Algebraic flux correction schemes for B-spline based finite elements

Delft Institute of Applied Mathematics
Numerical Analysis Group

Matthias Möller (m.moller@tudelft.nl)

May 21, 2015

Outline

1. Motivation

2. Mathematical building blocks
   - Introduction to isogeometric analysis
   - Principles of Algebraic Flux Correction

3. Applications
   - Constrained data projection
   - Constrained transport
   - Human brain development

4. Outlook

TUDelft

Vision

Unified simulation and design optimization toolkit

Multi-physics simulation

- complex coupled processes
- different models and scales
- different discretizations
- different resolutions

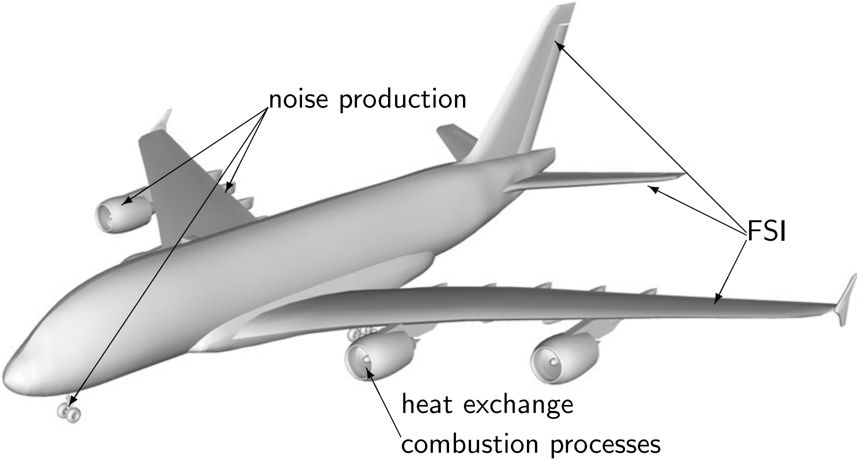
Multi-objective optimization

- (mutually excluding) goals
- adjoint based-optimization
- derivative-free methods
- hierarchical optimization

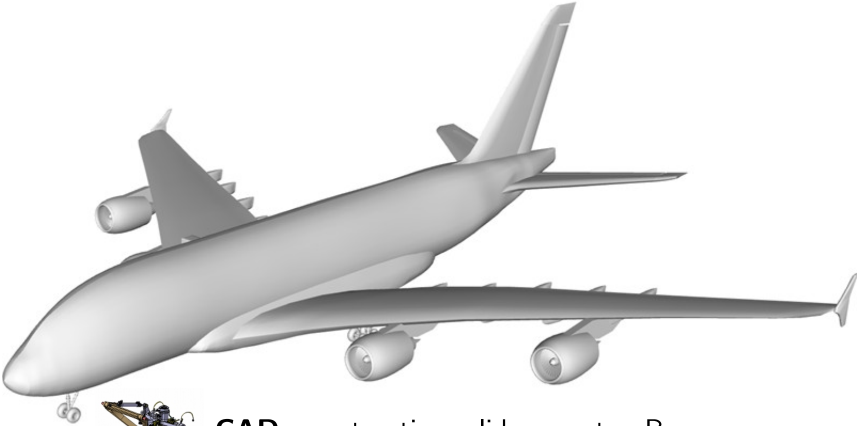
Commonalities

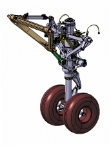
- **one common multi-component master geometry**
- amount of data requires parallel/distributed computing
- problem complexity requires HPC programming techniques

Multiple problems, one master geometry



Multiple problems, one master geometry

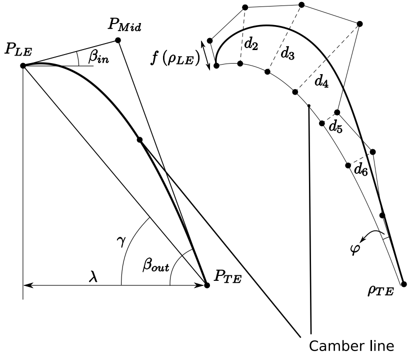


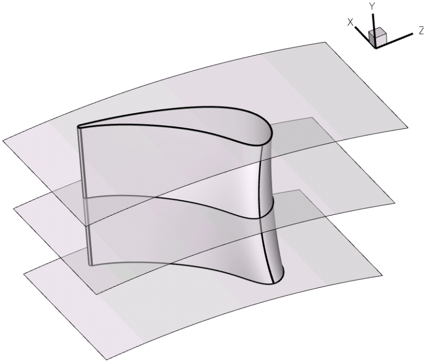


CAD: constructive solid geometry, B-rep, ...

- manual construction of individual parts in $\mathbb{R}^3$
- full scale optimization needs de-featuring

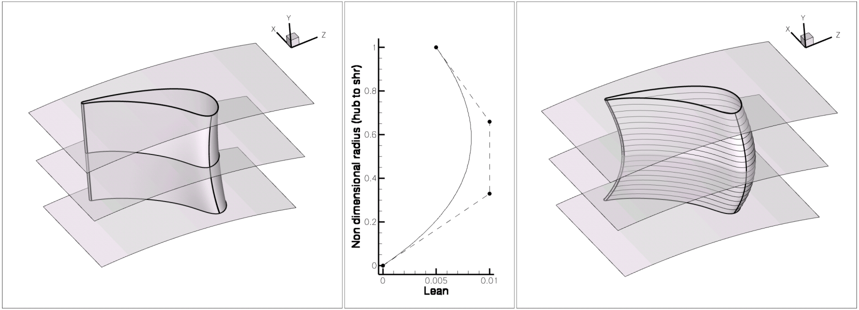
Axial turbine blade geometry generation[1]





[1] T. Verstraete. "CADO: a Computer Aided Design and Optimization Tool for Turbomachinery Applications". In: *Proc. of the 2nd International Conference on Engineering Optimization.* 2010.

Axial turbine blade geometry generation[1]



- direct control of parameters in k-dimensional design space
- moderate number of design parameters in optimization procedure

[1] T. Verstraete. "CADO: a Computer Aided Design and Optimization Tool for Turbomachinery Applications". In: *Proc. of the 2nd International Conference on Engineering Optimization*. 2010.

Isogeometric Analysis

The solution to all our problems?

Vision of the inventors[2]: **Full CAD-CAE integration**

1. Create *analysis-suitable* geometry in CAD tool (using NURBS, B-splines, T-splines, hierarchical splines, ...)
2. Apply established FEA-tools using the same spline basis functions as test and trial functions in the variational form
3. Perform automatic design optimization on the control points of the CAD geometry or, at best, in the design parameter space

[2] T.J.R. Hughes, J.A. Cottrell, and Y. Bazilevs. "Isogeometric Analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement". In: *Computer Methods in Applied Mechanics and Engineering* 194 (2005), pp. 4135–4195.

Isogeometric Analysis

A practical solution to some problems.

After 10 years of practical experience ...

- solid theoretical basis has been established
- succesfull application in fluid and solid mechanics, FSI, ...
- early-adoption in open-source and commercial codes (LSDYNA)

However, ...

- (manual) post-processing of CAD geometries is still required
- assembly of system matrices is more costly than in standard FEA
- global higher continuity in multi-patch scenario is challenging
- very little research pursued on efficient solution techniques
- even less research pursued on IgA for compressible flows

Isogeometric Analysis

A promising concept with large research potential!

IgA for convection-dominated and compressible flows

- algebraic stabilization methods based on matrix manipulations
- efficient assembly based on Fletcher's group formulation
- positivity-preserving time-stepping methods, e.g., SSP-RK

Efficient solution techniques

- outer discontinuous Galerkin multi-patch approach
- multi-level iterative solution algorithms

Heterogeneous HPC

- optimized single-patch compute-kernels for different platforms
- exploit compute-intensity and tensor-product construction of IgA

Outline

Spline space $\mathcal{S}(\Omega_0, p, \mathcal{M}, \mathcal{P})$

- **Polynomial space** of degree p over interval $\Omega_0 := [a,b] \subset \mathbb{R}$

$$\Pi^p([a,b]) := \{q(x) \in \mathcal{C}^\infty([a,b]) \,:\, q(x) = \sum_{i=0}^{p} c_i x^i, c_i \in \mathbb{R}\}$$

- **Polynomial spline** of degree p defined as $s : \Omega_0 \mapsto \mathbb{R}$ if

$$\begin{aligned} s\big|_{[x_i, x_{i+1}]} &\in \Pi^p([x_i, x_{i+1}]), \quad i = 1, \dots, k \\ \frac{d^j}{dx^j} s_{j-1}(x_i) &= \frac{d^j}{dx^j} s_i(x_i), \quad i = 2, \dots, k,\ j = 0, \dots, p - m_i \end{aligned}$$

for partition $\mathcal{P} := \{a = x_1 < \cdots < x_{p+1} = b\}$ of the interval Ω_0 and a set of positive integers $\mathcal{M} := \{1 \le m_i \le p+1\}$

B-splines

- **Knot vector** is sequence of non-decreasing coordinates $\xi_i \in [a,b] \subset \mathbb{R}$ in the parameter space $\Omega_0 = [a,b]$

$$\Xi = (\xi_1, \xi_2, \ldots, \xi_{n+p+1}), \quad \text{where}$$

  - $\xi_i \in \mathbb{R}$ is the i^{th} knot with knot index i
  - p is the polynomial order of the B-splines
  - n is the number of B-spline functions

- **Open knot vector** for $(\Omega_0, p, \mathcal{P}, \mathcal{M})$ is defined as

$$\Xi = (\underbrace{a, \ldots, a}_{p+1 \text{ times}}, \ldots, \underbrace{x_i, \ldots, x_i}_{m_i \text{ times}}, \ldots, \xi_n, \underbrace{b, \ldots, b}_{p+1 \text{ times}})$$

TU Delft

B-splines, cont'd

- **B-splines** of order p yield a basis of the spline space

$$\mathcal{S}(\Omega_0, p, \mathcal{M}, \mathcal{P}) = \mathcal{S}(\Xi, p) := \text{span}\{N_{i,p}(\xi), i = 1, \dots, n\}$$

- **Cox-de Boor recursion formula**

$$p = 0: \; N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leq \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$p > 0: \; N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi)$$

with „$\frac{0}{0} := 0$“

B-splines, cont'd

- **Derivative** of B-spline of order p is a B-spline of order $p-1$

$$\frac{d}{d\xi}N_{i,p}(\xi) = \frac{p}{\xi_{i+p}-\xi_i}N_{i,p-1}(\xi) - \frac{p}{\xi_{i+p+1}-\xi_{i+1}}N_{i+1,p-1}(\xi)$$

- Expression for $\mathbf{k}^{\text{th}}$ **derivative** of B-spline of order p

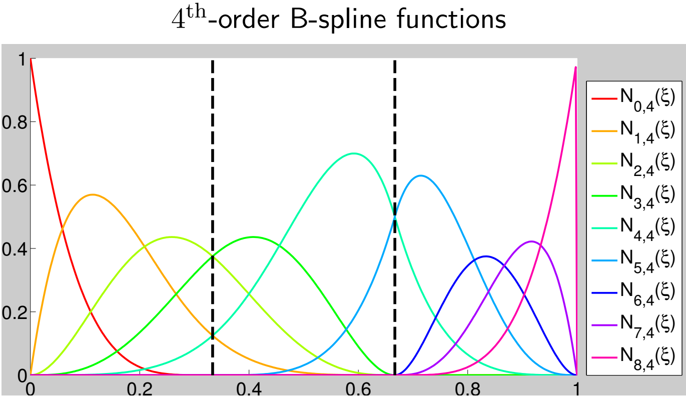
$$\frac{d^k}{d^k\xi}N_{i,p}(\xi) = \frac{p!}{(p-k)!}\sum_{j=0}^{k}\alpha_{k,j}N_{i+j,p-k}(\xi)$$

with recursively defined coefficients $\alpha_{k,j}$ [3]

[3] L. Piegl and W. Tiller. *The NURBS book*. Second edition. Springer, 1997.

TUDelft

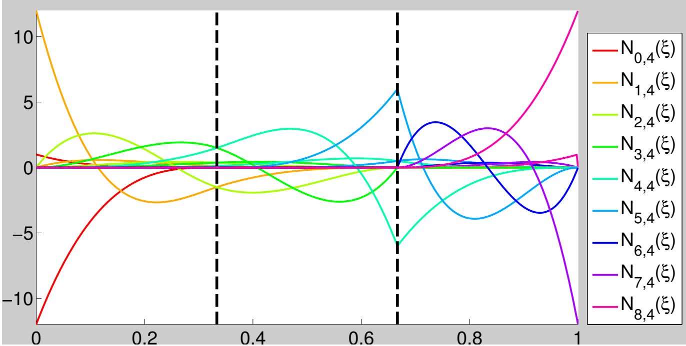
Nonuniform continuity at element boundaries



$$\Xi = (\underbrace{0,0,0,0,0}_{\text{discontinuity}}, \underbrace{1/3}_{C^3}, \underbrace{2/3, 2/3, 2/3}_{C^1}, \underbrace{1,1,1,1,1}_{\text{discontinuity}})$$

Nonuniform continuity at element boundaries

First derivatives of 4^{th}-order B-spline functions



$$\Xi = (\underbrace{0,0,0,0,0}_{\text{discontinuity}}, \underbrace{1/3}_{C^3}, \underbrace{2/3,2/3,2/3}_{C^1}, \underbrace{1,1,1,1,1}_{\text{discontinuity}})$$

Properties of B-splines

- B-splines form a **partition of unity**

$$\text{For each } \xi \in [a,b]: \quad \sum_{i=1}^{n} N_{i,p}(\xi) = 1 \quad \Rightarrow \quad \sum_{i=1}^{n} N'_{i,p}(\xi) = 0$$

- B-splines of order p have **compact support**

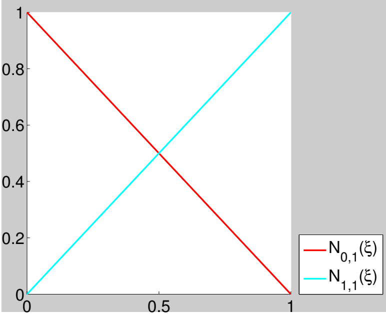
$$\text{supp } N_{i,p}(\xi) = [\xi_i, \xi_{i+p+1}), \quad i = 1, \dots, n$$

- B-splines are **strictly positive** over the interior of their support

$$N_{i,p}(\xi) > 0 \quad \text{for } \xi \in (\xi_i, \xi_{i+p+1}), \quad i = 1, \dots, n$$

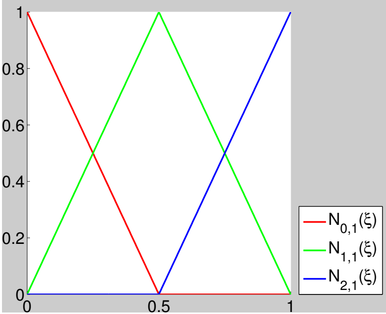
Knot insertion

$\Xi = (0, 0, 1, 1)$, $p = 1$



one element

$\Xi = (0, 0, 0.5, 1, 1)$, $p = 1$

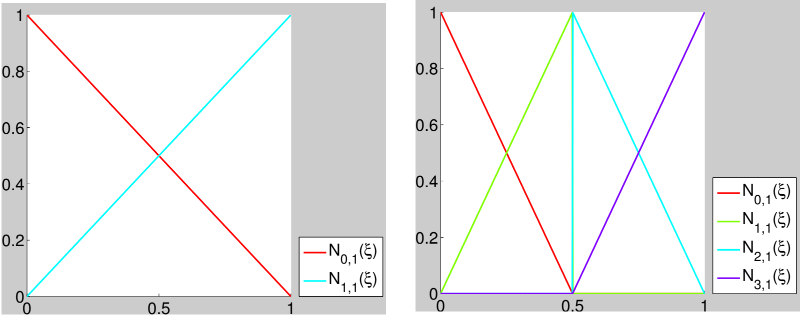


two elements, C^0-continuity

Equivalent to classical h-refinement for knot insertion with $m_{\text{new}} = p$

Knot insertion, cont'd

$\Xi = (0, 0, 1, 1),\ p = 1$ $\qquad$ $\Xi = (0, 0, 0.5, 0.5, 1, 1),\ p = 1$

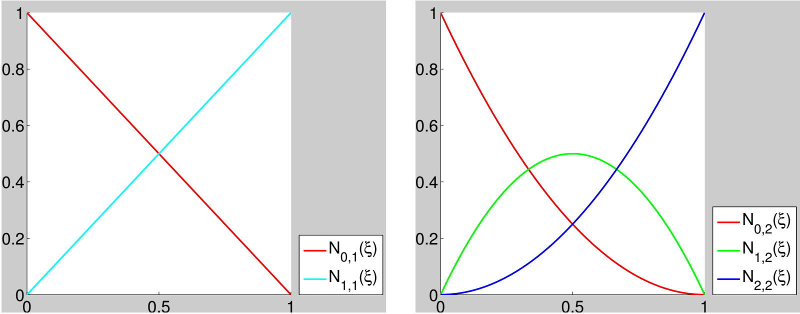


one element $\qquad$ two elements, discontinuity

Enables continuity reduction(!) for knot insertion with $m_{\text{new}} > p$

Order elevation

$\Xi = (0,0,1,1)$, $p = 1$ $\qquad$ $\Xi = (0,0,0,1,1,1)$, $p = 2$



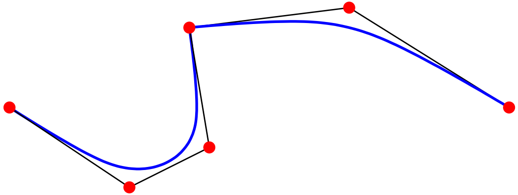
one element $\qquad$ one element

No negative values as with Lagrange basis functions for $p \geq 2$

B-spline curves

- **B-spline curves**: geometric mapping $\mathbf{G} : \Omega_0 \mapsto \Omega$

$$\mathbf{G}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{p}_i \quad \text{with control points } \mathbf{p}_i \in \mathbb{R}^d, d \geq 1$$



- **B-spline surfaces:** geometric mapping $\mathbf{G} : \Omega_0 \mapsto \Omega$

$$\mathbf{G}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{p}_i \quad \text{with control points } \mathbf{p}_i \in \mathbb{R}^d, d \geq 1$$

B-spline surfaces

- Multi-variate B-spline basis for $\Omega_0 = [a, b] \times [c, d]$

$$\mathcal{S}(\Xi, \mathcal{H}, p, q) = \text{span}\{N_{i,p}(\xi)N_{j,q}(\eta), i = 1..n, j = 1..m\}$$

- Geometric mapping $\mathbf{G} : \Omega_0 \mapsto \Omega$

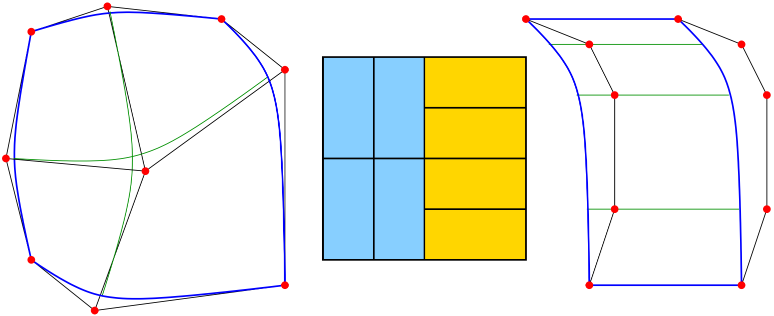
$$\mathbf{G}(\xi, \eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi)N_{j,q}(\eta)\mathbf{p}_{ij}, \quad \mathbf{p}_{ij} \in \mathbb{R}^d, d \geq 2$$

- Global basis functions (e.g., line-wise numbering)

$$N_a(\xi, \eta) = N_{i,p}(\xi)N_{j,q}(\eta), \quad a = n(j-1) + i$$

- Canonical generalization to higher dimensions including partition of unity, local support and strict positivity property

Multi-patch B-spline surfaces



- Possibility for different p/q and n/m-values per patch
- Use, e.g., Z-ordering for improved locality of DoFs
- Consider only single-patch domains in this talk

Ansatz spaces

- Construct ansatz space from B-spline basis functions

$$V_h = \mathsf{span}\{\varphi_a(\mathbf{x}) = N_a \circ \mathbf{G}^{-1}(\mathbf{x}),\ \mathbf{x} \in \Omega\}$$

- Approximate the solution the standard way

$$u(\mathbf{x}, t) \approx u_h = \sum_{a=1}^{N} \varphi_a(\mathbf{x}) u_a(t)$$

- Approximate fluxes by Fletcher's group formulation

$$f(u(\mathbf{x}, t)) \approx f_h = \sum_{a=1}^{N} \varphi_a(\mathbf{x}) f_a(t), \quad f_a(t) = f(u_a)$$

Principles of Algebraic Flux Correction

- **Discrete diffusion operators**[4] are symmetric $N \times N$ matrices with zero row and column sums

$$D = \{d_{ab}\}, \quad d_{ab} = d_{ba}, \quad d_{aa} := -\sum_{b \neq a} d_{ab}$$

- Decomposition into edge-contributions

$$[Du]_a = \sum_{b} d_{ab} u_b = d_{aa} u_a + \sum_{b \neq a} d_{ab} u_b = \sum_{b \neq a} f_{ab}$$

$$[Du]_b = \sum_{a \neq b} f_{ba}, \quad f_{ba} = -f_{ab}, \quad f_{ab} = d_{ab}(u_b - u_a)$$

[4] **KT2002**.

Row-sum mass lumping

- Consistent mass matrix

$$M = \{m_{ab}\}, \qquad m_{ab} = \int_\Omega \varphi_a(\mathbf{x})\varphi_b(\mathbf{x})d\mathbf{x}$$

- Row-sum lumped mass matrix

$$M_l := \mathsf{diag}(m_a), \qquad m_a = \sum_b m_{ab}$$

- Discrete mass diffusion operator

$$D = M - M_l, \qquad d_{ab} = m_{ab}, \quad d_{aa} := -\sum_{b \neq a} m_{ab}$$

TU Delft

Principles of Algebraic Flux Correction, cont'd

- Generic discrete system (strong imposition of bc's)

$$\begin{bmatrix} A_\Omega & A_\Gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} u_\Omega \\ u_\Gamma \end{bmatrix} = \begin{bmatrix} B_\Omega & B_\Gamma \\ 0 & I \end{bmatrix} \begin{bmatrix} g_\Omega \\ g_\Gamma \end{bmatrix}$$

with matrix coefficients

$$a_{aa} > 0, \forall a \quad a_{ab} \leq 0, \forall a, b \neq a \quad b_{ab} \geq 0, \forall a, b \tag{1}$$

- Local **discrete maximum principle** (proof[5])

$$\textbf{(??)} \wedge \sum_b a_{ab} = \sum_b b_{ab} \quad \Rightarrow \quad u_a^{\min} \leq u_a \leq u_a^{\max}$$

[5] **K2012a**.

Principles of Algebraic Flux Correction, cont'd

- Generic discrete system as before with matrix coefficients

$$a_{aa} > 0, \forall a \quad a_{ab} \leq 0, \forall a, b \neq a \quad b_{ab} \geq 0, \forall a, b \tag{2}$$

and strictly or irreducibly diagonally dominant matrix A. (3)

Then A is an **M-Matrix**, i.e. $A^{-1} \geq 0$.

- Global **discrete maximum principle** (proofs[6])

$$(\textbf{??}) \wedge (3) \wedge \sum_b a_{ab} = \sum_b b_{ab}, \forall a \quad \Rightarrow \quad \min g \leq u \leq \max g$$

[6] **K2012a**.

Outline

Constrained data projection[7]

- **High-order predictor:** consistent L_2-projections

$$\sum_b m_{ab} u_b^H = \int_\Omega \varphi_a f \, d\mathbf{x}$$

- **Low-order predictor:** lumped L_2-projections

$$m_a u_a^L = \int_\Omega \varphi_a f \, d\mathbf{x}$$

- Prelimited raw antidiffusive flux ($f_{ba} = -f_{ab}$)

$$f_{ab} = \begin{cases} m_{ab}(u_a^H - u_b^H), & \text{if } (u_a^H - u_b^H)(u_a^L - u_b^L) > 0 \\ 0, & \text{otherwise} \end{cases}$$

[7] D. Kuzmin et al. "Failsafe flux limiting and constrained data projections for equations of gas dynamics". In: *Journal of Computational Physics* 229.23 (2010), pp. 8766–8779. ISSN: 0021-9991. DOI: http://dx.doi.org/10.1016/j.jcp.2010.08.009. URL: http://www.sciencedirect.com/science/article/pii/S0021999110004468.

TU Delft

Constrained data projection, cont'd

- Bounds and antidiffusive increments

$$Q_a^{\pm} = \max_{b\neq a}\min (0, u_a^L - u_b^L), \qquad P_a^{\pm} = \sum_{b\neq a} \max\min (0, f_{ab})$$

- Nodal and edge-wise limiting coefficients

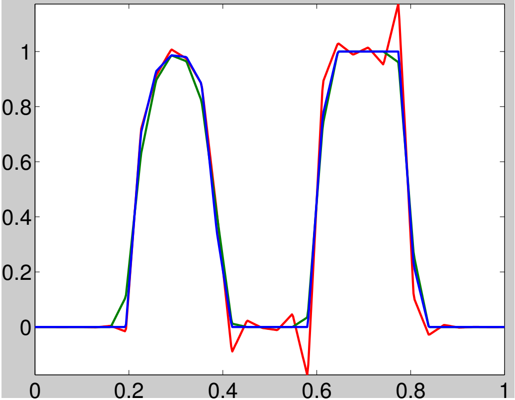
$$R_a^{\pm} = \frac{m_a Q_a^{\pm}}{P_a^{\pm}}, \qquad \alpha_{ab} = \begin{cases} \min(R_a^+, R_b^-), & \text{if } f_{ab} > 0 \\ \min(R_b^+, R_a^-), & \text{if } f_{ab} < 0 \end{cases}$$

- **Corrector:** constrained L_2-projection

$$u_a^\star = u_a^L + \frac{1}{m_a}\sum_{b\neq a} \alpha_{ab} f_{ab}, \quad 0 \leq \alpha_{ab} = \alpha_{ba} \leq 1$$

Test case: Semi-ellipse of McDonald

$p = 1, \, n = 32$



L_1-/L_2-errors

$||u^H - u||_1 = 0.0653$

$||u^L - u||_1 = 0.0684$

$||u^\star - u||_1 = 0.0606$

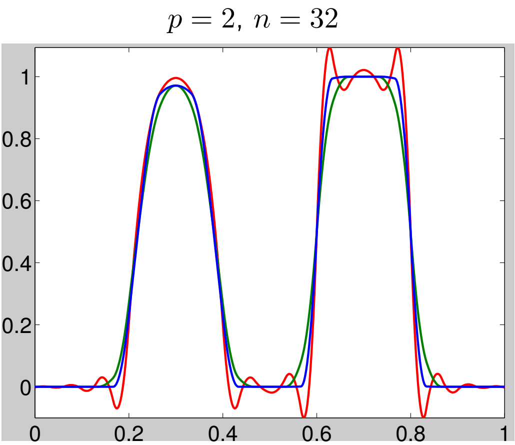
$||u^H - u||_2 = 0.1774$

$||u^L - u||_2 = 0.1686$

$||u^\star - u||_2 = 0.1677$

TUDelft

Test case: semi-ellipse of McDonald



L_1-/L_2-errors

$||u^H - u||_1 = 0.0351$

$||u^L - u||_1 = 0.0568$

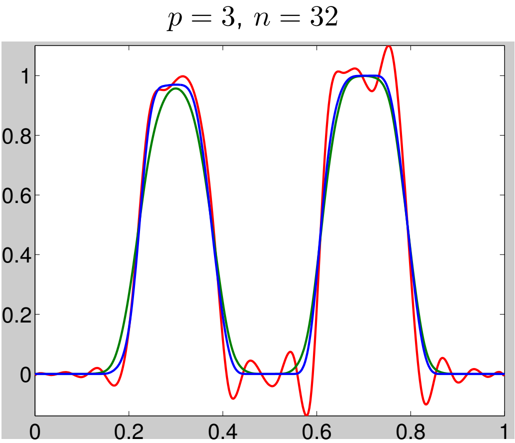
$||u^\star - u||_1 = 0.0352$

$||u^H - u||_2 = 0.0748$

$||u^L - u||_2 = 0.1121$

$||u^\star - u||_2 = 0.0892$

Test case: semi-ellipse of McDonald



L_1-/L_2-errors

$$\|u^H - u\|_1 = 0.0595$$

$$\|u^L - u\|_1 = 0.0788$$

$$\|u^\star - u\|_1 = 0.0615$$

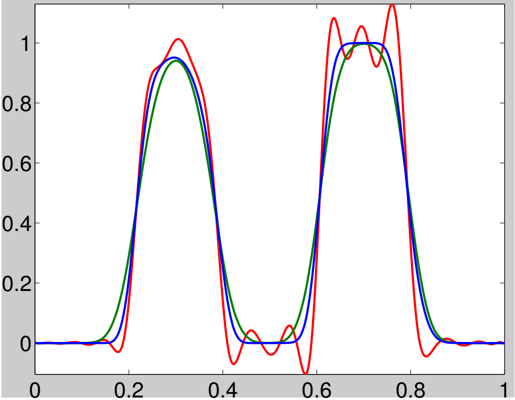
$$\|u^H - u\|_2 = 0.1213$$

$$\|u^L - u\|_2 = 0.1435$$

$$\|u^\star - u\|_2 = 0.1317$$

Test case: semi-ellipse of McDonald

$p = 4,\ n = 32$



L_1-/L_2-errors

$$||u^H - u||_1 = 0.0480$$

$$||u^L - u||_1 = 0.0896$$

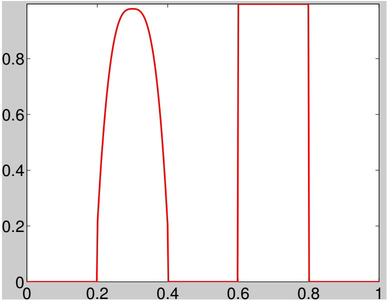
$$||u^\star - u||_1 = 0.0608$$

$$||u^H - u||_2 = 0.1014$$

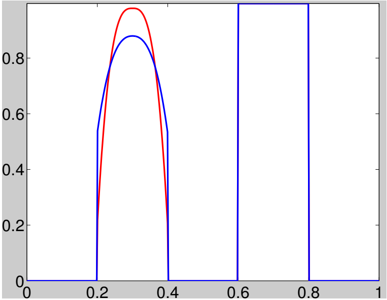
$$||u^L - u||_2 = 0.1505$$

$$||u^\star - u||_2 = 0.1252$$

Thought experiment: What are ideal knots?



Thought experiment: What are ideal knots?



L_1-/L_2-errors

$||u^H - u||_1 = 0.0037$

$||u^\star - u||_1 = 0.0220$

$||u^H - u||_2 = 0.0091$

$||u^\star - u||_2 = 0.0615$

- Deduce ideal knots from nodal correction factors or residual

$$\Xi = (0, 0, 0, 0, 0.1, 0.2, 0.2, 0.2, 0.2, 0.3, 0.4, 0.4, 0.4, 0.4, 0.5, 0.6, 0.6, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 0.8, 0.8, 0.9, 1, 1, 1, 1)$$

- Smoothness indicator is used to avoid peak clipping.

Constrained transport

- Stationary convection-diffusion equation

$$\begin{aligned} \nabla \cdot (\mathbf{v}u) - d\Delta u &= 0 \quad \text{in } \Omega \\ u &= g \quad \text{on } \Gamma_D \\ \nabla u \cdot \mathbf{n} &= 0 \quad \text{on } \Gamma_N \end{aligned}$$

- **High-order Galerkin scheme:** $(K+S)u^H = 0$

$$K = \{\mathbf{v}_b \cdot \int_\Omega \varphi_a \nabla \varphi_b \, d\mathbf{x}\}, \quad S = \{d \int_\Omega \nabla \varphi_a \cdot \nabla \varphi_b \, d\mathbf{x}\}$$

- **Low-order scheme:** $(K+D+S)u^L = 0$

$$D = \{d_{ab}\}, \quad d_{ab} := -\max\{k_{ab}, 0, k_{ba}\}$$

Constrained transport, cont'd

- **High-resolution scheme:** $(K + D + S)u^\star + \mathrm{lim}(-Du) = 0$
- Raw antidiffusive flux $\quad f_{ab} = d_{ab}(u_b - u_a) = -f_{ba}$
- Bounds and antidiffusive increments

$$Q^\pm_{a/b} = \sum_{b \neq a} \begin{matrix}\max\\ \min\end{matrix} (0, -/+f_{ab}), \qquad P^\pm_a = \sum_{b \neq a} \begin{matrix}\max\\ \min\end{matrix} (0, f_{ab})$$

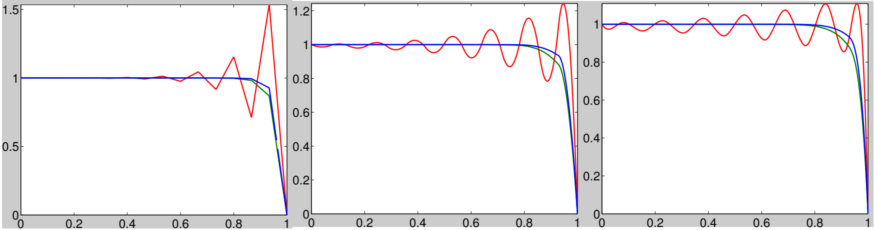
- Nodal and edge-wise limiting coefficients

$$R^\pm_a = \min\{1, \frac{Q^\pm_a}{P^\pm_a}\}, \qquad \alpha_{ab} = \begin{cases} R^+_a, & \text{if } f_{ab} > 0 \\ R^-_a, & \text{if } f_{ab} < 0 \end{cases}$$

TUDelft

Test case: convection-diffusion $\frac{v}{d} = 100$

$p = 1,\ n = 16$ $\qquad$ $p = 2,\ n = 16$ $\qquad$ $p = 3,\ n = 16$

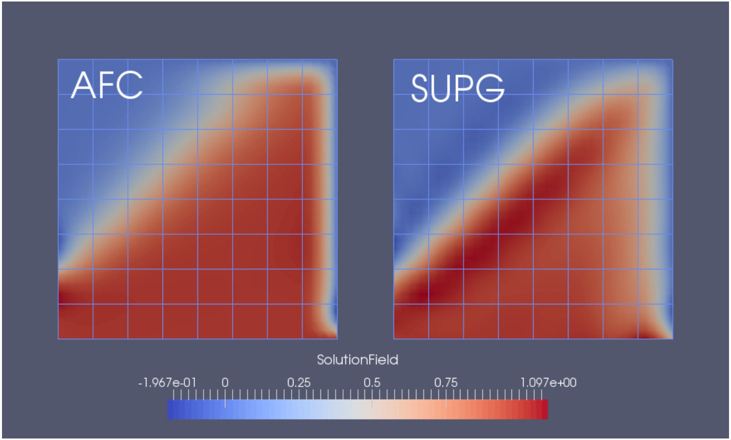


$\lVert u^H - u \rVert_1 = 0.0325$	$\lVert u^H - u \rVert_1 = 0.0415$	$\lVert u^H - u \rVert_1 = 0.0365$
$\lVert u^L - u \rVert_1 = 0.0056$	$\lVert u^L - u \rVert_1 = 0.0061$	$\lVert u^L - u \rVert_1 = 0.0059$
$\lVert u^\star - u \rVert_1 = 0.0028$	$\lVert u^\star - u \rVert_1 = 0.0031$	$\lVert u^\star - u \rVert_1 = 0.0029$

TU Delft

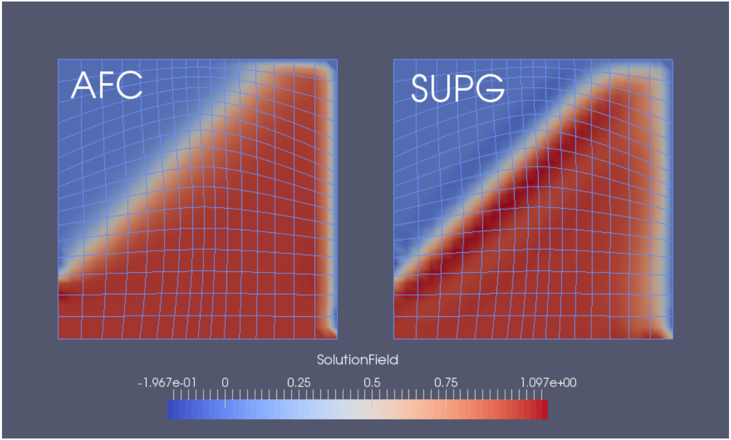
Test case: convection-diffusion

- Configuration: $\Omega = (0,1)^2, \quad d = 0.001, \quad \mathbf{v} = \frac{1}{\sqrt{2}}(1,1)^\top$
- MSc-project by Andrzej Jaeschke using G+SMO (JKU, Linz)



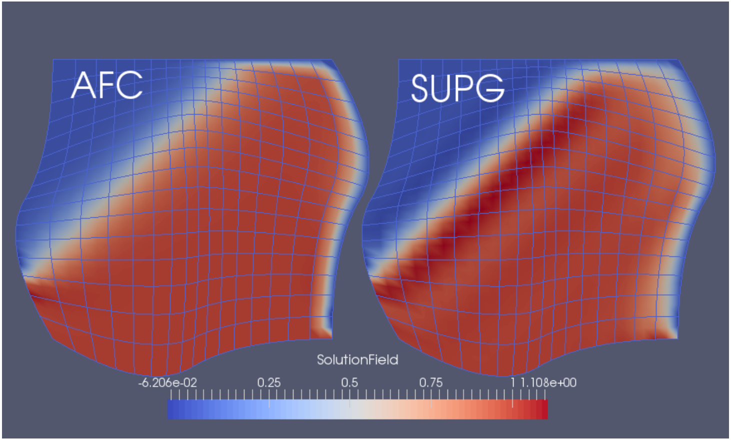
Test case: convection-diffusion

- Configuration: $\Omega = (0,1)^2, \quad d = 0.001, \quad \mathbf{v} = \frac{1}{\sqrt{2}}(1,1)^\top$
- MSc-project by Andrzej Jaeschke using G+SMO (JKU, Linz)



Test case: convection-diffusion

- Configuration: $\Omega = (0,1)^2, \quad d = 0.001, \quad \mathbf{v} = \frac{1}{\sqrt{2}}(1,1)^\top$
- MSc-project by Andrzej Jaeschke using G+SMO (JKU, Linz)



Human brain development[8]

- Gray-Scott reaction-diffusion model

$$\partial_t u + u\partial_t \log\sqrt{g_t} - d_1\Delta_{\mathcal{M}_t} u = f(1-u) - uv^2 \qquad \text{(inhibitor u)}$$
$$\partial_t v + v\partial_t \log\sqrt{g_t} - d_2\Delta_{\mathcal{M}_t} v = uv^2 - (f+k)v \qquad \text{(activator v)}$$

with determinant $\sqrt{g_t}$ of the surface parameterization

- Surface evolution model (morphogen-driven evolution)

$$\partial_t \mathcal{M} = h(u,v)\mathbf{n}, \qquad \text{e.g., } h(u,v) = Kv$$

- MSc-project by Jochen Hinz using Nutils (TU/e)

[8] J. Lefèvre and J.-F. Mangin. "A Reaction-Diffusion Model of Human Brain Development". In: *PLoS Computational Biology* 6.4 (2009).

Summary

- Algebraic flux correction concept has been generalized to higher-order approximations based on B-spline bases
- Original lowest-order approximation is naturally included
- Nodal correction factors/residual provide information to locally reduce 'inter-element' continuity by increasing knot multiplicity
- Peak clipping at smooth extrema is prevented by locally deactivating the flux limiter using the smoothness indicator[9]

[9] D. Kuzmin and F. Schieweck. "A parameter-free smoothness indicator for high-resolution finite element schemes". In: *Central European Journal of Mathematics* 11.8 (2013), pp. 1478–1488.

Current and future research

- Theoretical analysis of AFC principles for
  - arbitrary geometric mappings, NURBS basis functions, and
  - weakly imposed boundary conditions/multi-patch coupling
- Realization of more schemes of the AFC-type family
  - transport with highly anisotropic diffusion
- Implementation of an HPC IgA-framework in C++-11
  - exploit compute-intensity of high-order methods
  - accelerator support via expression templates for assembly
  - on-demand target-optimized compilation at run time

Acknowledgements: B. Jüttler (JKU, Linz), A. Mantzaflaris (RICAM, Linz), G. v. Zwieten (TU/e, Eindhoven), and A. Jaeschke & J. Hinz & J. v. Zwieten (TUD, Delft)