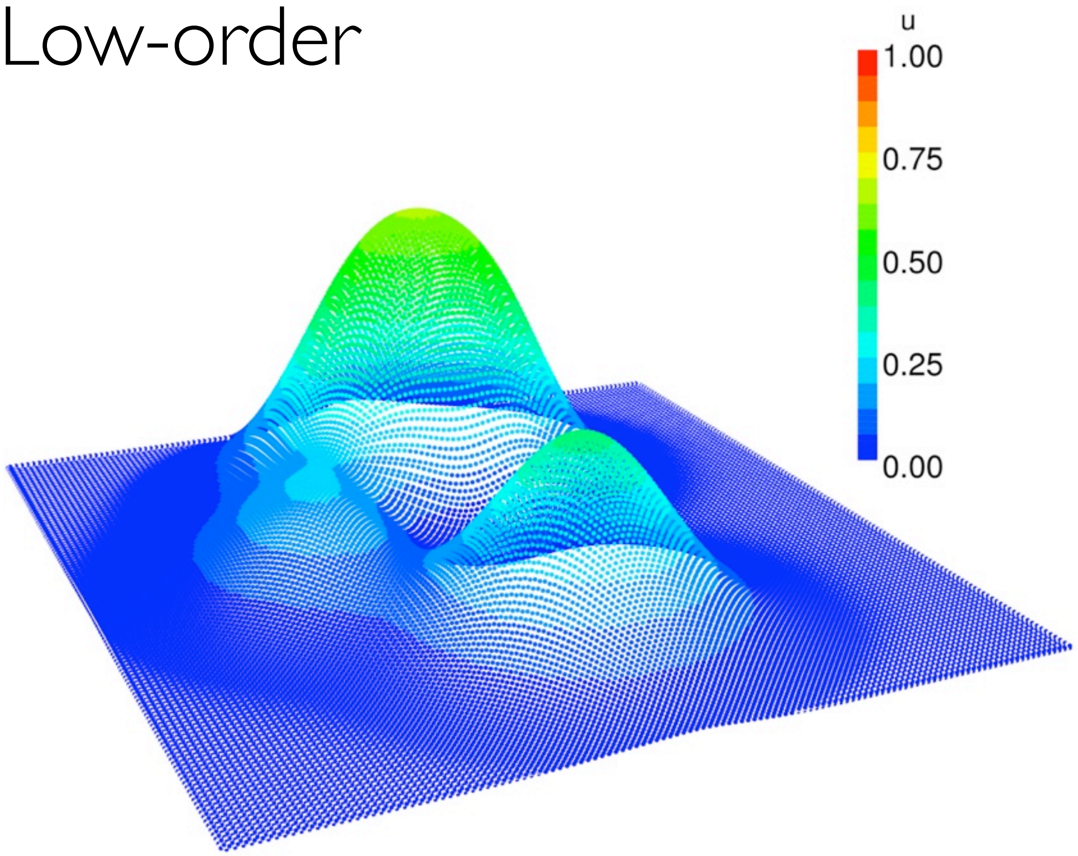
	p1	p2
Q_1	0.38	0.22
Q_1^{nc}	0.43	0.25

estimated order of accuracy

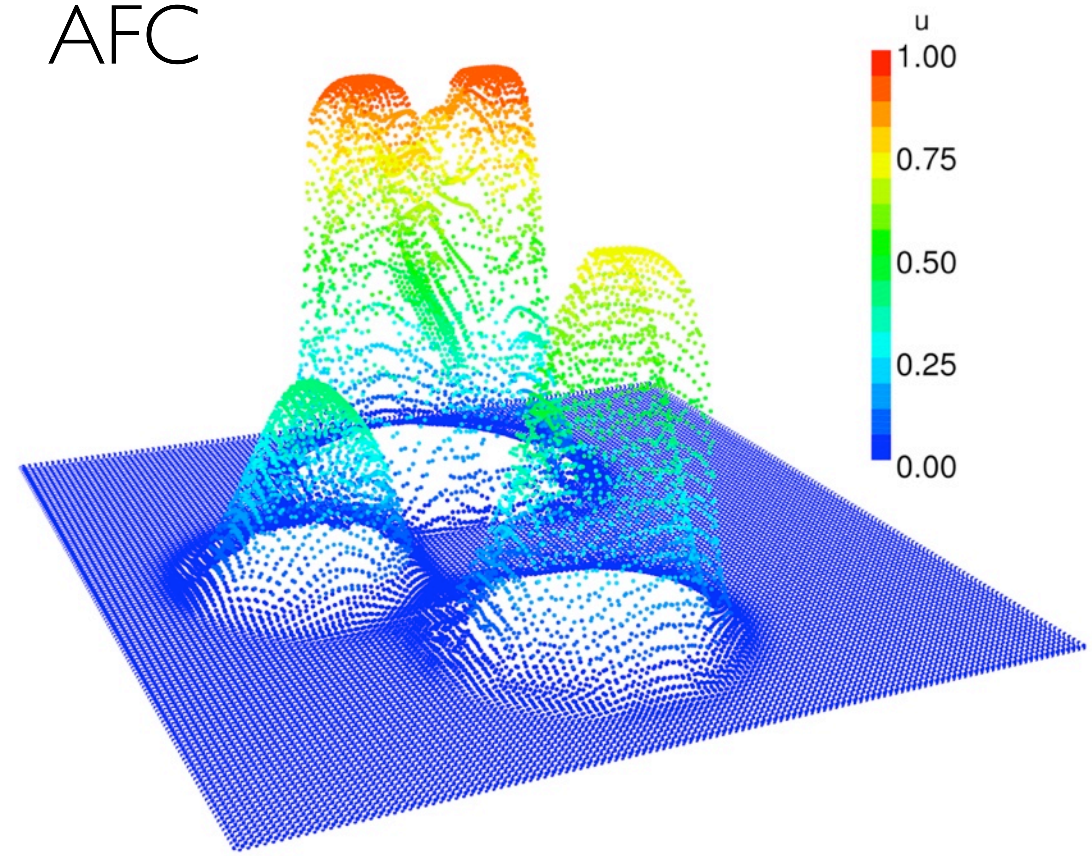
	p1	p2
Q_1	0.76	0.38
Q_1^{nc}	0.71	0.35

SBR: midpoint based Q_1^{nc} variant

Low-order



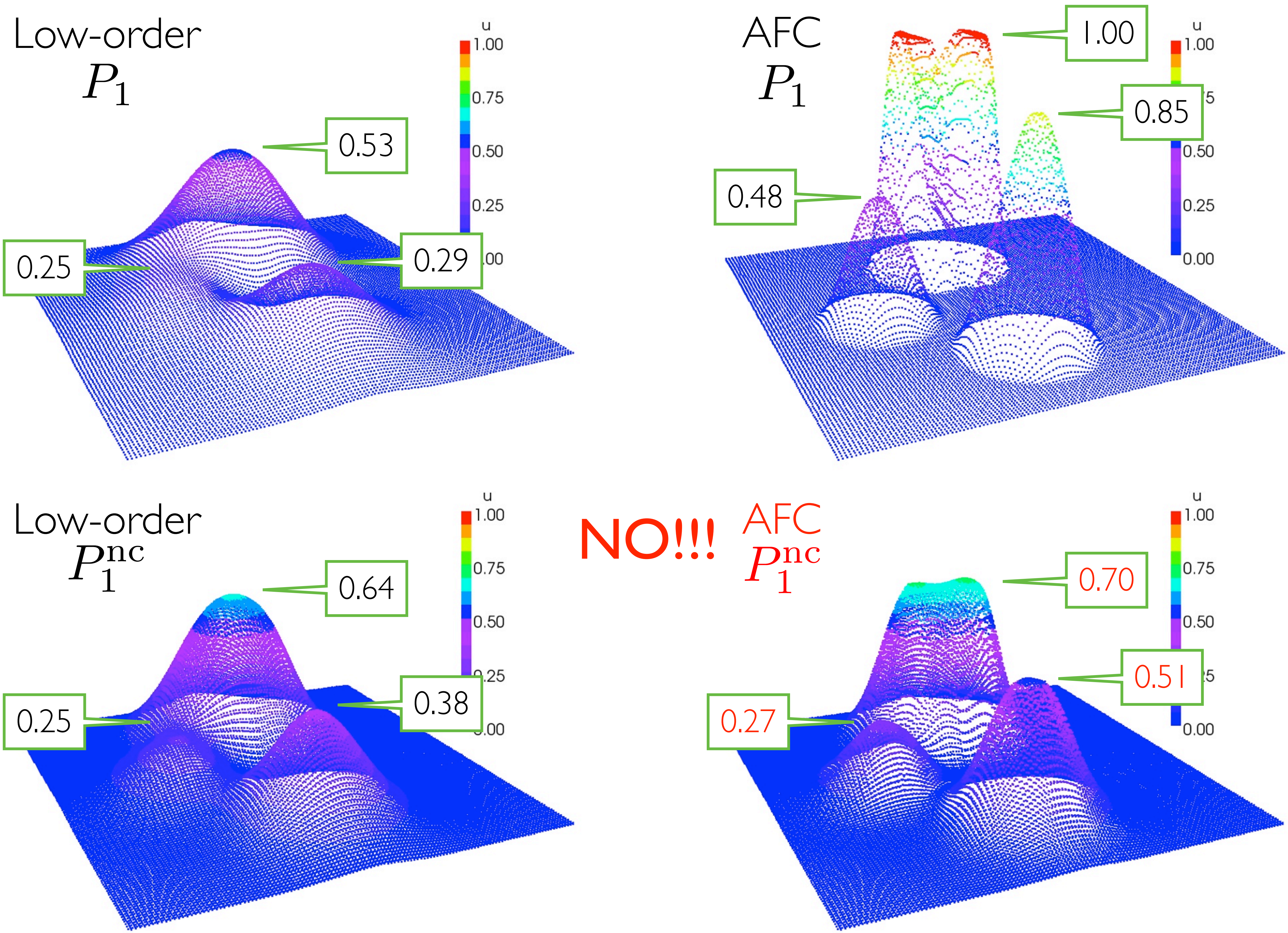
AFC



- The sign criteria on the coefficients of the mass matrix is a necessary condition for the application of the AFC machinery.
- Is it also a sufficient condition for AFC-type methods to work?

<u>Test</u>: nonconforming Crouzeix-Raviart elements on triangles lead to a diagonal consistent mass matrix with strictly positive coefficients!

SBR: linear *vs.* Crouzeix-Raviart elements$^{(\times)}$

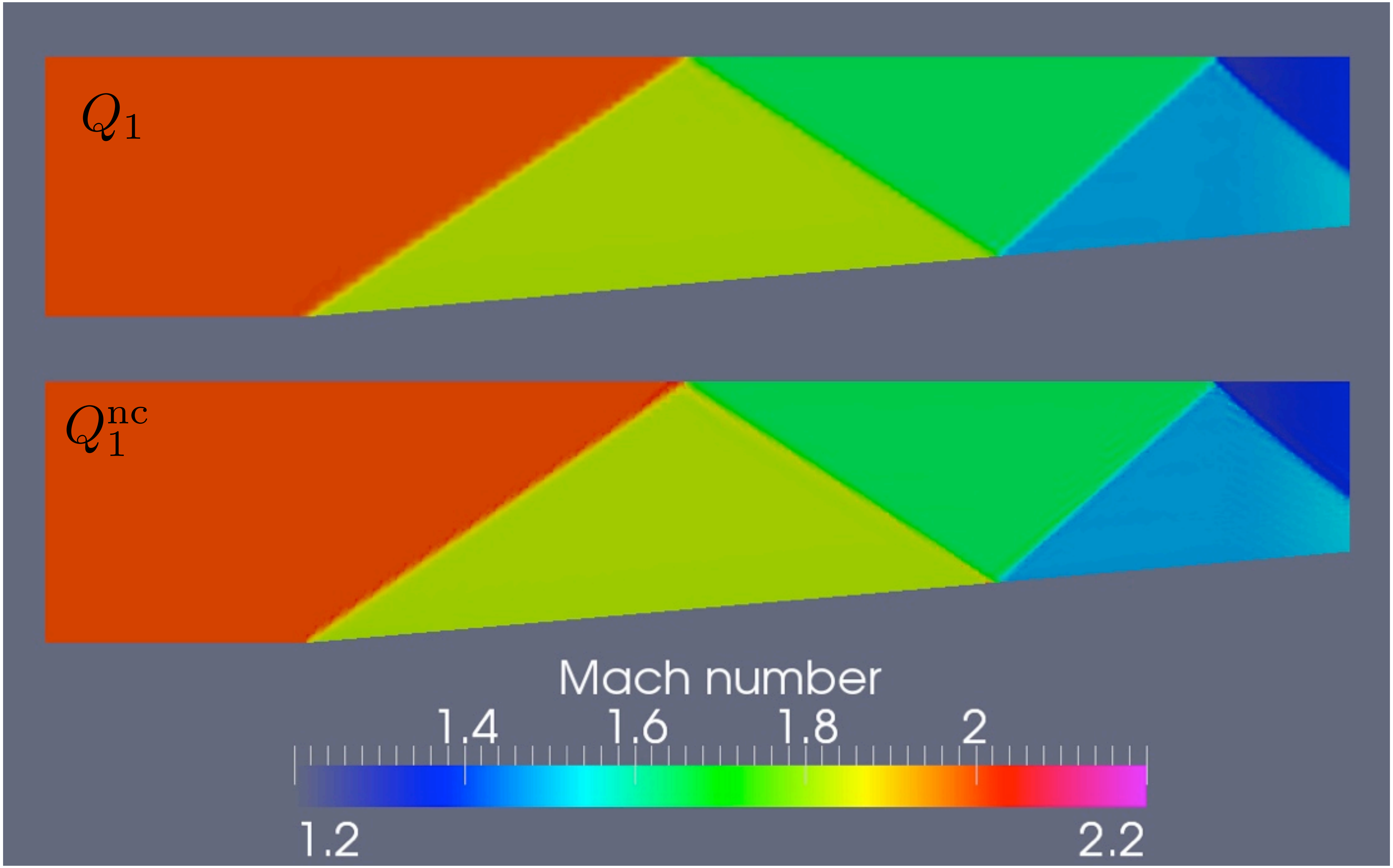


Tentative summary

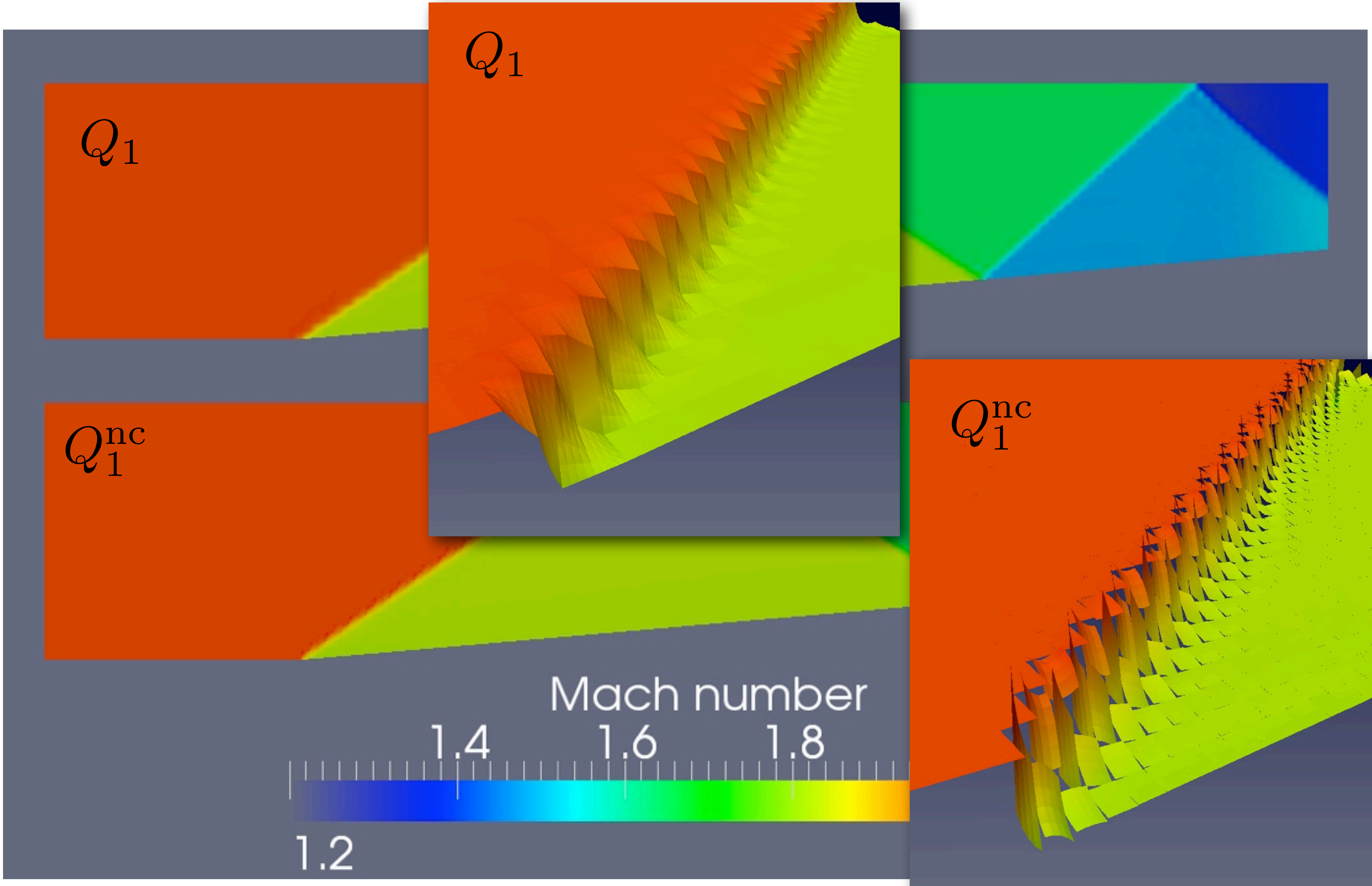
- Sign criteria seems to be a necessary and sufficient condition to ensure that AFC-type methods enforce upper and lower bounds.
- There is not (yet) a sufficient condition which guarantees that the AFC machinery produces accurate approximations in practice.

finite element	prerequisites	boundedness	accuracy
P1 and Q1	yes	yes	yes
integral meanvalue based Rannacher-Turek element	yes	yes	yes
midpoint based Rannacher-Turek element	no	no	no
Crouzeix-Raviart element	yes	yes	no

Linearized FCT for compressible Euler equations



Linearized FCT for compressible Euler equations



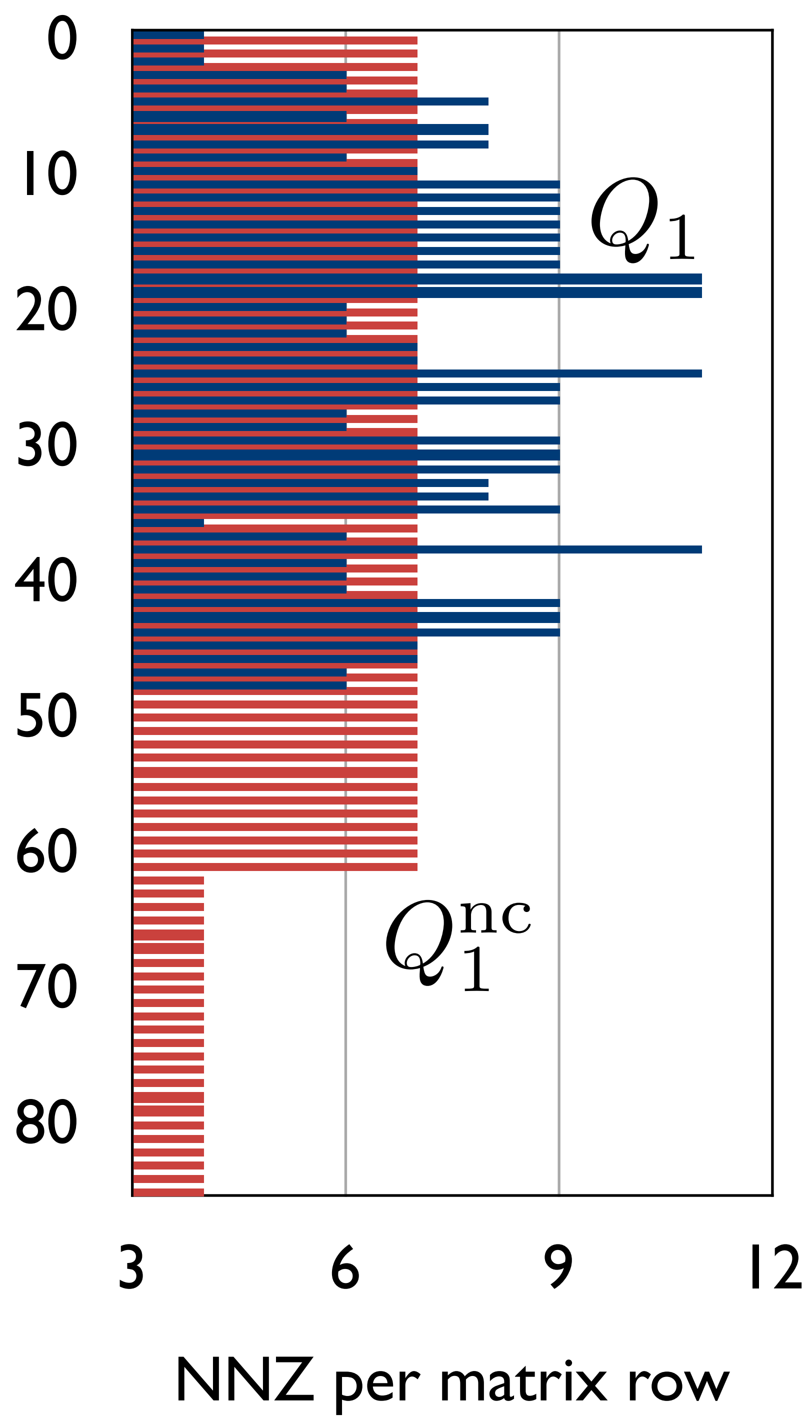
Algebraic Flux Correction

Part III: Efficiency and aspects of parallelization

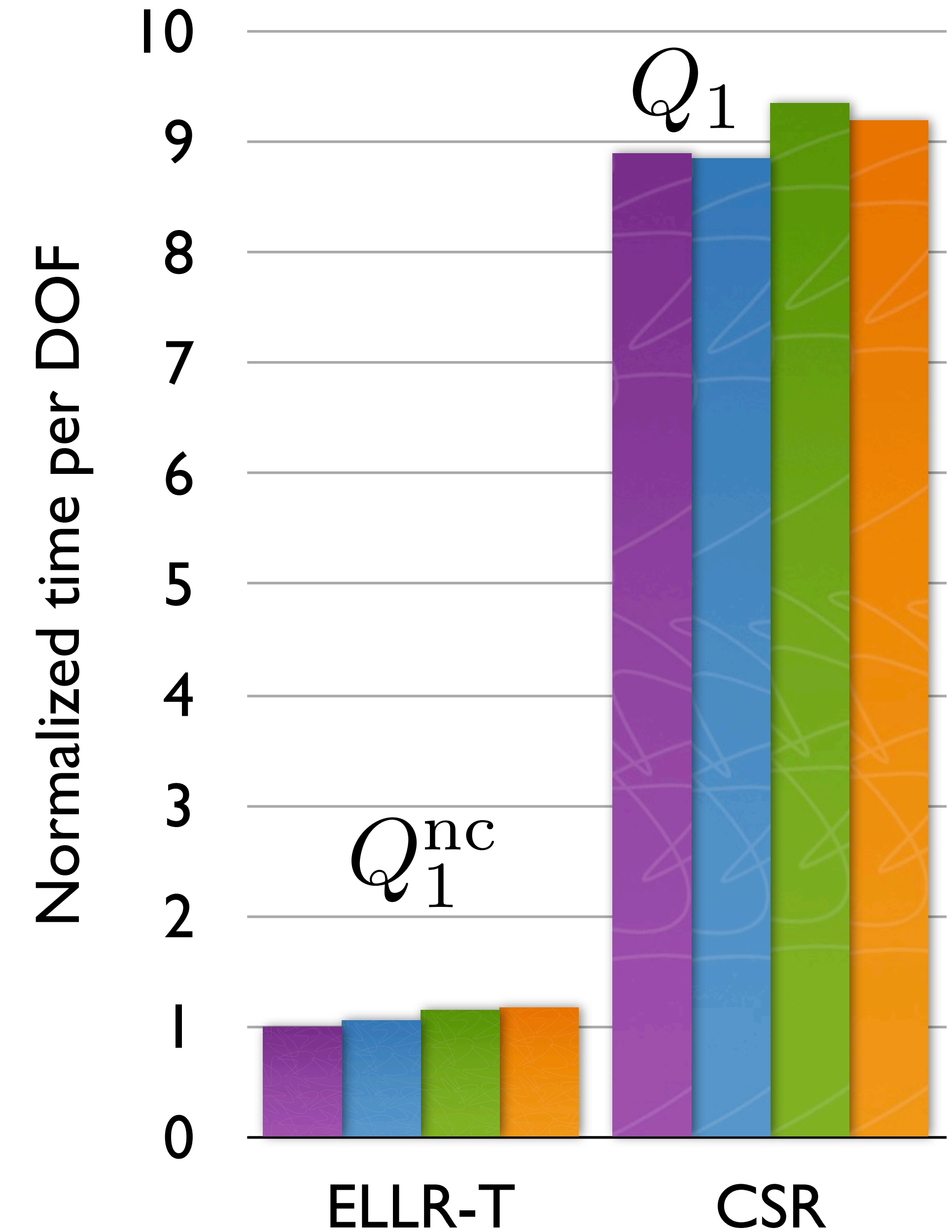
Efficient data structures
Edge-based assembly

Performance of SpMV-kernels

Matrix pattern for unstructured mesh



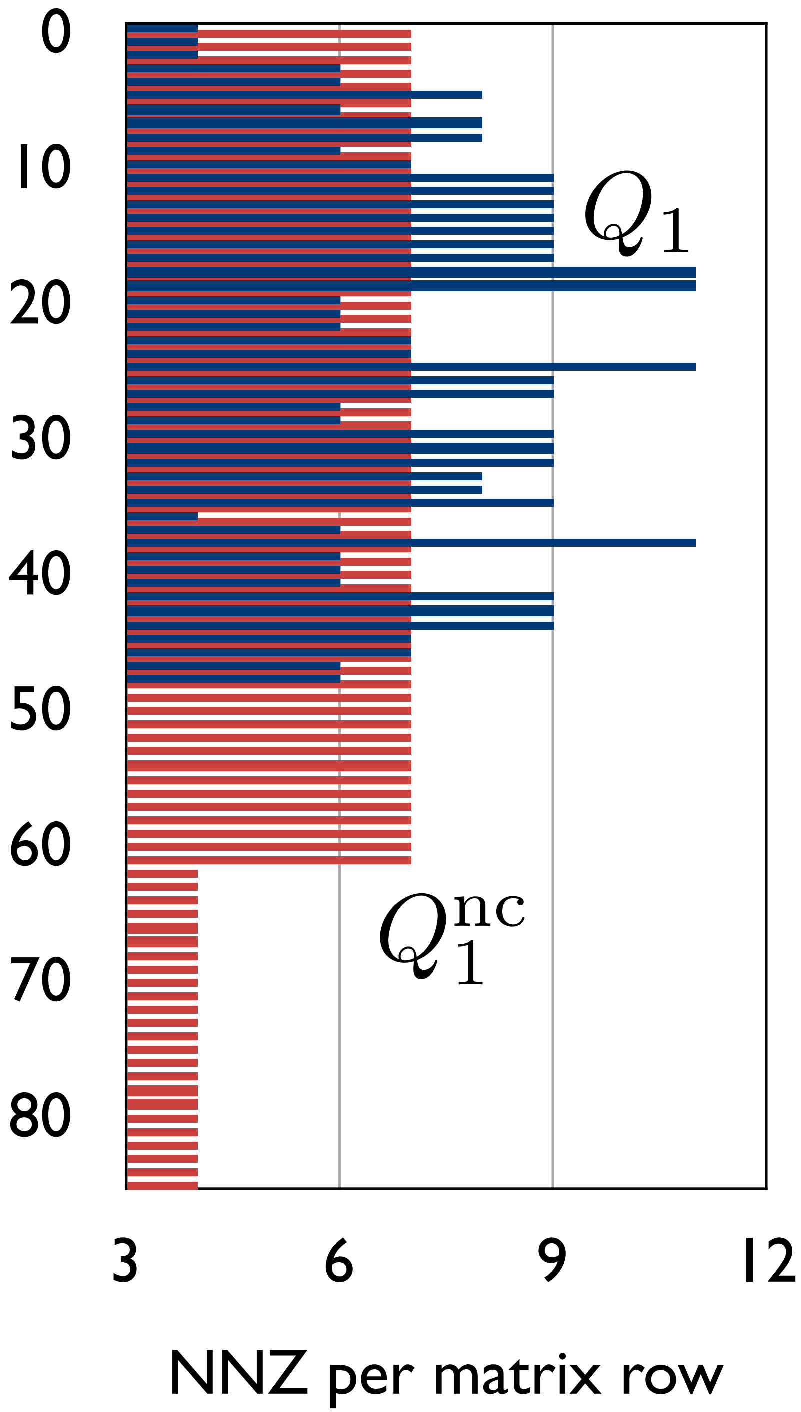
Tesla C2070 GPU (ECC off)
SpMV with different sorting strategies



results by M. Geveler and D. Ribbrock

Performance of SpMV-kernels

Matrix pattern for unstructured mesh



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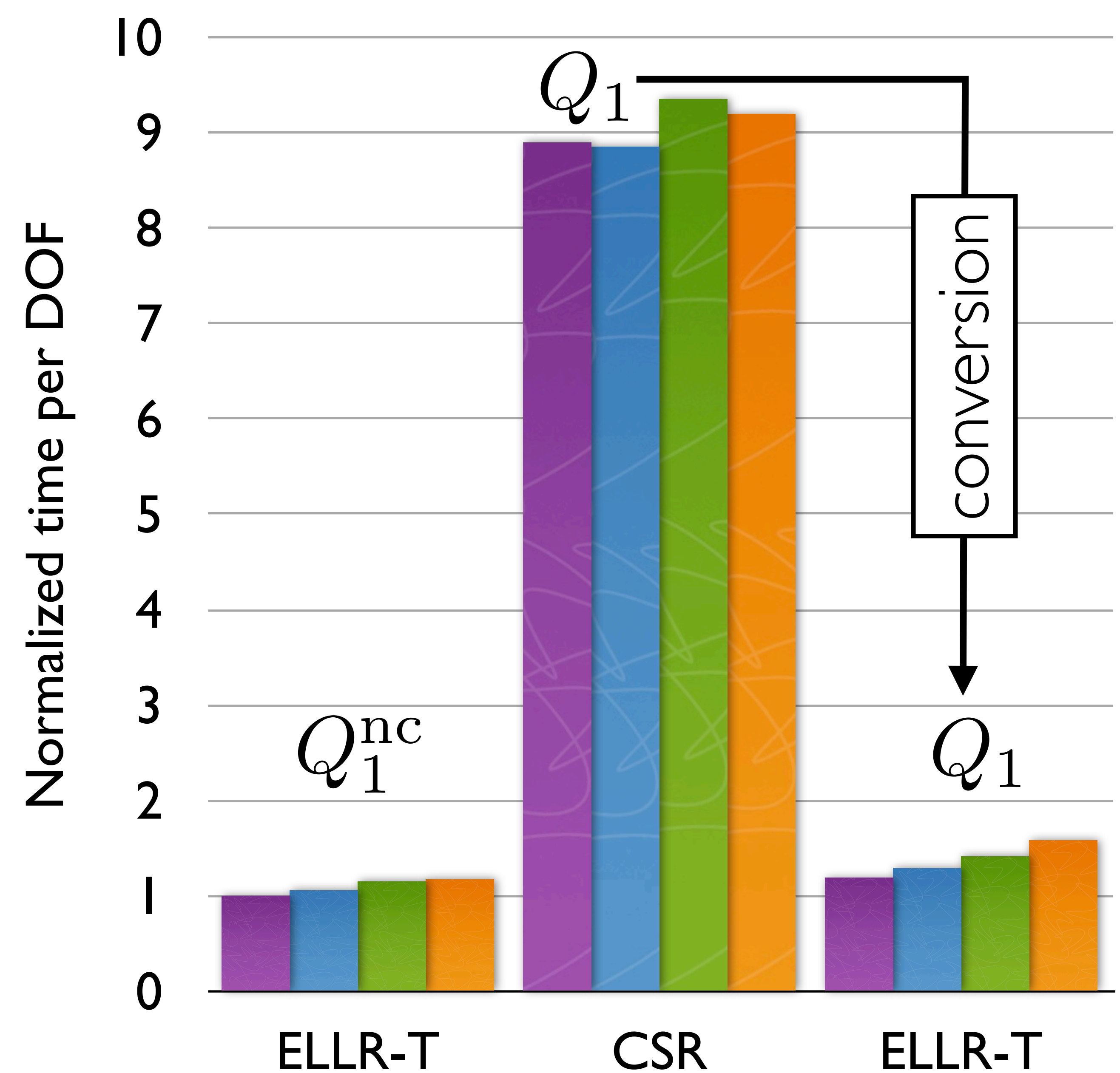

results by M. Geveler and D. Ribbrock

Model problem: $\partial_t u + \nabla \cdot \mathbf{f} = 0$

- FEM approximation $(w_h, \partial_t u_h + \nabla \cdot \mathbf{f}_h)_\Omega = 0$

$$\Leftrightarrow \quad (w_h, \partial_t u_h)_\Omega - (\nabla w_h, \mathbf{f}_h)_\Omega + \langle w_h, \mathbf{f}_h \cdot \mathbf{n} \rangle_\Gamma = 0 \qquad \forall w_h \in W_h$$

- Galerkin method edge-by-edge[(d)]

$$\forall i : \quad \sum_j m_{ij} \dot{u}_j - \sum_j \mathbf{c}_{ji} \cdot \mathbf{f}_j + \sum_j \mathbf{s}_{ij} \cdot \mathbf{f}_j = 0$$

$m_{ij} = (\varphi_i, \varphi_j)_\Omega$

$\mathbf{c}_{ij} = (\varphi_i, \nabla \varphi_j)_\Omega$

$\mathbf{s}_{ij} = \langle \varphi_i, \varphi_j \mathbf{n} \rangle_\Gamma$

$$\sum_j \mathbf{c}_{ij} = 0$$

- Discrete upwinding, Zalesak's limiter, etc. involve edge-by-edge loops

Model problem: $\partial_t u + \nabla \cdot \mathbf{f} = 0$

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$$-\mathbf{c}_{ii} = \sum_{j \neq i} \mathbf{c}_{ij}$$

$$\forall i: \quad \sum_j m_{ij} \dot{u}_j + \sum_{j \neq i} \underbrace{\mathbf{c}_{ij} \cdot \mathbf{f}_i - \mathbf{c}_{ji} \cdot \mathbf{f}_j}_{g_{ij}} + \sum_j \mathbf{s}_{ij} \cdot \mathbf{f}_j = 0$$

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- Discrete upwinding, Zalesak's limiter, etc. involve edge-by-edge loops

Generation of edge lists

$$\begin{bmatrix} 1 & 2 & \cdot & 3 & \cdot \\ 4 & 5 & \cdot & \cdot & 6 \\ \cdot & \cdot & 7 & 8 & \cdot \\ 9 & \cdot & 10 & 11 & \cdot \\ \cdot & 12 & \cdot & \cdot & 13 \end{bmatrix}$$

- Given matrix in CSR format

data	$\underline{1}$	2	3	$\underline{4}$	5	6	$\underline{7}$	8	$\underline{9}$	10	11	$\underline{12}$	13
colidx	1	2	4	1	2	5	3	4	1	3	4	2	5
rowidx	1	4	7	9	12	14							

- Edge data structure

Generation of edge lists

$$\begin{bmatrix} 1 & 2 & \cdot & 3 & \cdot \\ 4 & 5 & \cdot & \cdot & 6 \\ \cdot & \cdot & 7 & 8 & \cdot \\ 9 & \cdot & 10 & 11 & \cdot \\ \cdot & 12 & \cdot & \cdot & 13 \end{bmatrix}$$

```
sep ← rowidx; iedge=0
for i=1 to nrow do
    diagidx[i]=sep[i]++
    for ij=sep[i] to rowidx[i+1]-1 do
        j=colidx[ij]; ji=sep[j]++
        edgelist[++iedge][] ← {i,j,ij,ji}
```

- Given matrix in CSR format

data	$\underline{1}$	2	3	$\underline{4}$	5	6	$\underline{7}$	8	$\underline{9}$	10	11	$\underline{12}$	13
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colidx	1	2	4	1	2	5	3	4	1	3	4	2	5
rowidx	1	4	7	9	12	14							

- Edge data structure $\mathcal{E} = \{(i,j) : i < j \wedge \text{supp}(\varphi_i) \cap \text{supp}(\varphi_j) \neq \emptyset\}$

iedge	1	2	3	4
i	1	1	2	3
j	2	4	5	4
ij	2	3	6	8
ji	4	9	12	10

- Pointer to diagonal coefficients

irow	1	2	3	4	5
diagidx	1	5	7	11	13

Parallel edge-based assembly

- iterative solver
- matrix assembly
- vector assembly

- Edge-coloring of the sparsity graph

$$\mathcal{E} = \cup_{c=1}^{N_{\text{colors}}} \mathcal{E}_c$$

- Vizing's theorem (e.g., NTL algorithm)

$$d_G^{\max} \leq N_{\text{colors}} \leq d_G^{\max} + 1$$

- Parallel edge-by-edge loop

for c=1 **to** ncolors **do**
  forall the edges (i,j) $\in \mathcal{E}_c$ **do**
    $\mathrm{b}_i \mathrel{+}= \mathrm{g}_{ij}$
    $\mathrm{b}_j \mathrel{-}= \mathrm{g}_{ij}$

Parallel edge-based assembly

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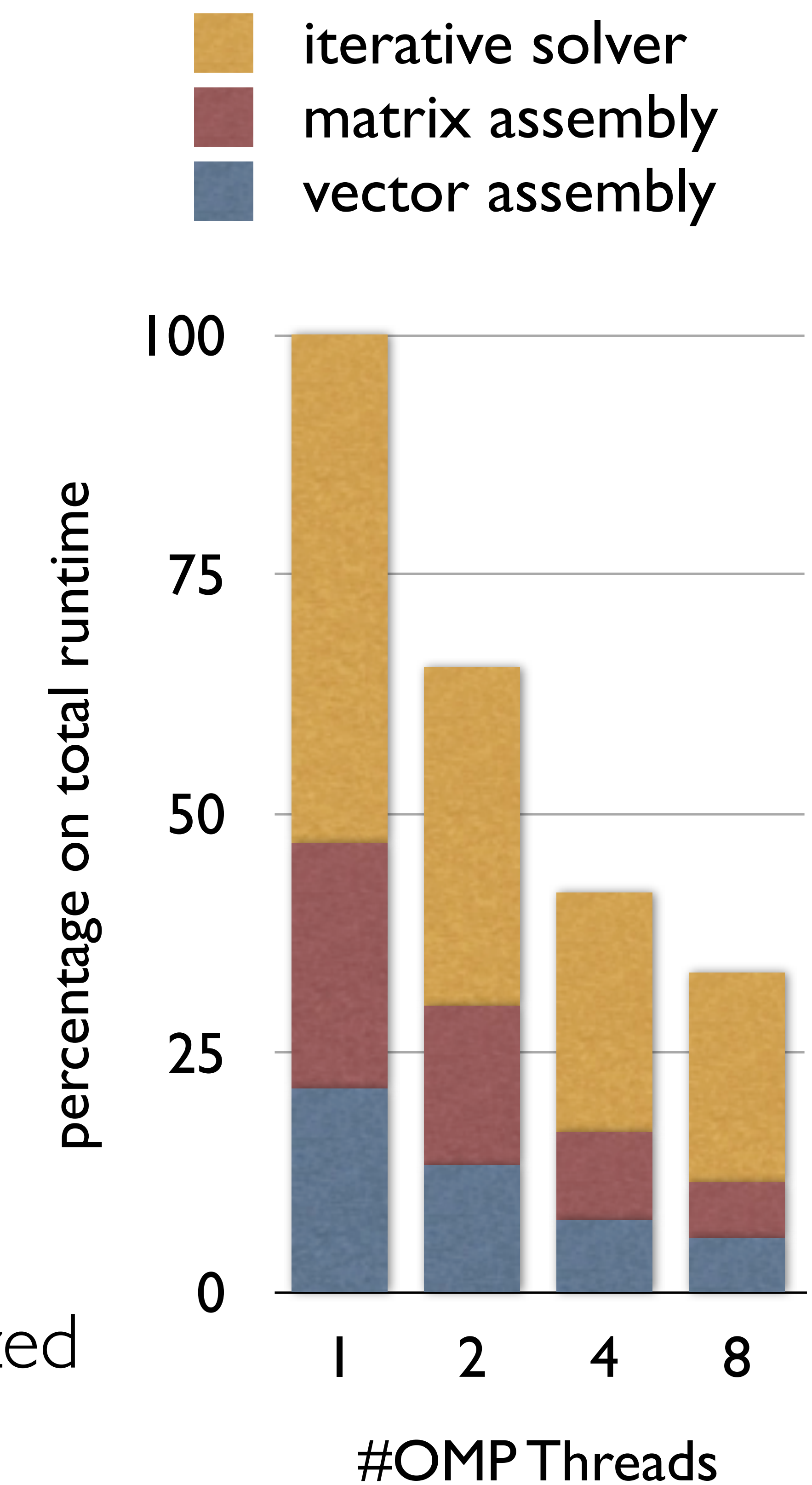
    $b_i \mathrel{+}= g_{ij}$
    $b_j \mathrel{-}= g_{ij}$

- Example: 2D Euler solver with linearized FCT, all nodal and edge-by-edge loops are parallelized



Conclusions and Outlook

- Algebraic flux correction schemes can be easily parallelized by regrouping the edges using edge-coloring techniques
- Nonconforming Rannacher-Turek element can be used within AFC-type methods (theoretical justification and numerics results)
- There is not (yet) a simple criterion that can be checked to estimate the accuracy of AFC-type methods a priori

If AFC works for Q_1^{nc} it may work for other approximations as well

  - composite finite elements, NURBS, ...

Appendix I

Weak imposition of boundary conditions

Boundary conditions

Convection-diffusion equation

$$\nabla \cdot (\mathbf{v}u - d\nabla u) = f \quad \text{in } \Omega$$
$$u = u_\mathrm{D} \quad \text{on } \Gamma_\mathrm{D}$$
$$(d\nabla u)\cdot \mathbf{n} = g \quad \text{on } \Gamma_\mathrm{N}$$

hyperbolic limit $d \to 0$

$$\nabla \cdot (\mathbf{v}u) = f \quad \text{in } \Omega$$
$$(\mathbf{v}u)\cdot \mathbf{n} = h \quad \text{on } \Gamma_\mathrm{in}$$

- Y. Basilevs, T. Hughes, *Weak imposition of Dirichlet boundary conditions in fluid mechanics*, Computers & Fluids 32 (1) 2007, 12-26
  - $\gamma = 1$ consistent, adjoint-consistent
  - $\gamma = -1$ consistent, adjoint-inconsistent

$$\beta_b = \frac{Cd}{h_b}$$

- E. Burman, *A penalty free non-symmetric Nitsche type method for the weak imposition of boundary conditions*, eprint arXiv:1106.5612v2 (Nov 2011)

$$\gamma = -1,\ \beta \equiv 0$$

Weak imposition of boundary conditions

$$
\begin{aligned}
&\int_{\Omega} -\nabla w_h \cdot (\mathbf{v} u_h - d\nabla u_h)\mathrm{d}\mathbf{x} + \int_{\Gamma} w_h(\mathbf{v} u_h) \cdot \mathbf{n}\mathrm{d}s \\
&- \int_{\Gamma_{\mathrm{D}}} w_h(d\nabla u_h) \cdot \mathbf{n}\mathrm{d}s - \int_{\Gamma_{\mathrm{D}}} (\gamma d\nabla w_h) \cdot \mathbf{n} u_h \mathrm{d}s \\
&- \int_{\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{in}}} (\mathbf{v} w_h) \cdot \mathbf{n} u_h \mathrm{d}s + \sum_{b=1}^{N_{\mathrm{eb}}} \int_{\Gamma_{\mathrm{D}} \cap \Gamma_b} \beta_b w_h u_h \mathrm{d}s \\
&= \int_{\Omega} w_h f \mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} w_h g \mathrm{d}s - \int_{\Gamma_{\mathrm{D}}} \gamma(d\nabla w_h) \cdot \mathbf{n} u_{\mathrm{D}} \mathrm{d}s \\
&- \int_{\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{in}}} (\mathbf{v} w_h) \cdot \mathbf{n} u_{\mathrm{D}} \mathrm{d}s + \sum_{b=1}^{N_{\mathrm{eb}}} \int_{\Gamma_{\mathrm{D}} \cap \Gamma_b} \beta_b w_h u_{\mathrm{D}} \mathrm{d}s
\end{aligned}
$$

$$
\gamma = \pm 1, \quad \beta_b = \frac{Cd}{h_b}
$$

Weak imposition of boundary conditions

$$
\int_{\Omega} -\nabla w_h \cdot (\mathbf{v} u_h - d\nabla u_h)\mathrm{d}\mathbf{x} + \int_{\Gamma} w_h (\mathbf{v} u_h) \cdot \mathbf{n}\mathrm{d}s
$$

$$
- \int_{\Gamma_{\mathrm{D}}} w_h (d\nabla u_h) \cdot \mathbf{n}\mathrm{d}s - \int_{\Gamma_{\mathrm{D}}} (\gamma d\nabla w_h) \cdot \mathbf{n} u_h \mathrm{d}s
$$

$$
- \int_{\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{in}}} (\mathbf{v} w_h) \cdot \mathbf{n} u_h \mathrm{d}s + \sum_{b=1}^{N_{\mathrm{eb}}} \int_{\Gamma_{\mathrm{D}} \cap \Gamma_b} \beta_b w_h u_h \mathrm{d}s
$$

$$
= \int_{\Omega} w_h f \mathrm{d}\mathbf{x} + \int_{\Gamma_{\mathrm{N}}} w_h g \mathrm{d}s - \int_{\Gamma_{\mathrm{D}}} \gamma (d\nabla w_h) \cdot \mathbf{n} u_{\mathrm{D}} \mathrm{d}s
$$

$$
- \int_{\Gamma_{\mathrm{D}} \cap \Gamma_{\mathrm{in}}} (\mathbf{v} w_h) \cdot \mathbf{n} u_{\mathrm{D}} \mathrm{d}s + \sum_{b=1}^{N_{\mathrm{eb}}} \int_{\Gamma_{\mathrm{D}} \cap \Gamma_b} \beta_b w_h u_{\mathrm{D}} \mathrm{d}s
$$

$$
\gamma = \pm 1, \quad \beta_b = \frac{Cd}{h_b}
$$

Weak imposition of boundary conditions

$$\int_\Omega -\nabla w_h \cdot (\mathbf{v}u_h - d\nabla u_h)\mathrm{d}\mathbf{x} + \int_\Gamma w_h(\mathbf{v}u_h)\cdot\mathbf{n}\mathrm{d}s$$

$$-\int_{\Gamma_\mathrm{D}} w_h(d\nabla u_h)\cdot\mathbf{n}\mathrm{d}s - \int_{\Gamma_\mathrm{D}} (\gamma d\nabla w_h)\cdot\mathbf{n}u_h\mathrm{d}s$$

$$-\int_{\Gamma_\mathrm{D}\cap\Gamma_\mathrm{in}} (\mathbf{v}w_h)\cdot\mathbf{n}u_h\mathrm{d}s + \sum_{b=1}^{N_\mathrm{eb}}\int_{\Gamma_\mathrm{D}\cap\Gamma_b} \beta_b w_h u_h\mathrm{d}s$$

$$= \int_\Omega w_h f\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathrm{N}} w_h g\mathrm{d}s - \int_{\Gamma_\mathrm{D}} \gamma(d\nabla w_h)\cdot\mathbf{n}u_\mathrm{D}\mathrm{d}s$$

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$$\gamma = \pm 1, \quad \beta_b = \frac{Cd}{h_b}$$

Weak imposition of boundary conditions

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$$\gamma = \pm 1, \quad \beta_b = \frac{Cd}{h_b}$$

Weak imposition of boundary conditions

$$
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&\int_\Omega -\nabla w_h \cdot (\mathbf{v} u_h - d\nabla u_h)\,\mathrm{d}\mathbf{x} + \int_\Gamma w_h (\mathbf{v} u_h) \cdot \mathbf{n}\,\mathrm{d}s \\
&- \int_{\Gamma_\mathrm{D}} w_h (d\nabla u_h) \cdot \mathbf{n}\,\mathrm{d}s - \int_{\Gamma_\mathrm{D}} (\gamma d \nabla w_h) \cdot \mathbf{n} u_h\,\mathrm{d}s \\
&- \int_{\Gamma_\mathrm{D} \cap \Gamma_\mathrm{in}} (\mathbf{v} w_h) \cdot \mathbf{n} u_h\,\mathrm{d}s + \sum_{b=1}^{N_\mathrm{eb}} \int_{\Gamma_\mathrm{D} \cap \Gamma_b} \beta_b w_h u_h\,\mathrm{d}s \\
&= \int_\Omega w_h f\,\mathrm{d}\mathbf{x} + \int_{\Gamma_\mathrm{N}} w_h g\,\mathrm{d}s - \int_{\Gamma_\mathrm{D}} \gamma (d\nabla w_h) \cdot \mathbf{n} u_\mathrm{D}\,\mathrm{d}s \\
&- \int_{\Gamma_\mathrm{D} \cap \Gamma_\mathrm{in}} (\mathbf{v} w_h) \cdot \mathbf{n} u_\mathrm{D}\,\mathrm{d}s + \sum_{b=1}^{N_\mathrm{eb}} \int_{\Gamma_\mathrm{D} \cap \Gamma_b} \beta_b w_h u_\mathrm{D}\,\mathrm{d}s
\end{aligned}
$$

$$
\gamma = \pm 1, \quad \beta_b = \frac{Cd}{h_b}
$$

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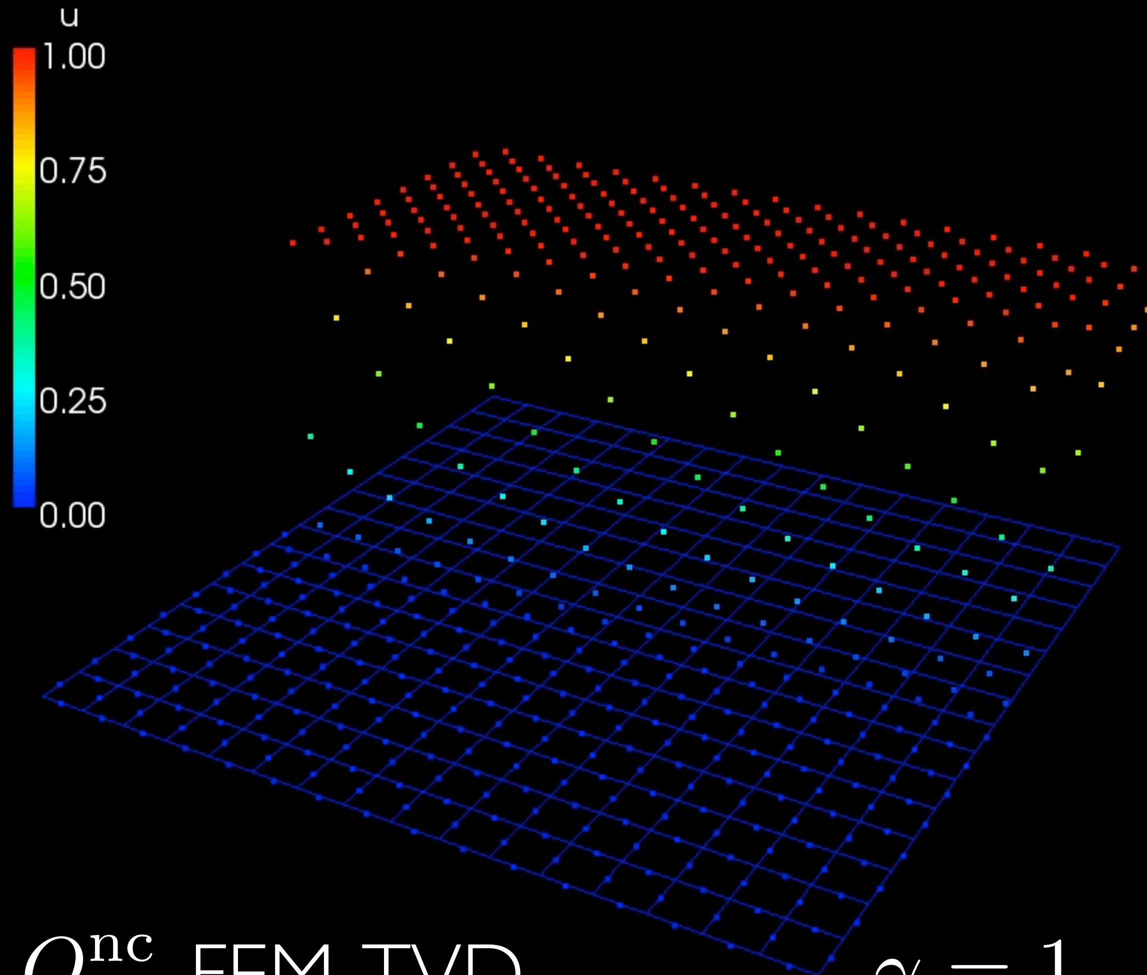
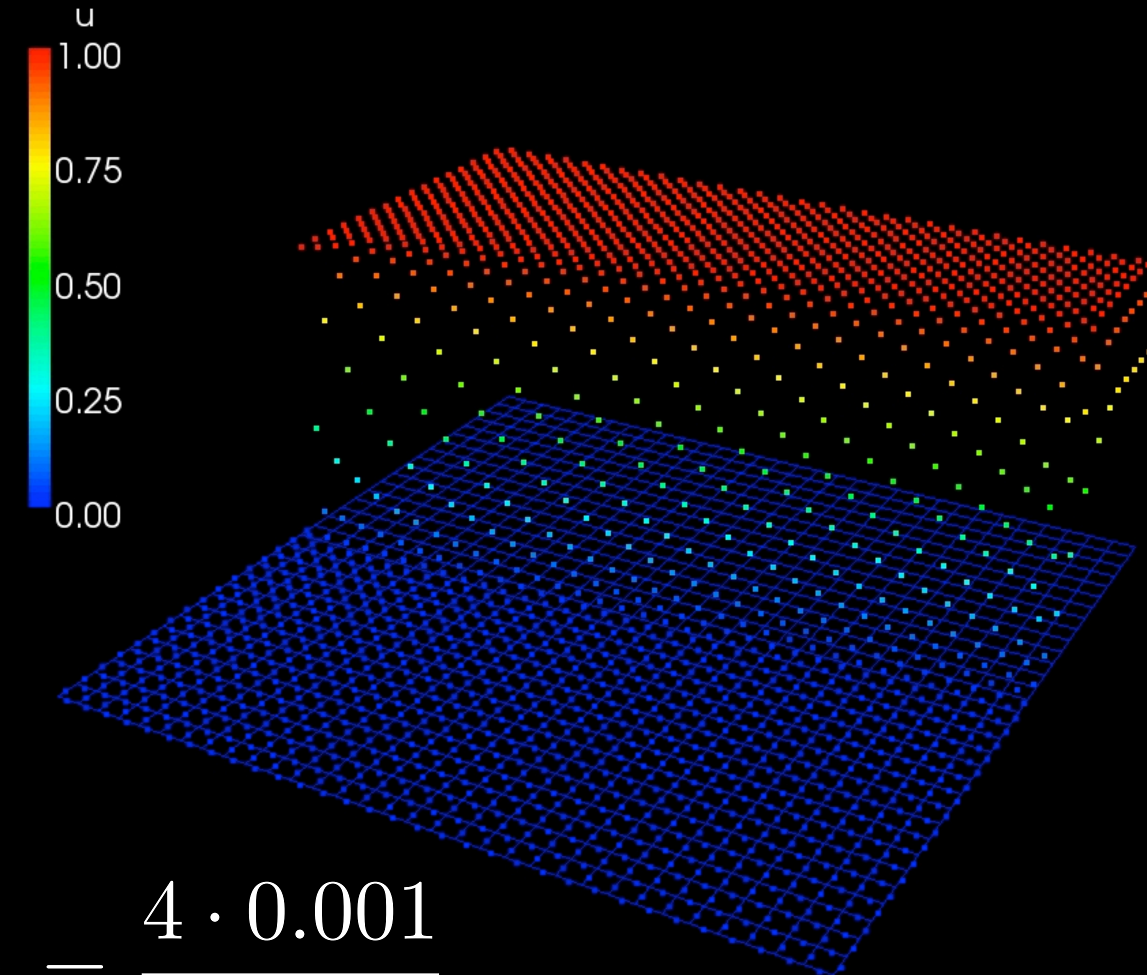
$$\int_\Omega -\nabla w_h \cdot (\mathbf{v} u_h - d\nabla u_h)\mathrm{d}\mathbf{x} + \int_\Gamma w_h(\mathbf{v} u_h)\cdot \mathbf{n}\mathrm{d}s$$

$$-\int_{\Gamma_\mathrm{D}\cap\Gamma_\mathrm{in}} (\mathbf{v} w_h)\cdot \mathbf{n} u_h \mathrm{d}s \qquad \int_{\Gamma_\mathrm{out}} w_h \mathbf{v}\cdot\mathbf{n} u_h \mathrm{d}s$$

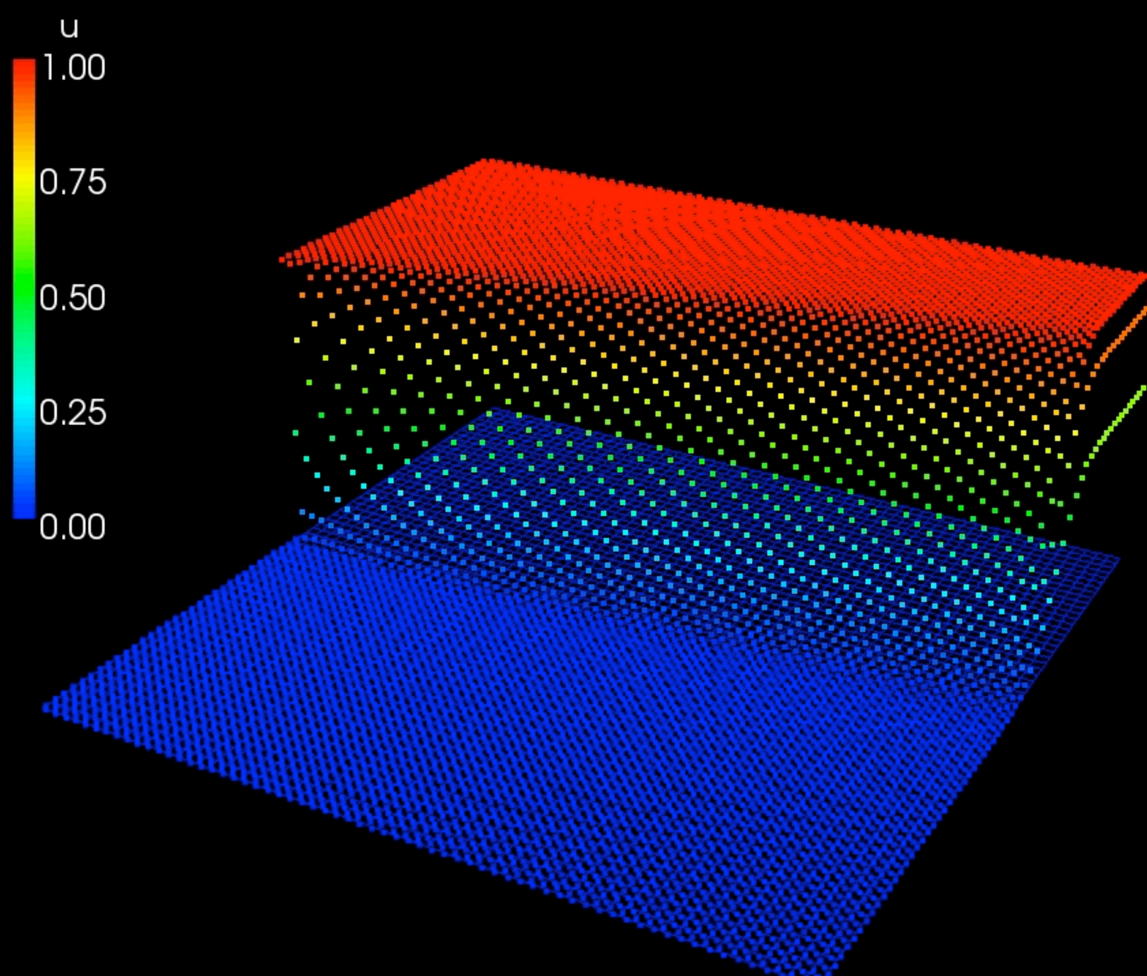
$$= \int_\Omega w_h f \mathrm{d}\mathbf{x}$$

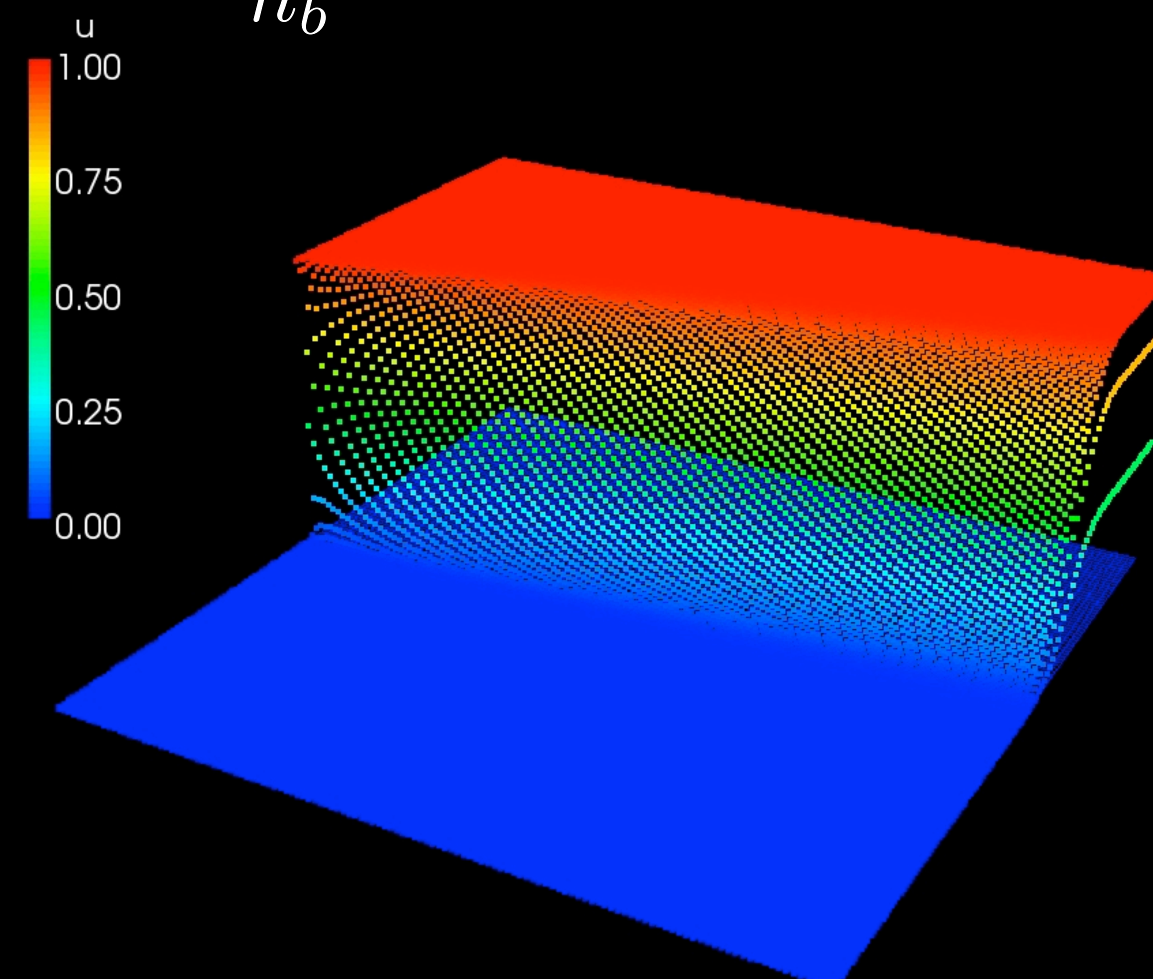
$$-\int_{\Gamma_\mathrm{D}\cap\Gamma_\mathrm{in}} (\mathbf{v} w_h)\cdot \mathbf{n} u_\mathrm{D} \mathrm{d}s$$

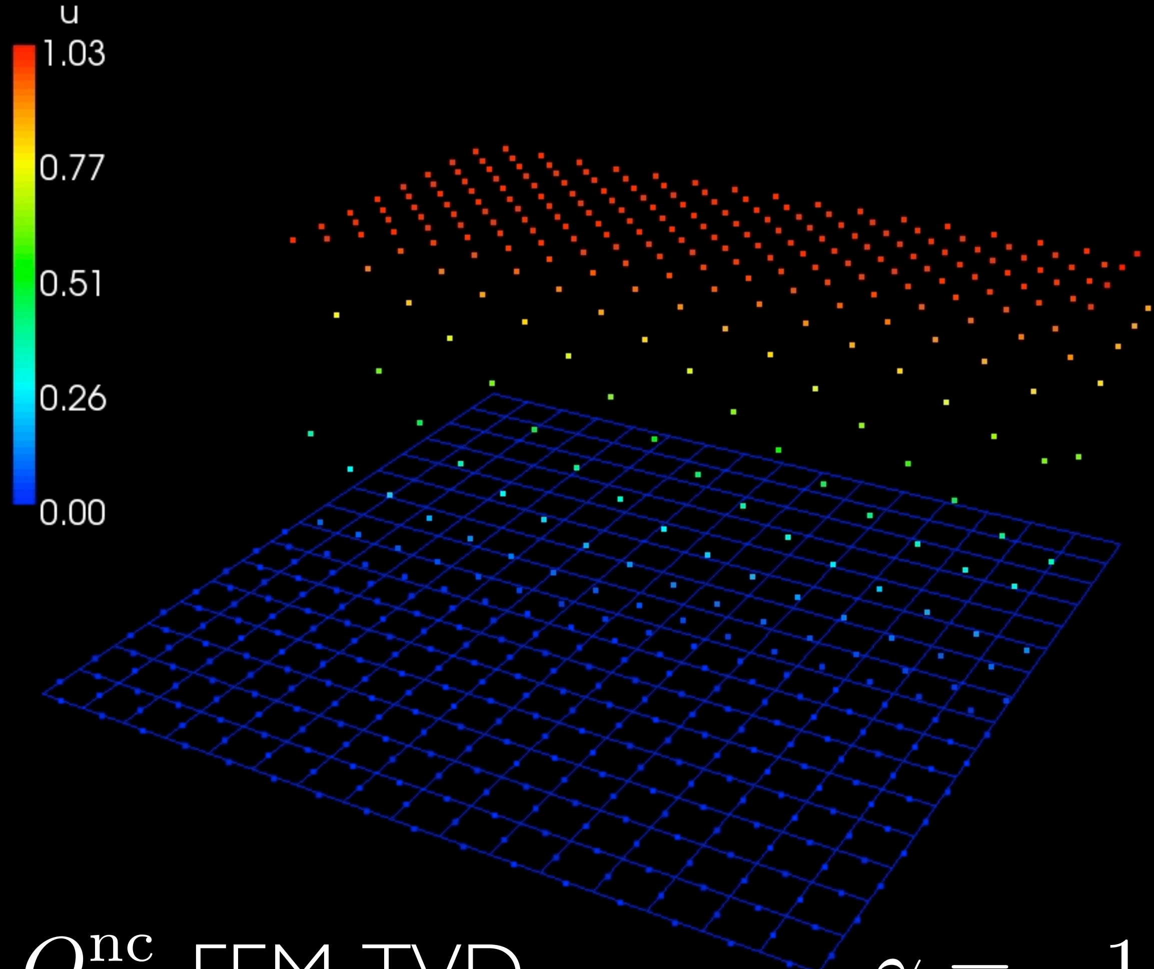


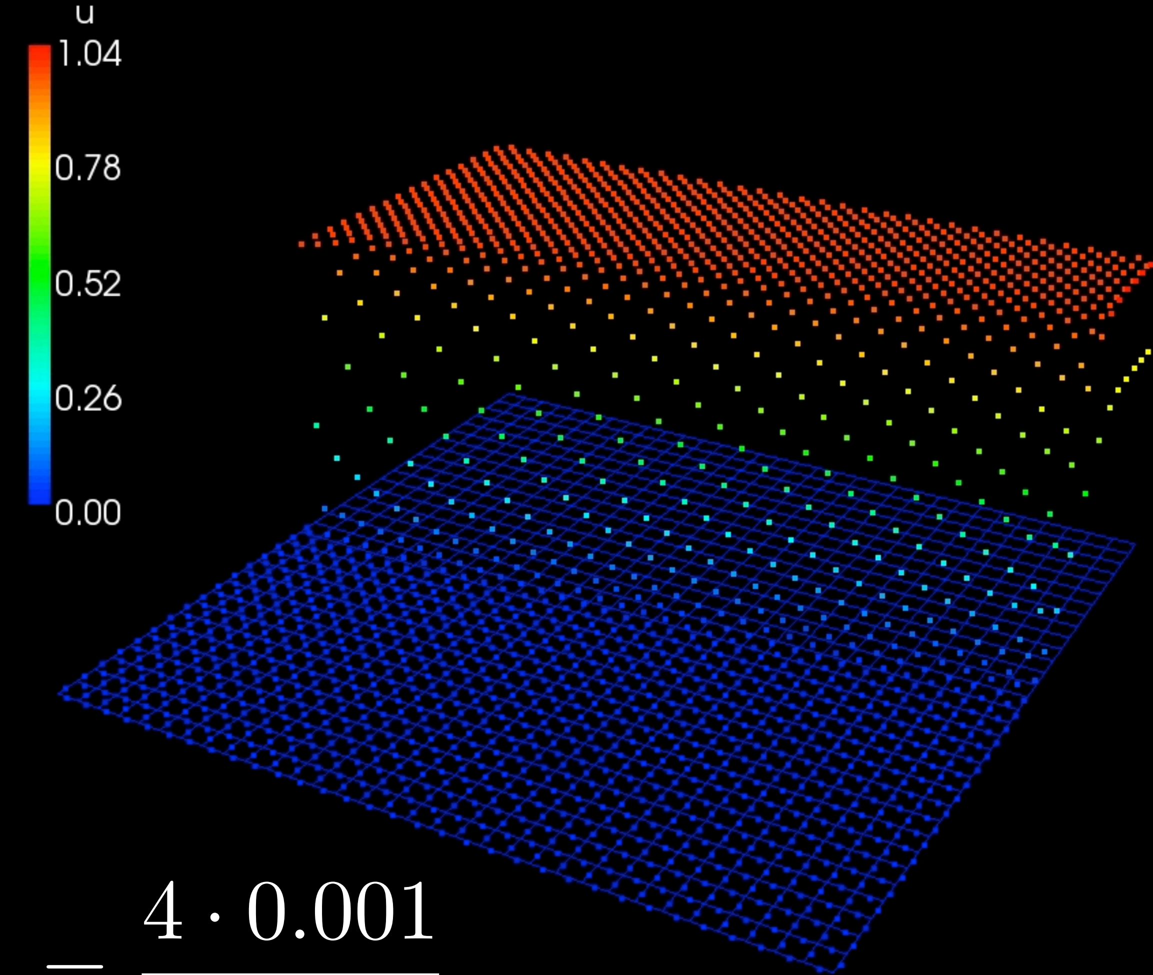


Q_1^{nc} FEM-TVD $\gamma = 1, \qquad \beta_b = \dfrac{4 \cdot 0.001}{h_b}$

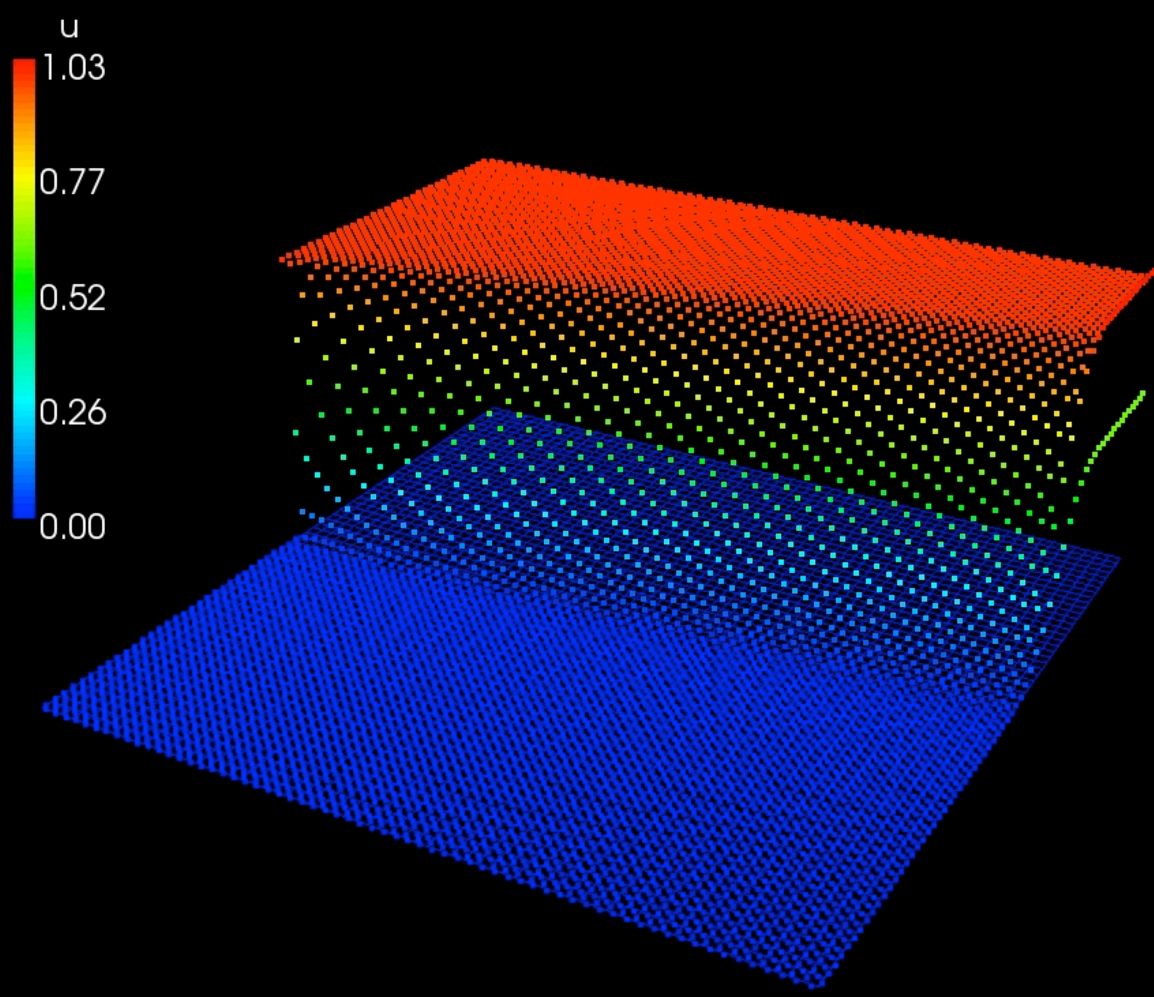


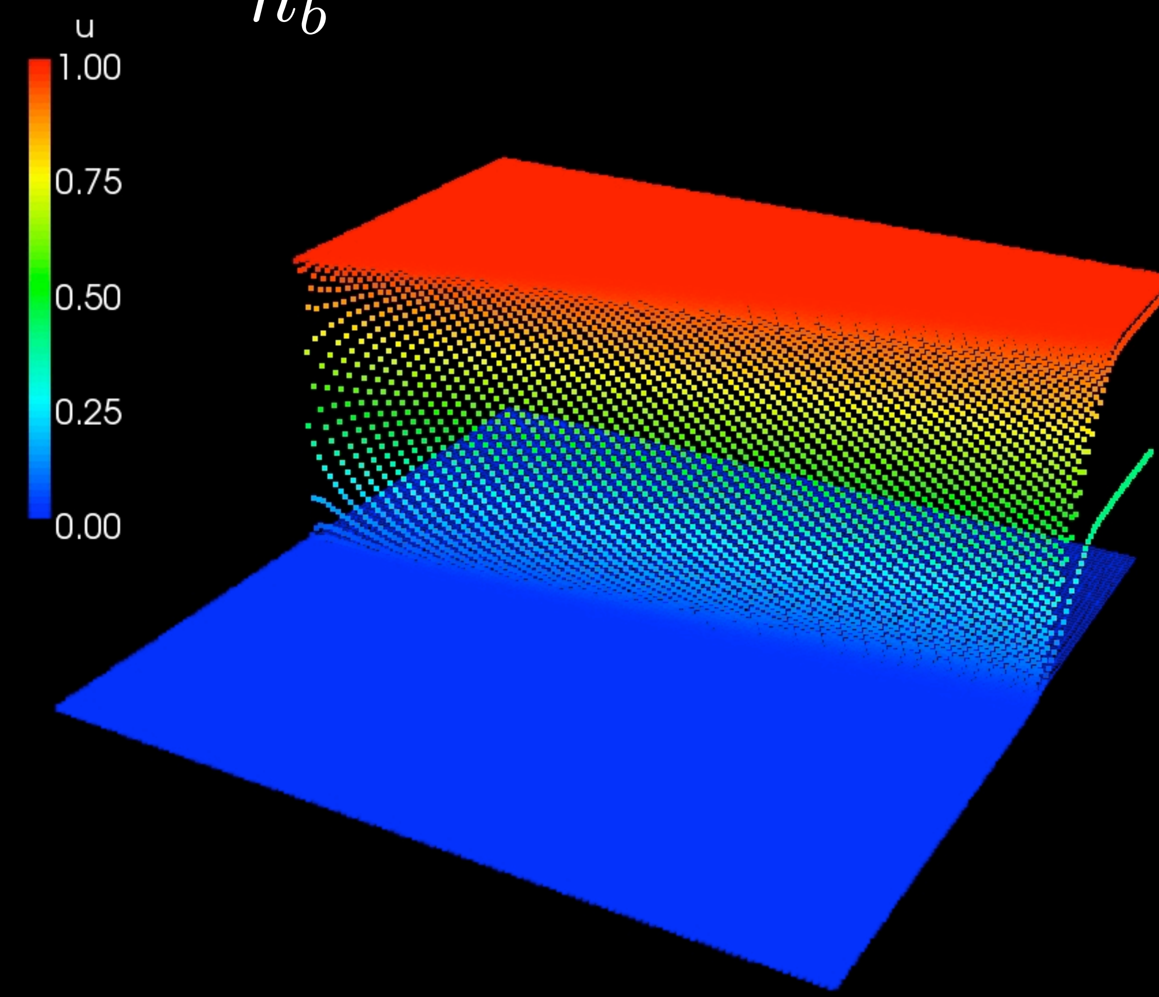


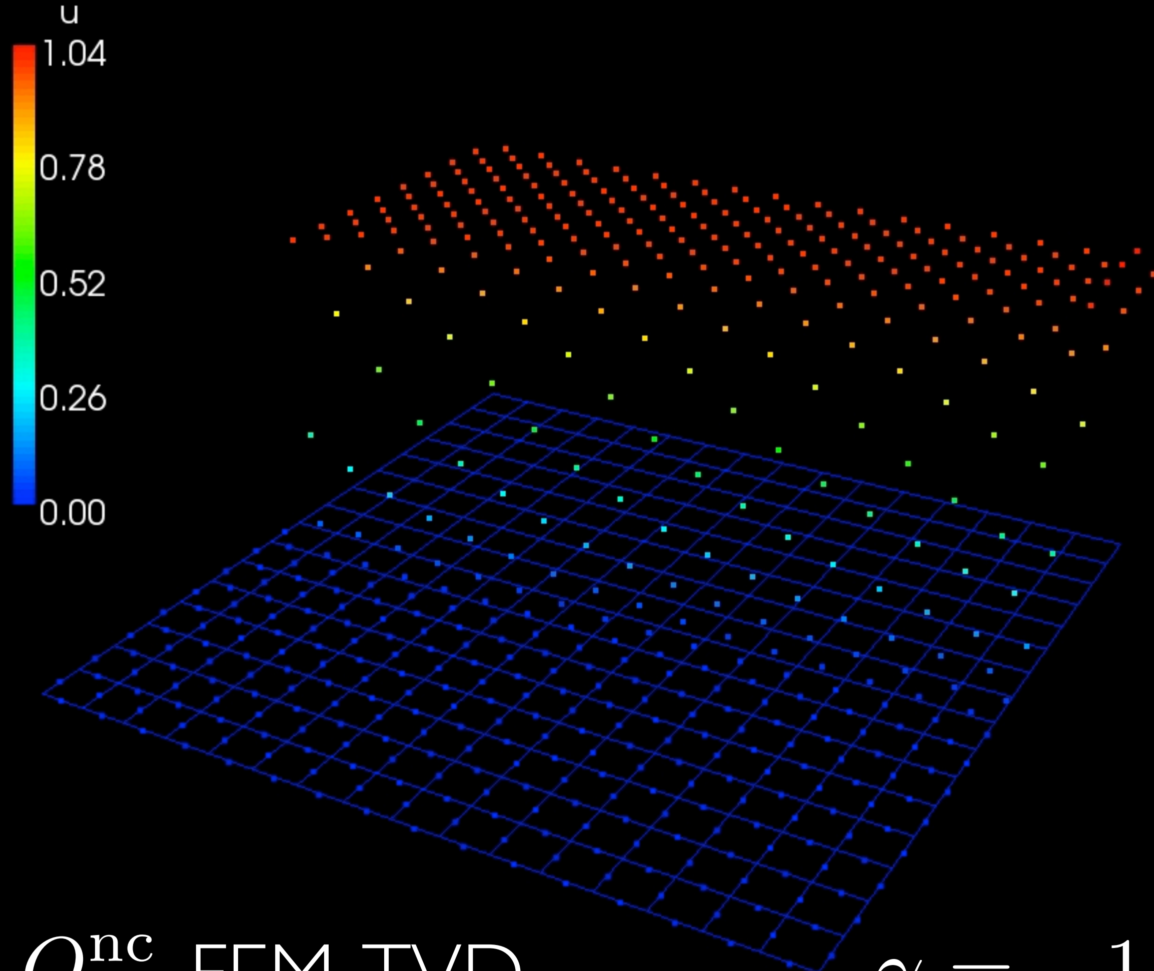


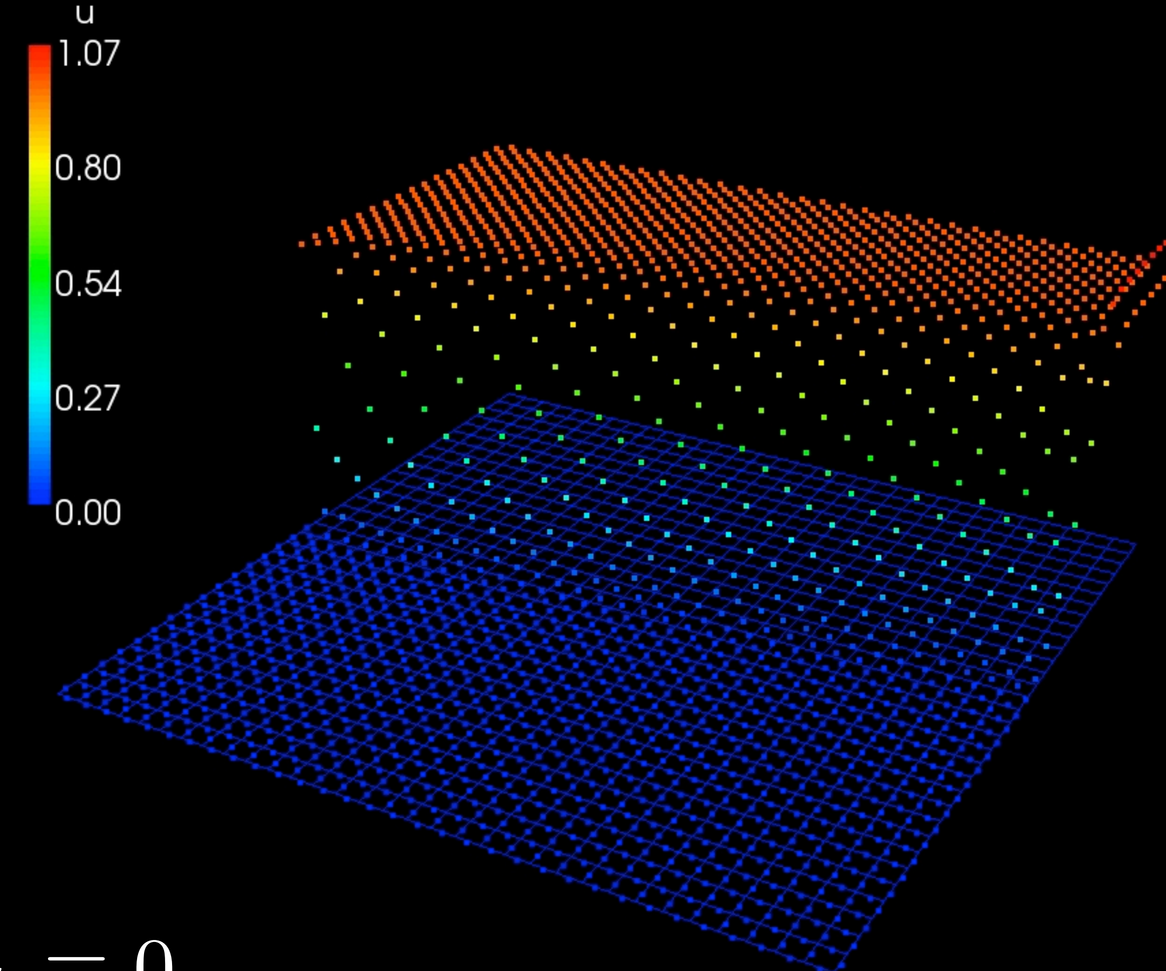


Q_1^{nc} FEM-TVD $\gamma = -1, \quad \beta_b = \dfrac{4 \cdot 0.001}{h_b}$

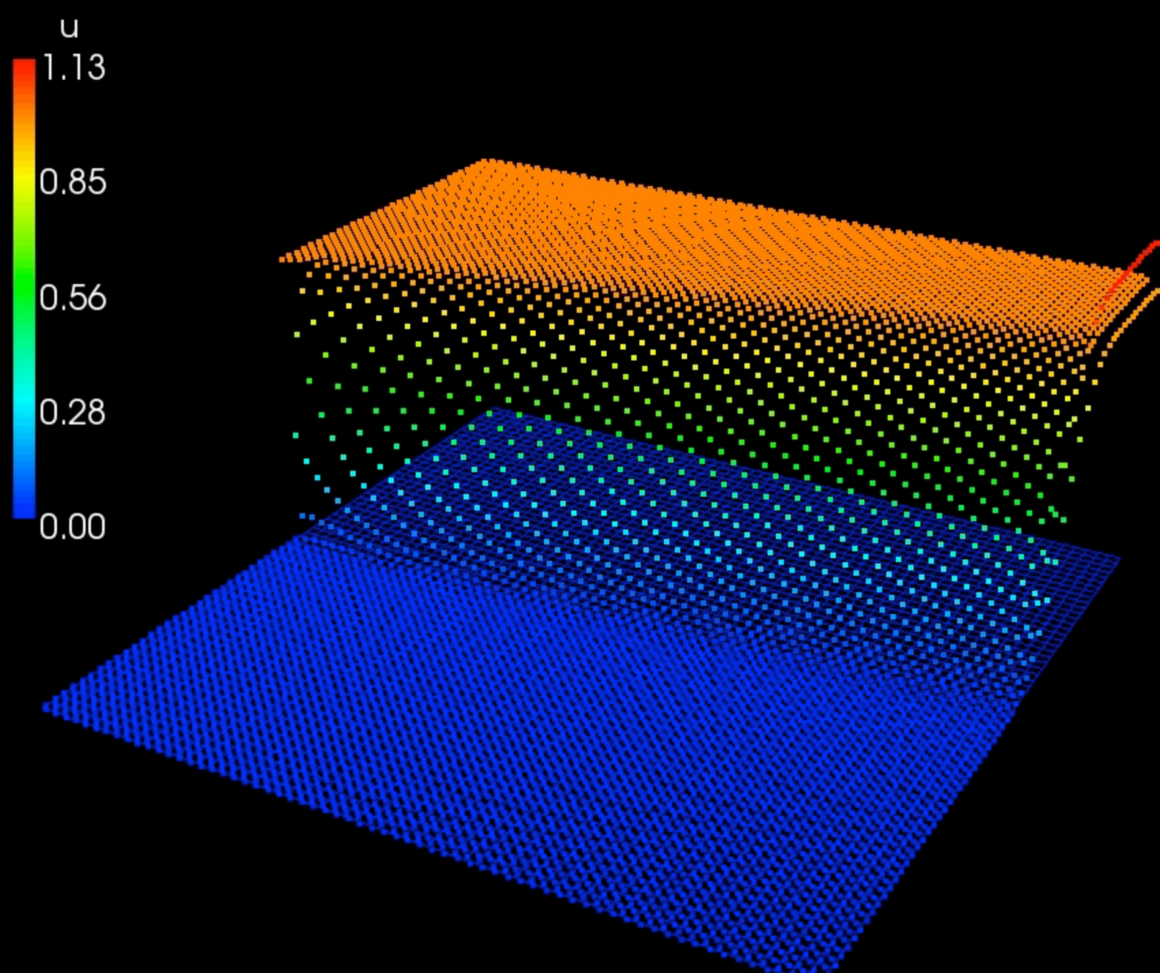


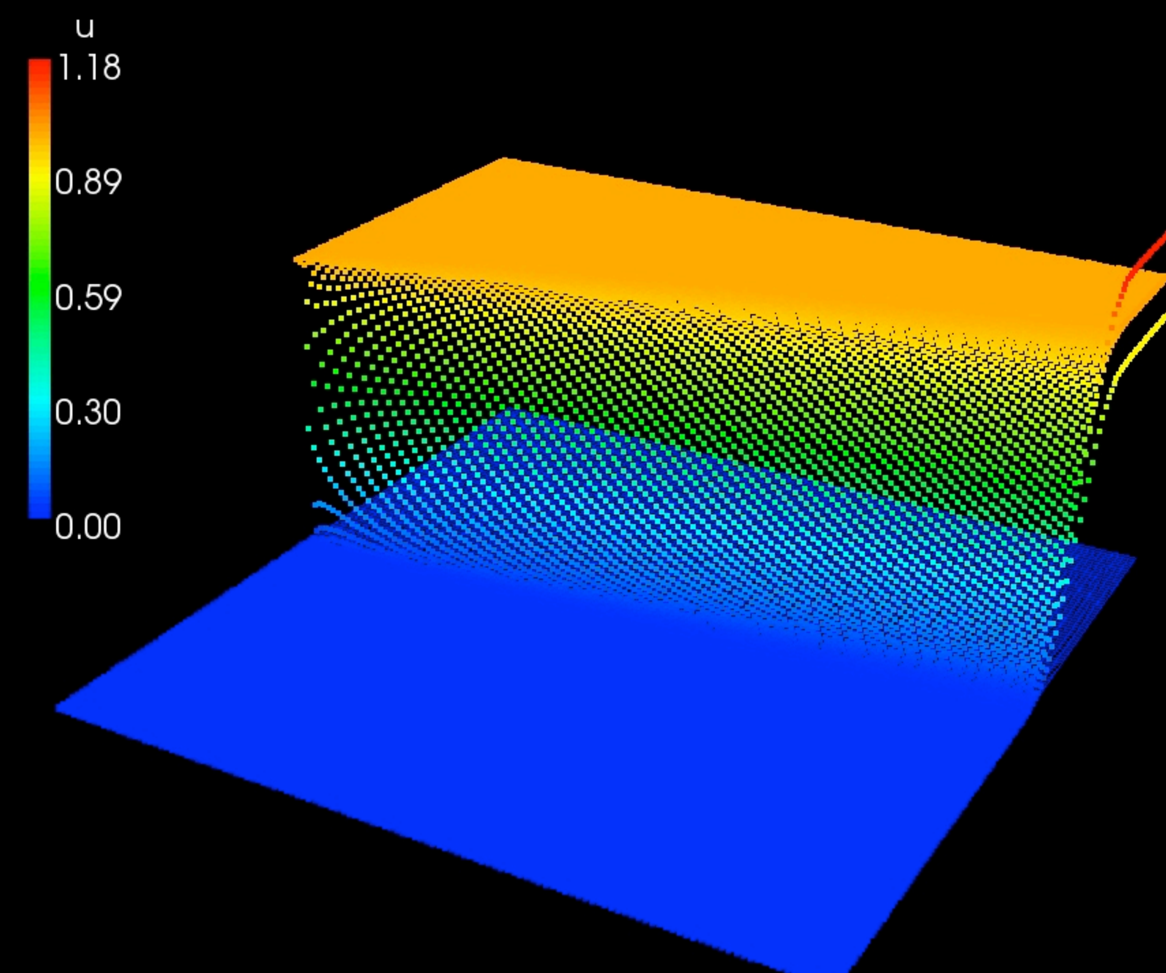






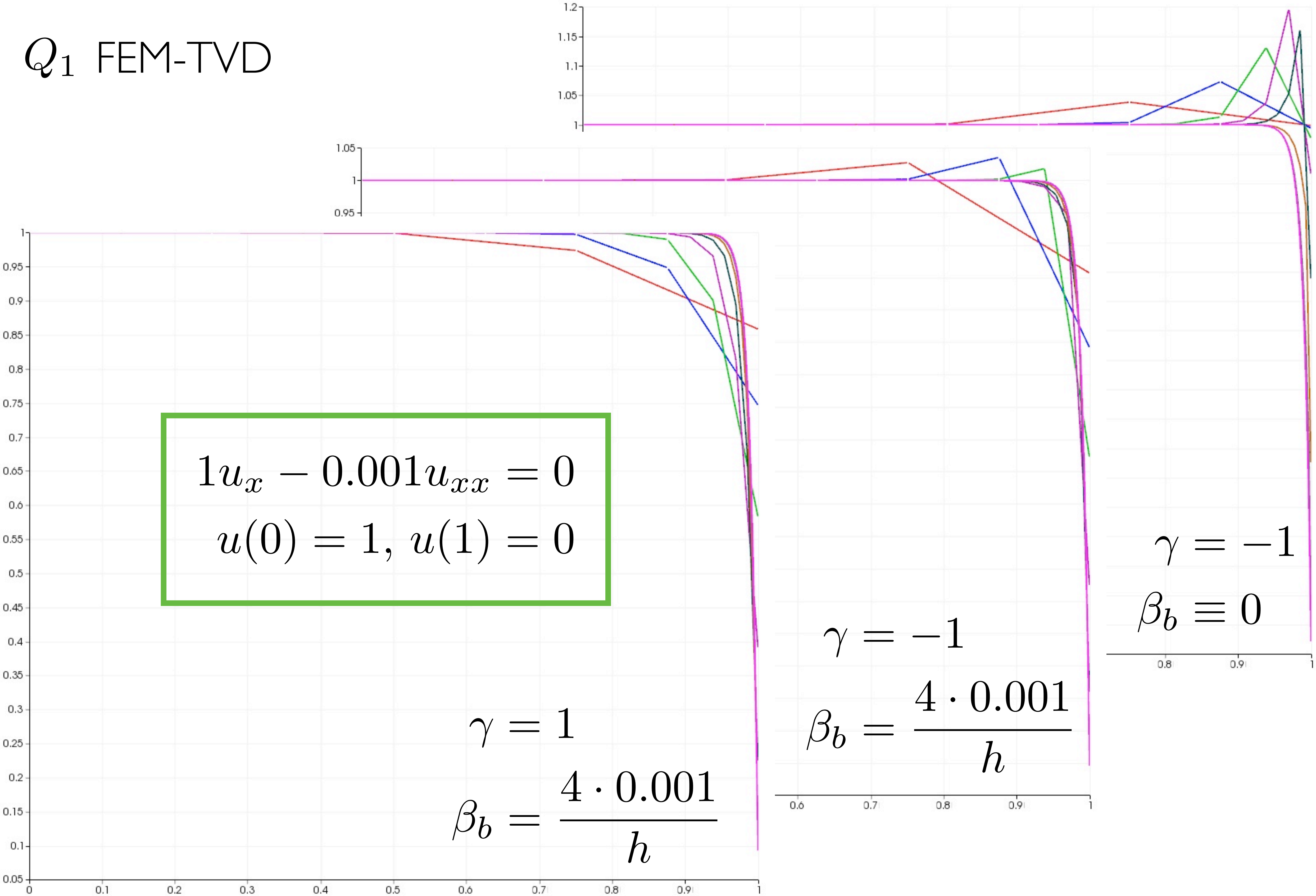
Q_1^{nc} FEM-TVD $\gamma = -1, \quad \beta_b \equiv 0$





1D convection-diffusion equation

Q_1 FEM-TVD



$$1u_x - 0.001u_{xx} = 0$$
$$u(0) = 1,\; u(1) = 0$$

$\gamma = 1$, $\beta_b = \dfrac{4 \cdot 0.001}{h}$

$\gamma = -1$, $\beta_b = \dfrac{4 \cdot 0.001}{h}$

$\gamma = -1$, $\beta_b \equiv 0$

Appendix II

Zalesak's multidimensional flux limiter

Zalesak's limiter: $P_i^{\pm} := 0, \quad Q_i^{\pm} := u_i^L \quad \forall \texttt{nodes} \; i$

- Compute the sums of positive/negative antidiffusive fluxes

$$\begin{aligned} P_i^+ &\mathrel{+}= \max\{0, \; f'_{ij}\} & P_i^- &\mathrel{+}= \min\{0, \; f'_{ij}\} \\ P_j^+ &\mathrel{-}= \max\{0, -f'_{ij}\} & P_j^- &\mathrel{-}= \min\{0, -f'_{ij}\} \end{aligned} \qquad \forall \texttt{edges} \; (i,j)$$

- Determine the distance to the local maximum/minimum values

$$\begin{aligned} Q_i^+ &= \max\{Q_i^+, u_j^L\} & Q_i^- &= \min\{Q_i^-, u_j^L\} \\ Q_j^+ &= \max\{Q_j^+, u_i^L\} & Q_j^- &= \min\{Q_j^-, u_i^L\} \end{aligned} \qquad \forall \texttt{edges} \; (i,j)$$

- Evaluate the nodal correction factors for the net increment

$$R_i^{\pm} = \min\left\{1, Q_i^{\pm}/P_i^{\pm}\right\} \quad \forall \texttt{nodes} \; i$$

- Check the sign of the prelimited antidiffusive flux and multiply it by

$$\alpha_{ij} = \begin{cases} \min\{R_i^+, R_j^-\}, & \texttt{if} \;\; f'_{ij} > 0 \\ \min\{R_i^-, R_j^+\}, & \texttt{if} \;\; f'_{ij} < 0 \end{cases} \qquad \forall \texttt{edges} \; (i,j)$$

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