Multiwavelet troubled cell indicator for discontinuity detection of discontinuous Galerkin schemes

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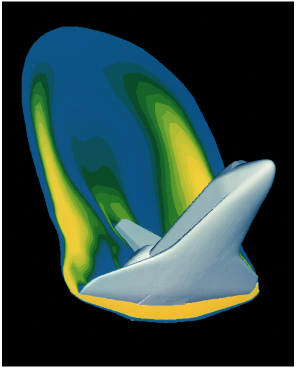
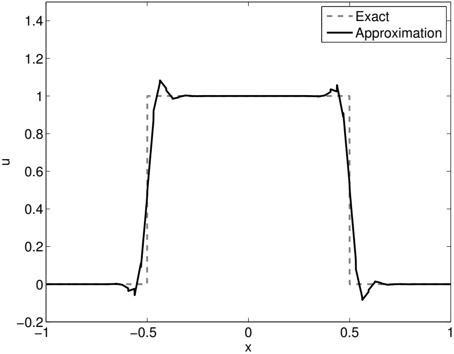
Motivation



Flow around Space Shuttle



Solution linear advection equation

Outline

Outline

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Discontinuous Galerkin

Hyperbolic partial differential equation:

$$u_t + f(u)_x = 0; \quad x \in [-1, 1], \; t \geq 0.$$

- DG approximation: for $x \in I_j$, write,

$$u_h(x) = \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j), \quad \xi_j = \frac{2}{\Delta x}(x - x_j)$$

- approximation space: orthonormal Legendre polynomials

$$\int_{-1}^{1} \phi_\ell(x)\phi_m(x)dx = \delta_{\ell m}$$

- k: highest polynomial degree of the approximation

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Limiters

Limiter:

- Helps to control spurious oscillations
- Reduces polynomial order in nonsmooth regions
- May flatten local extrema (diffusive property)

Troubled cell indicator:

- Helps to limit at discontinuities only

Troubled cell indicators

Examples of troubled cell indicators for DG:

- minmod based TVB limiter
  (Cockburn and Shu, Math. Comput. 1989)
- KXRCF indicator
  (Krivodonova et al., Appl. Numer. Math. 2004)
- Harten's subcell resolution
  (Qiu and Shu, SIAM J. Sci. Comput. 2005)

These indicators use local information (neighbouring cells)
Our new multiwavelet approach uses global information

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Multiwavelets

Multiwavelets (Alpert, SIAM J. Math. Anal. 1993):

- specific set of piecewise polynomials
- based on orthonormal Legendre polynomials
- possible to decompose function into several levels

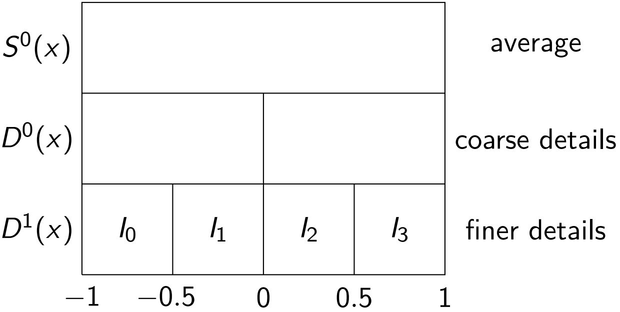
Relation between DG and multiwavelets (2^n elements):

$$u_h(x) = \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) = S^0(x) + \sum_{m=0}^{n-1} D^m(x)$$

Multiwavelet decomposition

Example uses $n = 2$: 4 elements on $[-1, 1]$

$$u_h(x) = \sum_{j=0}^{3} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) = S^0(x) + \sum_{m=0}^{n-1} D^m(x), \quad n - 1 = 1$$



Regions where multiwavelet contributions are continuous

Both $D^1(x)$ and $u_h(x)$: continuous on $I_0, \ldots, I_3$

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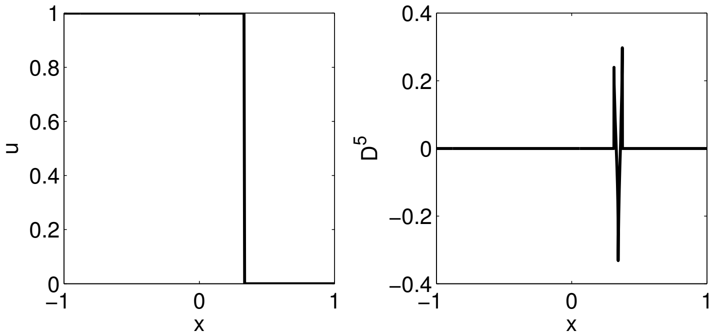
Discontinuous example

Most details are visible in $D^{n-1}(x)$

Example: use $n = 6$: 2^6 elements



Multiwavelet approximation $D^5(x)$ of square wave

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Multiwavelet troubled cell indicator

- Troubled cells: focus on highest level $D^{n-1}(x)$
- Compute absolute average $\bar{D}_j^{n-1}$ on element I_j
- Element I_j is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max\left\{\bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1\right\}, C \in [0, 1]$$

- Indication of troubled cells: compute $D^{n-1}(x)$ only

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Choice of C

I_j is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max\left\{\bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1\right\}, C \in [0, 1]$$

Parameter C: defines strictness of indicator,

- $C = 0$: every element is detected
- $C = 0.2$: select largest 80% of averages
- $C = 0.8$: select largest 20% of averages

Multiwavelet troubled cell indicator

- Global detector, more accurate than local detector
  (Zaide and Roe, 20th AIAA CFD Conf. 2011)

- Limiter: mechanism to control limited regions
  Now: troubled cell indicator as switch

- Moment limiter (Krivodonova, J. Comput. Phys. 2007)

  Only a choice, other limiters possible

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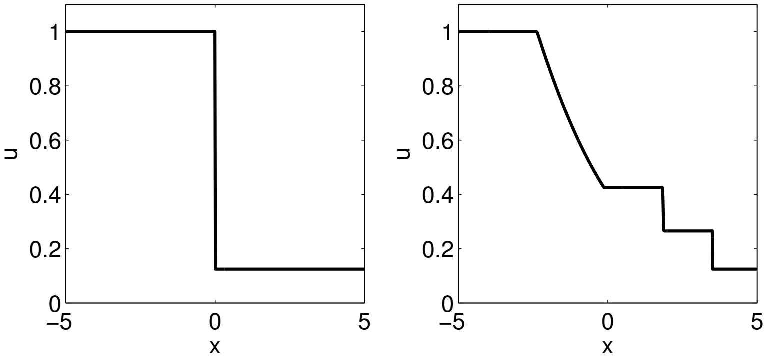
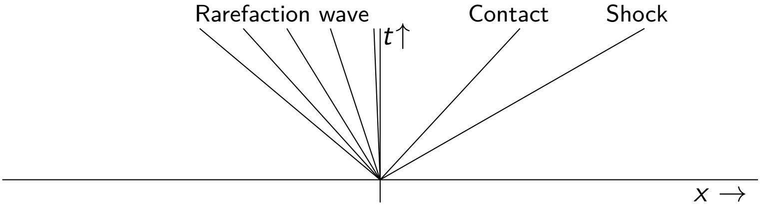
Sod's shock tube (J. Comput. Phys. 1978)



Density in Sod's shock tube at $T = 0$ (left) and $T = 2$ (right)

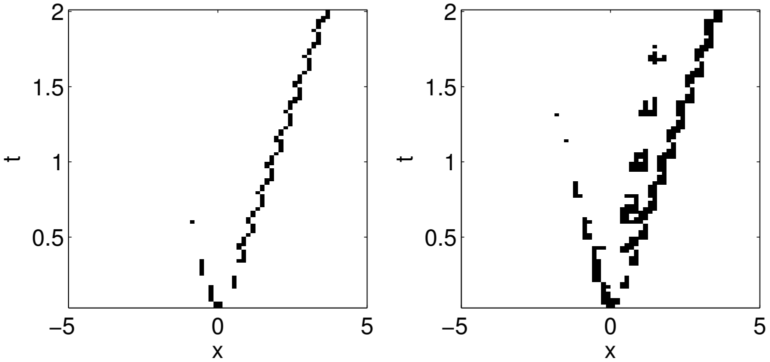
Sod: time history

Results: focus on detected troubled cells



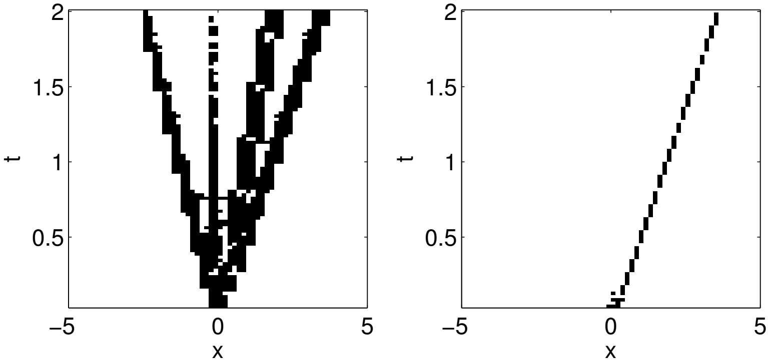
Time history of troubled cells

Sod: detected troubled cells



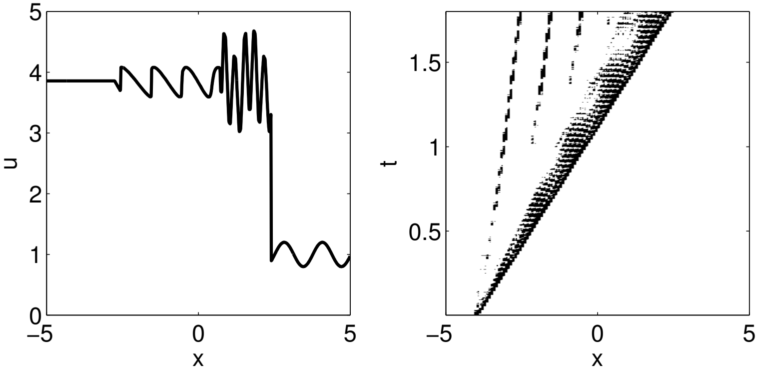
Detected troubled cells, $C = 0.9$ (left) or $C = 0.5$ (right)

Sod: detected troubled cells



Detected troubled cells, $C = 0.1$ (left) or KXRCF (right)

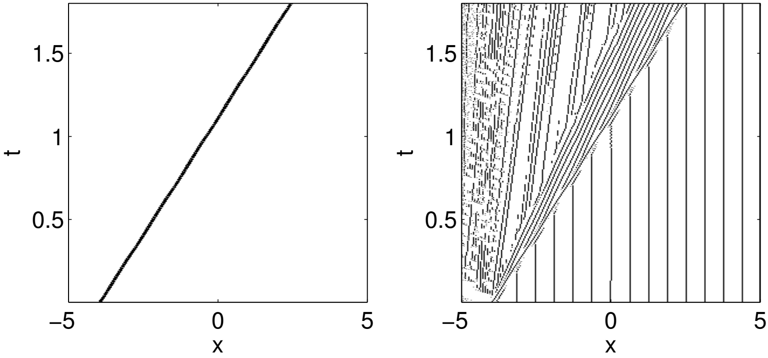
Sine entropy wave



Solution (left) and detected troubled cells, $C = 0.05$ (right)

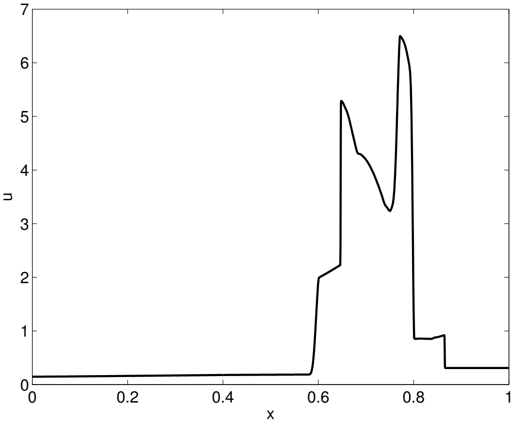
(Shu and Osher, J. Comput. Phys. 1989)

Sine entropy wave



KXRCF (left) and Harten's (right) troubled cell indicator

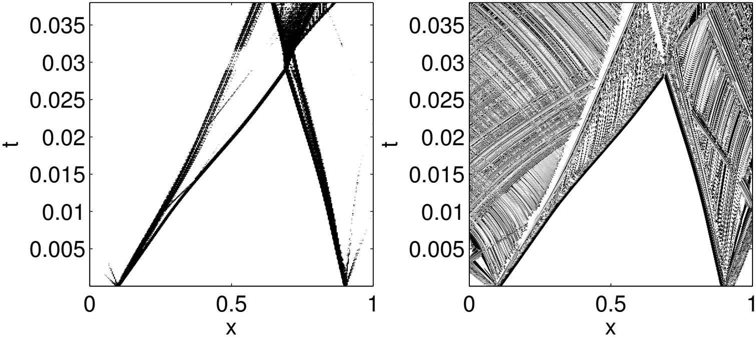
Blast waves



Solution at $T = 0.038$

(Woodward and Colella, J. Comput. Phys. 1984)

Blast: detected troubled cells



Detected troubled cells, $C = 0.001$ (left) or Harten (right)

Two-dimensional approach

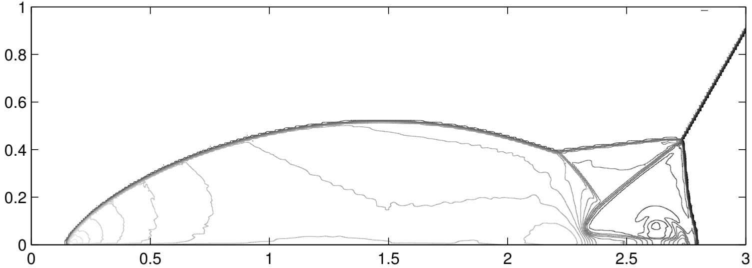
In two-dimensions, the multiwavelet expansion is:

$$S^0(x,y) + \sum_{m_x=0}^{n_x-1} \sum_{m_y=0}^{n_y-1} \left\{ D^{\alpha,\mathbf{m}}(x,y) + D^{\beta,\mathbf{m}}(x,y) + D^{\gamma,\mathbf{m}}(x,y) \right\}$$

number of elements: $2^{n_x} \times 2^{n_y}$

- α mode: multiwavelets in y-direction
- β mode: multiwavelets in x-direction
- γ mode: multiwavelets both x- and y-direction

Double Mach reflection

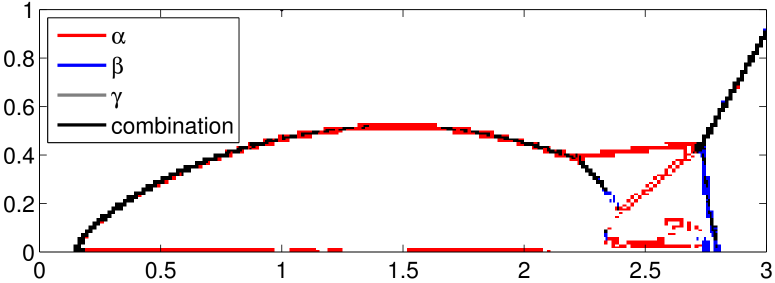


Density contours using $C = 0.05$

$T = 0.2$, $\Delta x = \Delta y = \frac{1}{128}$, $k = 1$

(Woodward and Colella, J. Comput. Phys. 1984)

Detected troubled cells



Detected troubled cells at $T = 0.2,\ C = 0.05$

Different troubled cells are detected by modes

Computation time

Compare computation time, double Mach reflection:

- More accurate result: don't limit continuous regions
- Decrease of computation time

	limit everywhere	$C = 0.05$
512×128	57	50
1024×256	493	441

Computation time in minutes, $T = 0.2$, $k = 1$

Outline

Conclusion

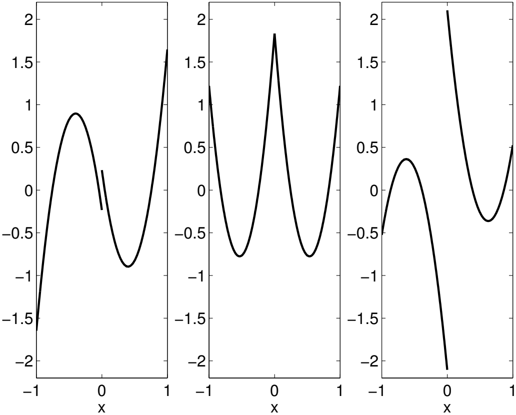
- Global troubled cell indicator, switch in limiter
- Multiwavelet decomposition: $D^{n-1}(x)$ detects discontinuity
- Parameter C defines strictness of detector
- More accurate than existing detectors
- Two-dimensional detection in different modes
- Decrease of computation time

Future work:

→ How to choose parameter C

→ Applying to unstructured meshes

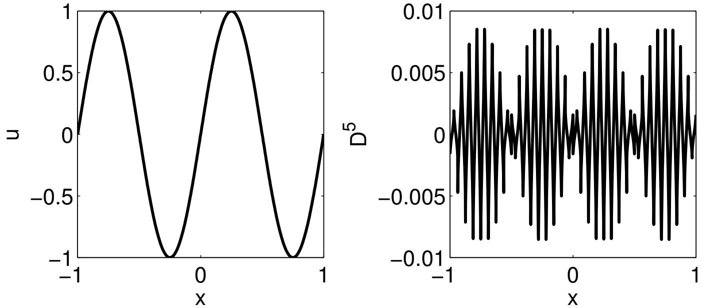
Multiwavelets



Multiwavelet basis belonging to $k = 2$

Basis spans piecewise polynomials on $[-1, 0] \cup [0, 1]$, degree ≤ 2

Continuous example



Multiwavelet approximation $D^5(x)$ of $\sin(2\pi x)$, 2^6 elements