Distributed time-parallel solution of transient problems with MGRIT and p-multigrid methods

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Model problem and research aim

Our aim is to develop an efficient solution strategy for model problems of the form

$$\begin{aligned} \partial_t u(\mathbf{x},t) - \kappa \Delta u(\mathbf{x},t) &= f && \mathbf{x} \in \Omega,\ t \in [0,T] \\ u(\mathbf{x},t) &= g && \mathbf{x} \in \Gamma,\ t \in [0,T] \\ u(\mathbf{x},0) &= u^0(\mathbf{x}) && \mathbf{x} \in \Omega \end{aligned}$$

For us, efficient means the following

- robust with respect to h and p, and the number of patches
- computationally efficient throughout all problem sizes
- good strong and weak scaling in the number of processors and time steps

We treat the temporal problem using a **multigrid-reduction-in-time** approach with two-level θ time-stepping and apply a p-**multigrid** solver for the spatial problem.

Motivation

It is well known that solving the linear systems that arise from isogeometric discretizations is a challenge. A recent paper by Gervasio *et al.* quantifies the (exponential) growth of the condition number of the mass matrix M (left) and the stiffness matrix K (right).

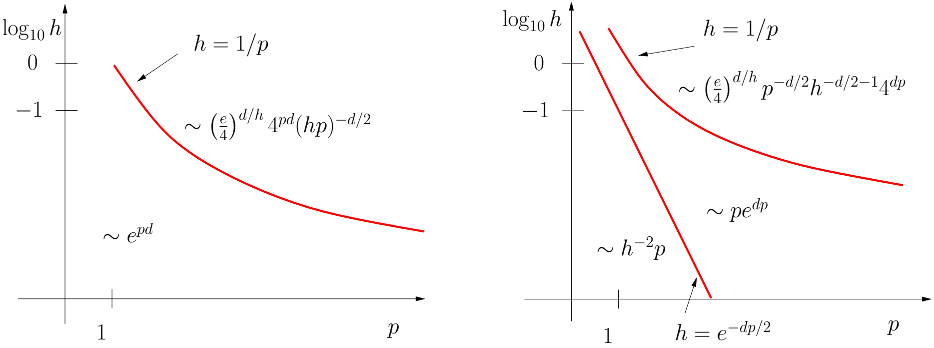
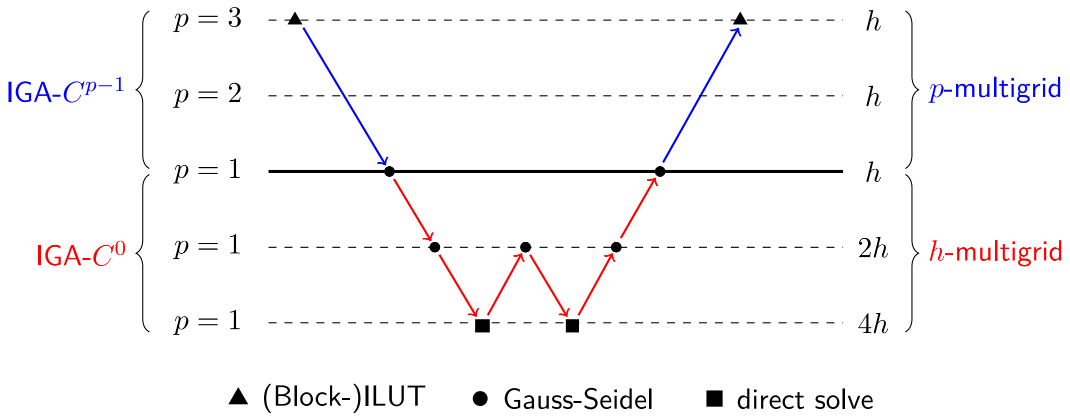


Table 3 from Gervasio *et al.*, doi:10.1007/s10915-020-01204-1

Our strategy for solving the spatial problem $-\kappa \Delta u = f$

1. Start at $\mathcal{V}_{h,p}$ and perform a direct projection to $\mathcal{V}_{h,1}$ using a p-multigrid solver with ILUT smoother (single-patch case) and block-ILUT smoother (multi-patch case)
2. Solve the problem at level $p = 1$ using classical h-multigrid with Gauss-Seidel smoother



Numerical efficiency – single-patch case

Our approach is robust with respect to h and p for tensor-product B-splines (top)

	$p=2$		$p=3$		$p=4$		$p=5$	
	ILUT	GS	ILUT	GS	ILUT	GS	ILUT	GS
$h=2^{-6}$	4	30	3	62	3	176	3	491
$h=2^{-7}$	4	29	3	61	3	172	3	499
$h=2^{-8}$	5	30	3	60	3	163	3	473
$h=2^{-9}$	5	32	3	61	3	163	3	452

as well as for locally refined THB-splines (bottom)

	$p=2$		$p=3$		$p=4$		$p=5$	
	ILUT	GS	ILUT	GS	ILUT	GS	ILUT	GS
$h=2^{-4}$	6	17	8	47	7	177	10	1033
$h=2^{-5}$	6	16	7	44	8	182	7	923
$h=2^{-6}$	6	17	5	43	6	201	12	1009

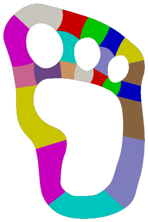
Numerical efficiency – multi-patch case

Our approach is moreover robust with respect to the number of patches, whereby the block-ILUT smoother requires less multigrid iterations than the global ILUT smoother.

	$p=2$ # patches			$p=3$ # patches			$p=4$ # patches			$p=5$ # patches		
	4	16	64	4	16	64	4	16	64	4	16	64
$h=2^{-5}$	$3(5)$	$4(7)$	$4(9)$	$3(5)$	$3(7)$	$4(11)$	$2(4)$	$2(6)$	$4(-)$	$2(4)$	$2(6)$	$-(-)$
$h=2^{-6}$	$3(5)$	$3(5)$	$4(7)$	$3(5)$	$3(7)$	$4(10)$	$3(6)$	$2(7)$	$3(11)$	$3(5)$	$3(7)$	$3(10)$
$h=2^{-7}$	$3(5)$	$3(5)$	$3(5)$	$3(5)$	$3(6)$	$3(8)$	$3(5)$	$2(6)$	$3(10)$	$-(5)$	$6(7)$	$3(11)$

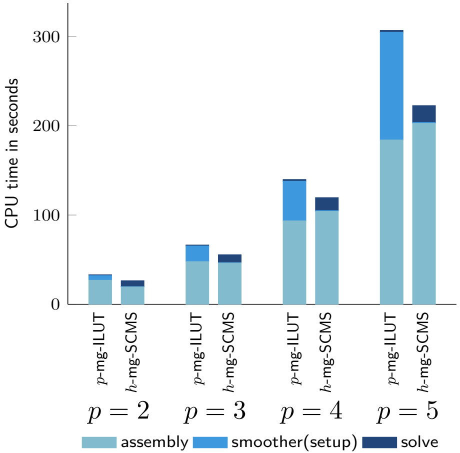
The Yeti footprint benchmark

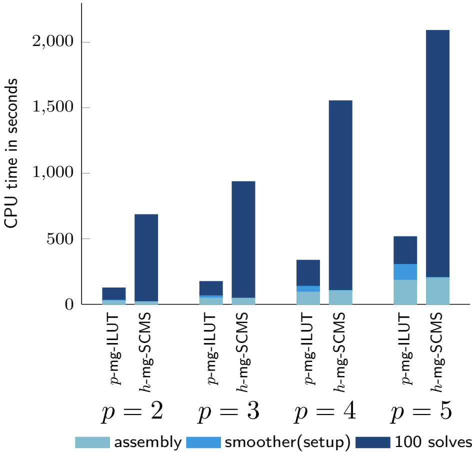
	$p=2$		$p=3$		$p=4$		$p=5$	
	block	global	block	global	block	global	block	global
$h=2^{-3}$	4	5	2	4	2	4	2	4
$h=2^{-4}$	4	8	3	5	3	5	2	4
$h=2^{-5}$	4	78	3	6	3	5	3	5



Computational efficiency

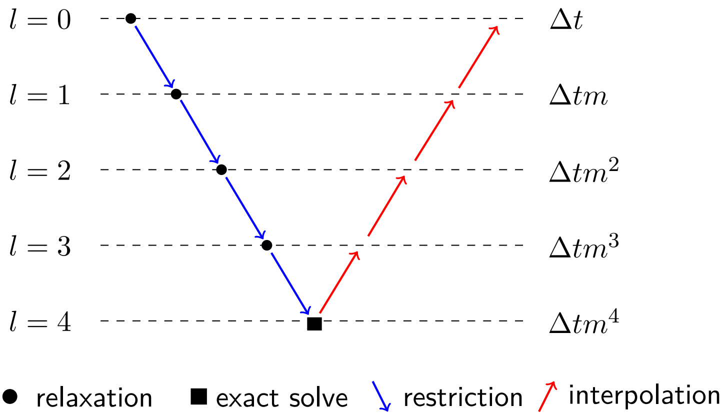
Although our approach has higher setup times than h-multigrid with subspace corrected mass smoother [Takac, 2017] (left) it outperforms the latter for multiple solves (right)





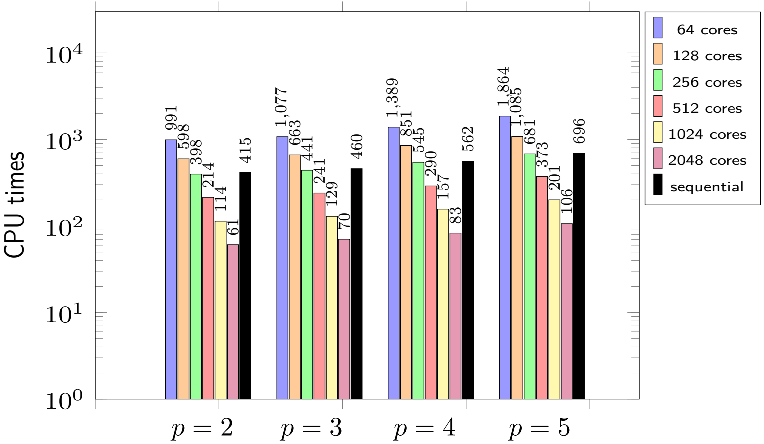
Multigrid-reduction-in-time

The temporal problem is solved with the multigrid-reduction-in-time algorithm by Friedhoff *et al.*, whereby the fully implicit backward Euler scheme is used on all temporal levels l.



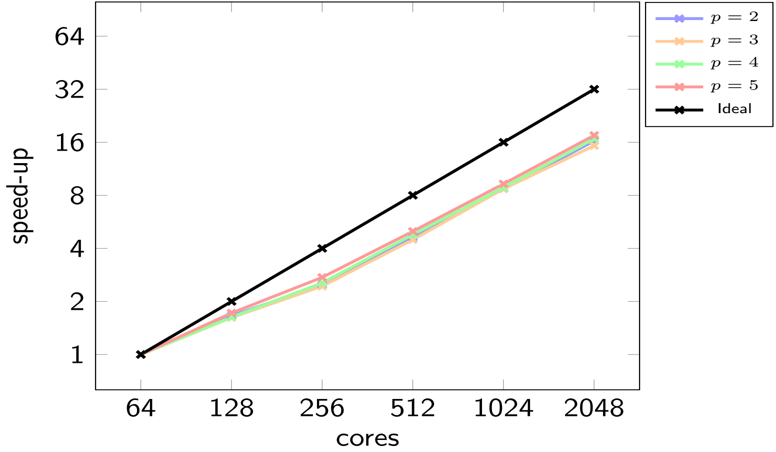
Computational efficiency – strong scaling

Heat equation with $h = 2^{-6}$ spatial resolution solved for $N_t = 10.000$ time steps with backward Euler method on 128 Xeon Gold 6130 CPUs (2.10GHz, 96GB, 16 cores)



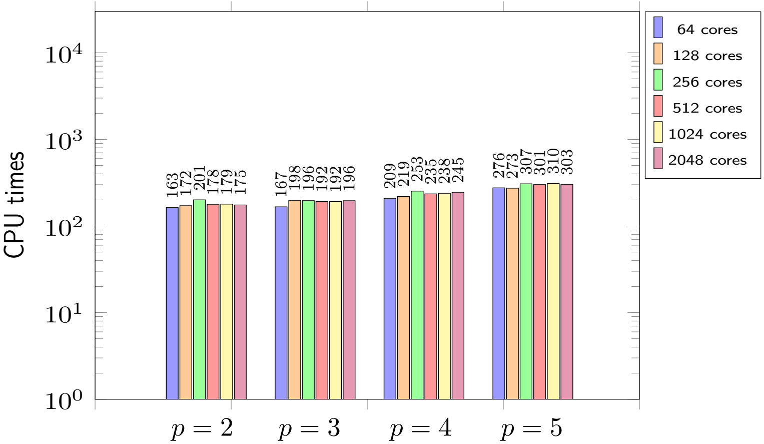
Computational efficiency – speed up

Heat equation with $h = 2^{-6}$ spatial resolution solved for $N_t = 10.000$ time steps with backward Euler method on 128 Xeon Gold 6130 CPUs (2.10GHz, 96GB, 16 cores)



Computational efficiency – weak scaling

Heat equation with $h = 2^{-6}$ spatial resolution solved for $N_t = \text{cores}/64 \cdot 1.000$ time steps with backward Euler method on 128 Xeon Gold 6130 CPUs (2.10GHz, 96GB, 16 cores)



Conclusions and further reading

Multigrid-reduction-in-time together with p-**multigrid and (block-)ILUT smoothing** yields an efficient solution strategy for transient problems discretized by IGA

- R.Tielen, M. Möller, D. Göddeke and C. Vuik: *p-multigrid methods and their comparison to h-multigrid methods within Isogeometric Analysis*, CMAME, Vol 372 (2020)
- R. Tielen, M. Möller and C. Vuik: *A block ILUT smoother for multipatch geometries in Isogeometric Analysis*, In: Springer INdAM Series, Springer, 2021
- R. Tielen, M. Möller and C. Vuik: *Multigrid Reduced in Time for Isogeometric Analysis*, Submitted to: Proceedings of the Young Investigators Conference 2021.
- R. Tielen, M. Möller and C. Vuik: *Combining p-multigrid and multigrid reduced in time methods to obtain a scalable solver for Isogeometric Analysis*, arXiv:2107.05337
- R. Tielen *p-Multigrid Methods for Isogeometric Analysis*, PhD thesis. To be defended Oct 2021.

Thanks for your interest in our work! Contact: `M.Moller@tudelft.nl`