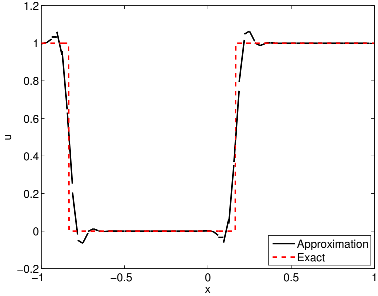
Automated parameters for troubled-cell indicators using outlier detection

Thea Vuik
Delft University of Technology

Collaboration with Jennifer Ryan, University of East Anglia

June 24, 2015

Introduction: Nonlinear hyperbolic PDE's



Limiters:

- Too few elements: oscillatory approximation
- Too many elements: too diffusive, computationally expensive

Which elements need limiting? Troubled-cell indicator

Outline

1. Building blocks: DG and multiwavelets
2. Multiwavelet troubled-cell indicator (with parameter)
3. Outlier detection
4. Results

Outline

1. Building blocks: DG and multiwavelets
2. Multiwavelet troubled-cell indicator (with parameter)
3. Outlier detection
4. Results

Discontinuous Galerkin method

$$\begin{cases} u_t + f(u)_x = 0, & x \in [-1, 1], \quad t > 0, \\ u(x, 0) = u_0(x), & x \in [-1, 1]. \end{cases}$$

- Discretize $[-1, 1]$ into 2^n elements
- Approximation space V_h^k: kth-degree piecewise polynomials
- Approximate u by $u_h \in V_h^k$

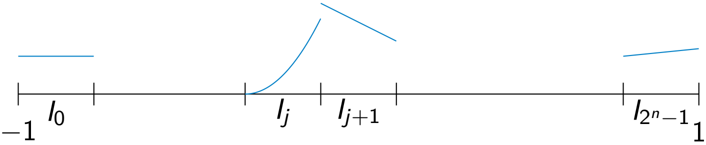
- Multiply PDE by $v_h \in V_h^k$, integrate over I_j
- Integrate by parts

TUDelft

DG approximations and multiwavelets

Global DG approximation, 2^n elements on $[-1, 1]$:

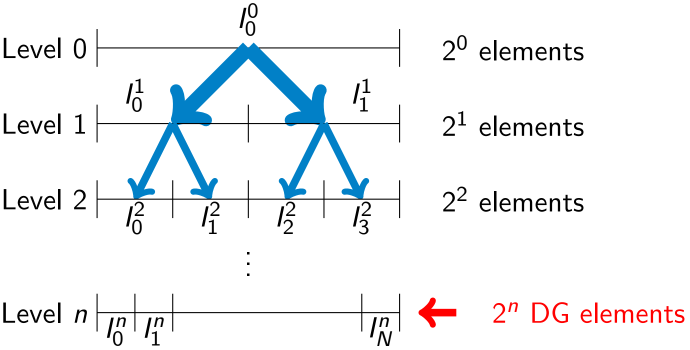
$$u_h(x) = \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j)$$



Corresponding multiwavelet expansion:

$$u_h(x) = \underbrace{\sum_{\ell=0}^{k} s_{\ell 0}^{0} \phi_\ell(x)}_{\text{global average}} + \sum_{m=0}^{n-1} \underbrace{\sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^{m} \psi_{\ell j}^{m}(x)}_{\text{finer details}}$$

Multiresolution idea



$$V_n^k = \{f : f \in \mathbb{P}^k(I_j^n), j = 0, \dots, 2^n - 1\}$$

$$V_0^k \subset V_1^k \subset \cdots \subset V_n^k \subset \cdots$$

(Alpert, SIAM J. Math. Anal. 1993)

Scaling functions and DG basis

DG basis functions:

- Orthonormal Legendre polynomials
- Basis for V_0^k : scaling function basis
- Basis functions for V_n^k: dilation and translation

$$\phi_{\ell j}^n(x) = 2^{n/2}\phi_\ell(2^n(x+1) - 2j - 1),$$

$$\ell = 0, \dots, k, \; j = 0, \dots, 2^n - 1, \; x \in I_j^n$$

(Archibald et al., Appl. Num. Math. 2011)

TUDelft

Multiwavelets

Multiwavelet space W_m^k:

- Orthogonal complement of V_m^k in V_{m+1}^k:

$$V_m^k \oplus W_m^k = V_{m+1}^k, \quad W_m^k \perp V_m^k, \quad W_m^k \subset V_{m+1}^k$$

- V_n^k can be split into $n+1$ orthogonal subspaces:

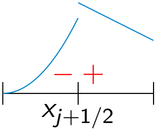
$$V_n^k = V_0^k \oplus W_0^k \oplus W_1^k \oplus \cdots \oplus W_{n-1}^k$$

Split up $f \in V_n^k$ into different levels:

$$u_h(x) = \sum_{\ell=0}^{k} s_{\ell 0}^{0} \phi_\ell(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^{m} \psi_{\ell j}^{m}(x)$$

Jumps in DG approximations

$$u_h(x) = \sum_{\ell=0}^{k} s_{\ell 0}^{0} \phi_\ell(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^{m} \psi_{\ell j}^{m}(x)$$

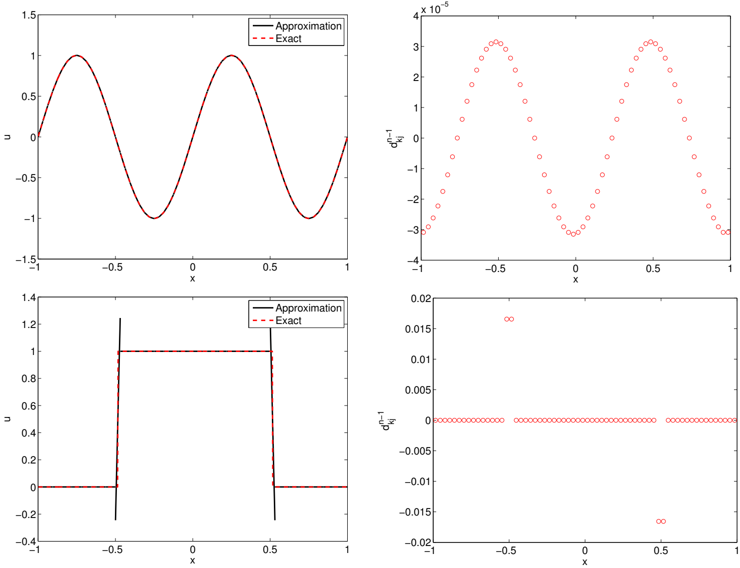


Coefficient $d_{\ell j}^{n-1}$: measures jump in (derivatives) approximation

$$d_{\ell j}^{n-1} = \sum_{m=0}^{k} c_{m\ell}^{n} \left(u_h^{(m)}(x_{j+1/2}^{+}) - u_h^{(m)}(x_{j+1/2}^{-}) \right),$$

where

$$c_{m\ell}^{n} = \frac{2^{(-n+1)m}}{m!} \cdot \int_0^1 x^m \psi_\ell(x)\, dx.$$



Approximation $\qquad$ d_{kj}^{n-1}

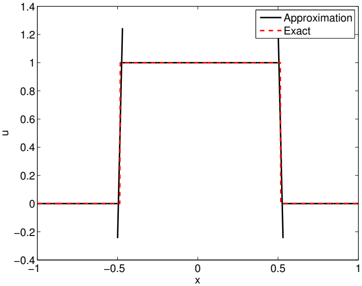
Outline

1. Building blocks: DG and multiwavelets
2. Multiwavelet troubled-cell indicator (with parameter)
3. Outlier detection
4. Results

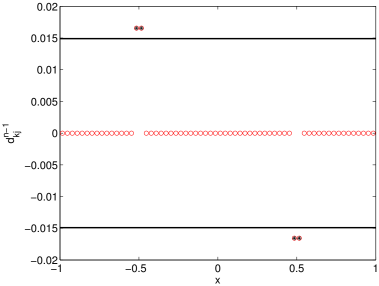
Original approach

Detect elements I_j and I_{j+1} if

$$|d_{kj}^{n-1}| > C \cdot \max_j |d_{kj}^{n-1}|, C \in [0,1].$$

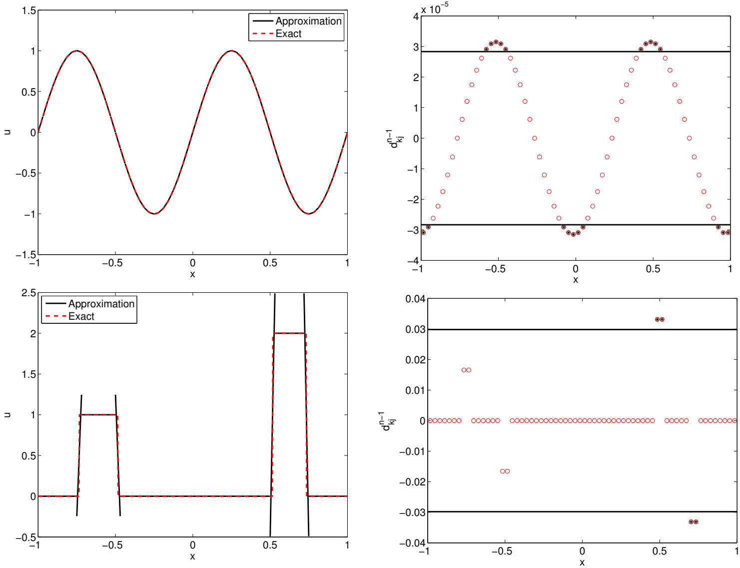


Approximation



Detected, $C = 0.9$

64 elements, $k = 1$



Approximation

Detected, $C = 0.9$

How to choose C?

Outline

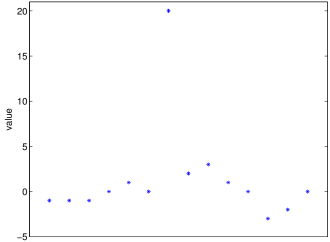
1. Building blocks: DG and multiwavelets
2. Multiwavelet troubled-cell indicator (with parameter)
3. Outlier detection
4. Results

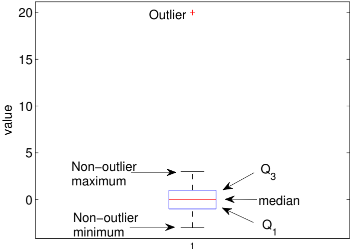
Outlier detection

d_{kj}^{n-1}:

- vector containing jumps over element boundaries
- coefficient big compared to neighbors: detect

$\Rightarrow$ Boxplot approach



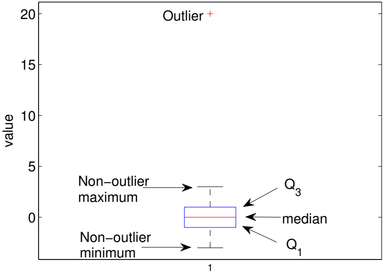


(Tukey, 1977)

Boxplot

$$\mathbf{d} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 20 \\ 2 \\ 3 \\ 1 \\ 0 \\ -3 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{d}^s = \begin{pmatrix} -3 \\ -2 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 3 \\ 20 \end{pmatrix}$$

- 25th and 75th percentiles: $Q_1 = -1, \quad Q_3 = 1$
- Lower bound: $Q_1 - 3(Q_3 - Q_1) = -7$
- Upper bound: $Q_3 + 3(Q_3 - Q_1) = 7$



Whisker length

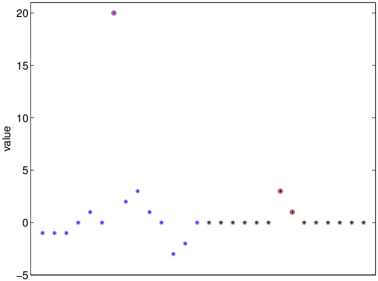
$$d_j < Q_1 - W \cdot (Q_3 - Q_1) \text{ or } d_j > Q_3 + W \cdot (Q_3 - Q_1)$$

Whisker length 3:

- Coverage of 99.9998%
- Normally distributed: 0.0002% detected asymptotically
- Few false positives if data well behaved
- Continuous function: no elements are detected!

(Hoaglin et al., J. Amer. Statist. Assoc. (1986))

Local information



- Divide global vector in local vectors
- Apply boxplot approach for each local vector
- Ignore 'outliers' near split boundaries

TU Delft

Outline

1 Building blocks: DG and multiwavelets

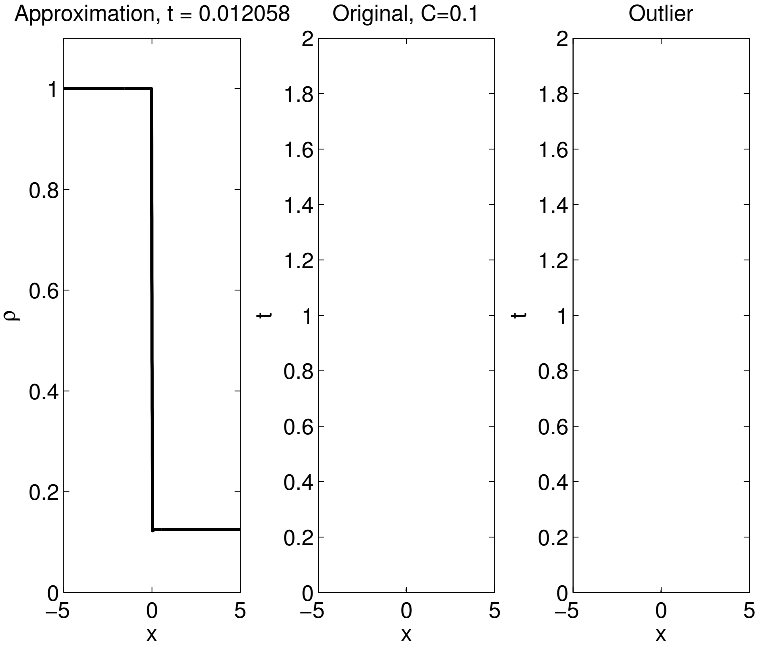
2 Multiwavelet troubled-cell indicator (with parameter)

3 Outlier detection

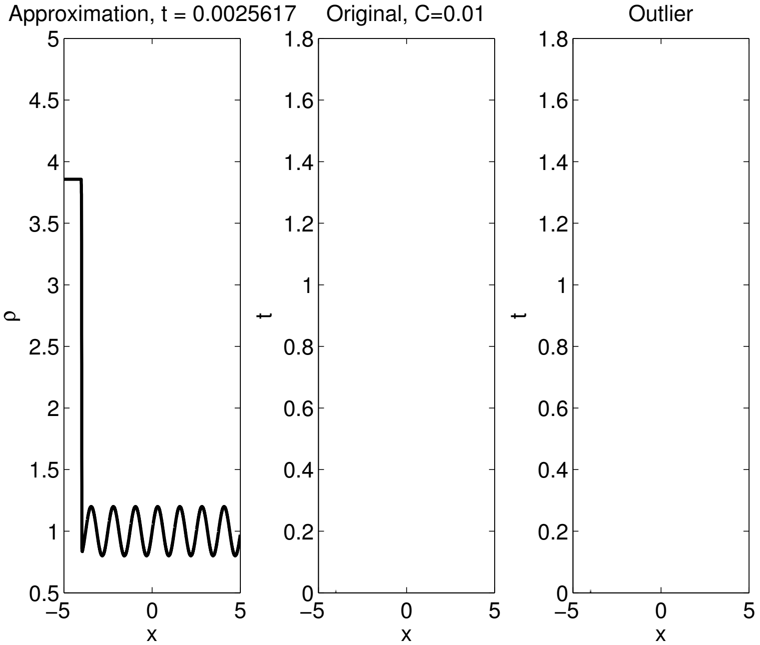
4 Results

Applications

- Apply original indicator with optimal parameter C
- Compare with outlier-detected results (no parameter)
- Euler equations: Sod, sine-entropy wave



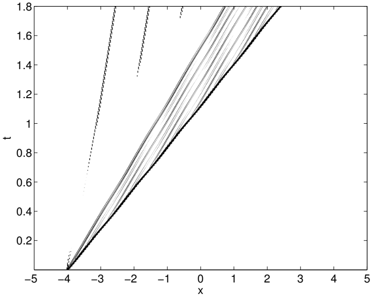
Approximation, t = 0.012058
Original, C=0.1
Outlier
ρ
t
t
x
x
x


Approximation, t = 0.0025617
Original, C=0.01
Outlier
ρ
t
t
x
x
x
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−5
0
5

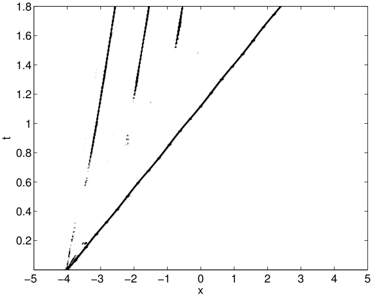
Minmod-based TVB indicator

$$u^-_{j+\frac{1}{2}} = \bar{u}_j + \tilde{u}_j, \quad \tilde{u}_j = \sum_{\ell=1}^{k} u_j^{(\ell)} \phi_\ell(1)$$

$$u^+_{j-\frac{1}{2}} = \bar{u}_j - \tilde{\tilde{u}}_j, \quad \tilde{\tilde{u}}_j = -\sum_{\ell=1}^{k} u_j^{(\ell)} \phi_\ell(-1)$$



$M = 100$

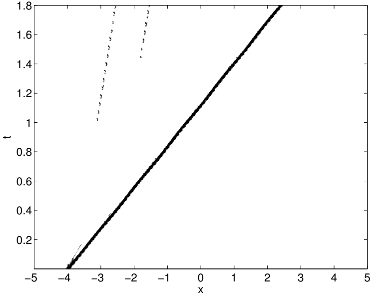


Outlier

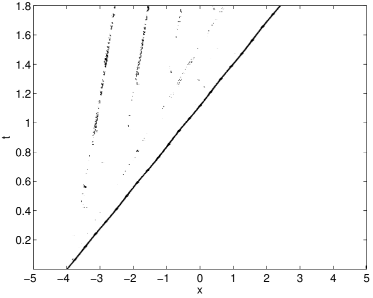
KXRCF detector

Jump across inflow edge:

$$\mathcal{I}_j = \left| \int_{\partial I_j^-} (u_h|_{I_j} - u_h|_{I_{n_j}}) ds \right|$$



Threshold equal to 1



Outlier

TU Delft

Conclusion and future research

- Original troubled-cell indicator: problem-dependent parameter
- Outlier-detection technique using boxplots
- Local-vector approach
- Parameters no longer needed!

- General meshes