Multiwavelet troubled-cell indicator for discontinuity detection of discontinuous Galerkin schemes

Thea Vuik
Delft University of Technology

Collaboration with Jennifer Ryan, University of East Anglia

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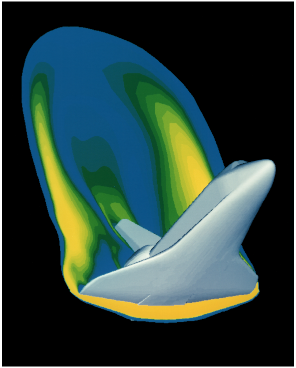
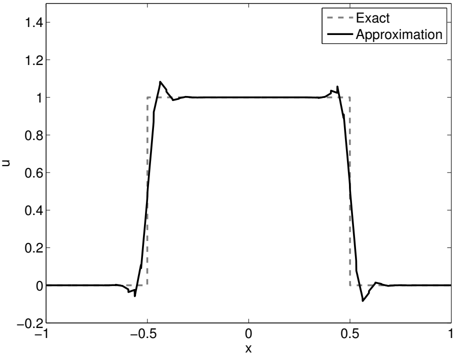
Motivation



Flow around Space Shuttle



Solution linear advection equation

Outline

Outline

1. Discontinuous Galerkin
2. Limiters and troubled-cell indicators
3. Multiwavelets
4. Multiwavelet troubled-cell indicator
5. Numerical examples (1d Euler equations)
6. Numerical example (2d Euler equations)
7. Conclusion

Discontinuous Galerkin

Hyperbolic partial differential equation:

$$u_t + f(u)_x = 0; \quad x \in [-1, 1],\ t \geq 0.$$

- DG approximation: for $x \in I_j$, write,

$$u_h(x) = \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j), \quad \xi_j = \frac{2}{\Delta x}(x - x_j)$$

- approximation space: orthonormal Legendre polynomials

$$\int_{-1}^{1} \phi_\ell(x)\phi_m(x)dx = \delta_{\ell m}$$

- k: highest polynomial degree of the approximation

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Limiters

Limiter:

- Helps to control spurious oscillations
- Reduces polynomial order in nonsmooth regions
- May flatten local extrema (diffusive property)

Troubled-cell indicator:

- Helps to limit at discontinuities only

Troubled-cell indicators

Examples of troubled-cell indicators for DG:

- minmod-based TVB limiter
  (Cockburn and Shu, Math. Comput. 1989)
- KXRCF indicator
  (Krivodonova et al., Appl. Numer. Math. 2004)
- Harten's subcell resolution
  (Qiu and Shu, SIAM J. Sci. Comput. 2005)

These indicators use local information (neighboring cells)
Multiwavelet approach: global and local information

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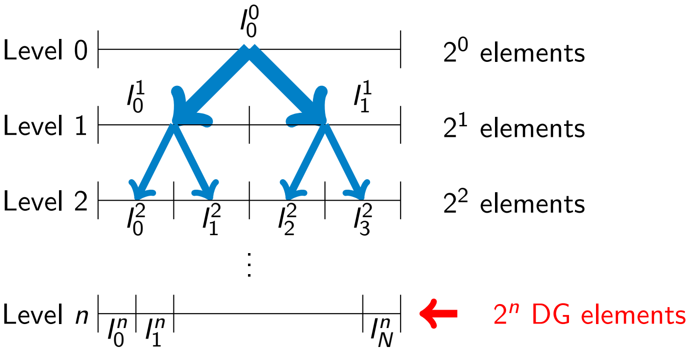
Multiresolution idea

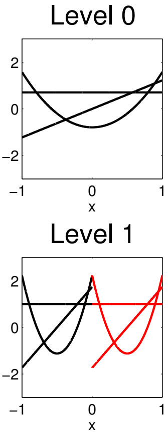


$$V_n^k = \{f : f \in \mathbb{P}^k(I_j^n), j = 0, \ldots, 2^n - 1\}$$
$$I_j^n = (-1 + 2^{-n+1} j, -1 + 2^{-n+1}(j+1)]$$

$V_0^k \subset V_1^k \subset \cdots \subset V_n^k \subset \cdots$

(Alpert, SIAM J. Math. Anal. 1993)

Scaling functions and DG basis



DG basis functions:

- Orthonormal Legendre polynomials
- Basis for V_0^k : scaling function basis
- Basis functions for V_n^k: dilation and translation

$$\phi_{\ell j}^n(x) = 2^{n/2}\phi_\ell(2^n(x+1) - 2j - 1),$$

$$\ell = 0, \dots, k,\ j = 0, \dots, 2^n - 1,\ x \in I_j^n$$

(Archibald et al., Appl. Num. Math. 2011)

Multiwavelets

$$V_m^k = \{f : f \in \mathbb{P}^k(I_j^m), j = 0, \ldots, 2^m - 1\}$$

Multiwavelet space W_m^k:

- Orthogonal complement of V_m^k in V_{m+1}^k:

$$V_m^k \oplus W_m^k = V_{m+1}^k, \quad W_m^k \perp V_m^k, \quad W_m^k \subset V_{m+1}^k$$

- V_n^k can be split into $n+1$ orthogonal subspaces:

$$V_n^k = V_0^k \oplus W_0^k \oplus W_1^k \oplus \cdots \oplus W_{n-1}^k$$

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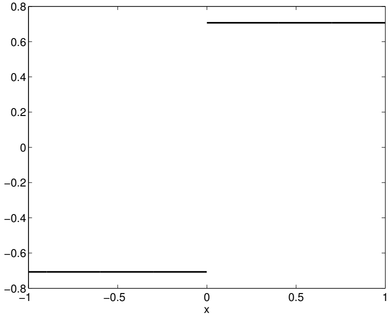
Example: Haar wavelet

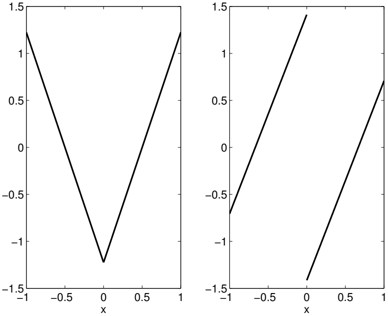
W_0^k:

- Subset of V_1^k
- Basis: piecewise polynomials on $I_0^1 = [-1, 0]$ and $I_1^1 = [0, 1]$
- $k = 0$: Haar wavelets



Haar wavelets, level 0

Multiwavelets



Multiwavelet basis, $k = 1$

Formulae for $x \in (0,1)$:

$$\psi_0(x) = \sqrt{\frac{3}{2}}(-1+2x), \text{ even in } 0$$

$$\psi_1(x) = \sqrt{\frac{1}{2}}(-2+3x), \text{ odd in } 0$$

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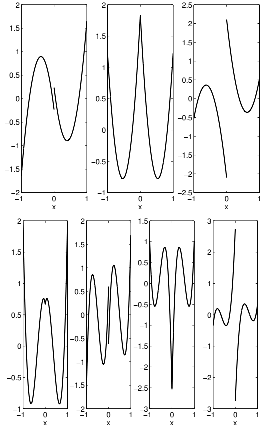
Multiwavelets



Multiwavelets, $k = 2$ (top) and $k = 3$ (bottom)

$$\psi_0(x) = \frac{1}{3}\sqrt{\frac{1}{2}}(1 - 24x + 30x^2)$$

$$\psi_1(x) = \frac{1}{2}\sqrt{\frac{3}{2}}(3 - 16x + 15x^2)$$

$$\psi_2(x) = \frac{1}{3}\sqrt{\frac{5}{2}}(4 - 15x + 12x^2)$$

$$\psi_0(x) = \sqrt{\frac{15}{34}}(1 + 4x - 30x^2 + 28x^3)$$

$$\psi_1(x) = \sqrt{\frac{1}{42}}(-4 + 105x - 300x^2 + 210x^3)$$

$$\psi_2(x) = \frac{1}{2}\sqrt{\frac{35}{34}}(-5 + 48x - 105x^2 + 64x^3)$$

$$\psi_3(x) = \frac{1}{2}\sqrt{\frac{5}{42}}(-16 + 105x - 192x^2 + 105x^3)$$

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Multiwavelets and DG

$$V_n^k = V_0^k \oplus W_0^k \oplus W_1^k \oplus \cdots \oplus W_{n-1}^k$$

Relation between DG and multiwavelets (2^n elements):

$$\begin{aligned} u_h(x) &= \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) \\ &= S^0(x) + D^0(x) + D^1(x) + \ldots + D^{n-1}(x) \end{aligned}$$

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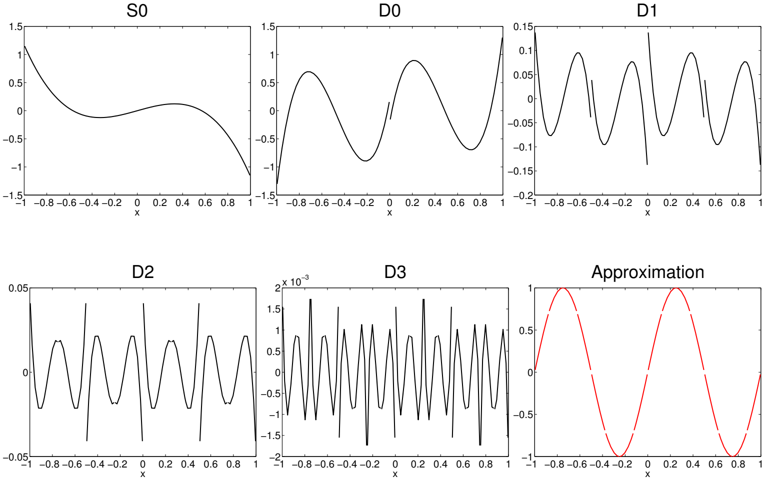
Continuous example: $\sin(2\pi x), n = 4, k = 3$

Projection on DG basis, multiwavelet decomposition



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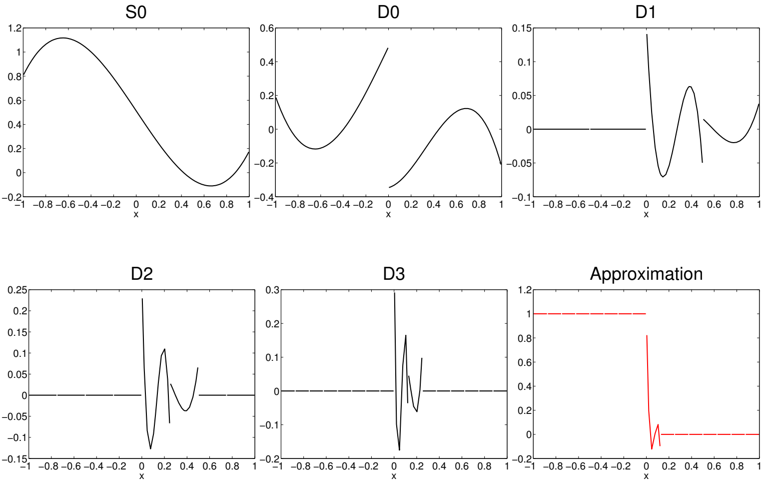
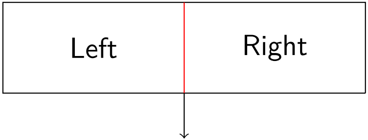
Square wave, $n = 4, k = 3$

Projection on DG basis, multiwavelet decomposition



Highest level

- D^{n-1} constructed using $\mathbf{d}^{n-1} = (d_0^{n-1} \cdots d_k^{n-1})^\top$
- Jump between cells: $\Delta\mathbf{u} = ([u_h]^{(0)} \cdots [u_h]^{(k)})^\top$



$$\mathbf{d}^{n-1}, \Delta\mathbf{u}$$

$$\mathbf{d}^{n-1} = A\Delta\mathbf{u},$$

where

$$A(\ell+1, r+1) = 2^{-\frac{n-1}{2}} \frac{2^{(-n+1)r}}{r!} \int_0^1 x^r \psi_\ell(x) dx.$$

Highest level

This means that D^{n-1}:

- Measures jumps in approximation (derivatives) at element boundaries;
- Can be used for detection of discontinuities (in derivatives).

Outline

Multiwavelet troubled-cell indicator

- Troubled cells: focus on highest level $D^{n-1}(x)$
- Compute absolute average $\bar{D}_j^{n-1}$ on element I_j
- Element I_j is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max\left\{\bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1\right\}, C \in [0, 1]$$

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Choice of C

I_j is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max\left\{\bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1\right\}, C \in [0, 1]$$

Parameter C: defines strictness of indicator,

- $C = 0$: every element is detected
- $C = 0.2$: select largest 80% of averages
- $C = 0.8$: select largest 20% of averages

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Multiwavelet troubled-cell indicator

Applications: Euler equations

- Local detector: shock in different locations
  (Zaide and Roe, 20th AIAA CFD Conf. 2011)

  Our indicator: combines local and global nature

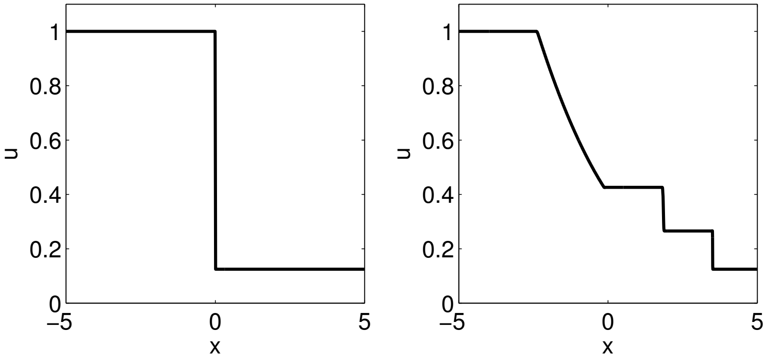
- Limiter: mechanism to control limited regions
  Now: troubled-cell indicator as switch

- Moment limiter (Krivodonova, J. Comput. Phys. 2007)

  Only a choice, other limiters possible

Outline

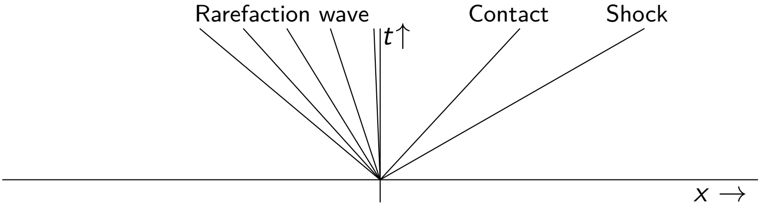
Sod's shock tube (J. Comput. Phys. 1978)



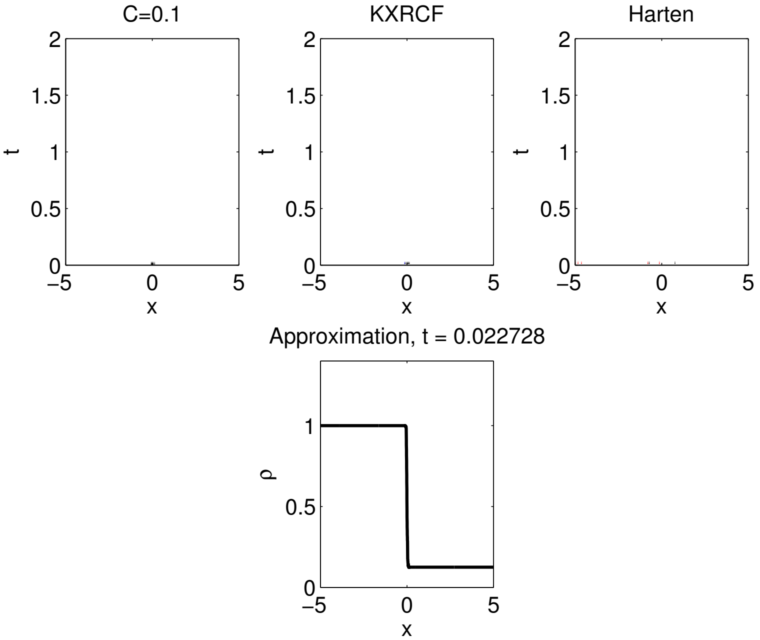
Density in Sod's shock tube at $T = 0$ (left) and $T = 2$ (right)

Sod: time history

Results: focus on detected troubled cells



Time history of troubled cells


C=0.1
KXRCF
Harten
Approximation, t = 0.022728
t
x
ρ

Sod: percentages

Percentages of detected troubled cells, 256 elements:

	$k=2$		$k=3$	
	Ave	Max	Ave	Max
$C=0.1$	3.1507	7.0312	**2.5654**	6.6406
KXRCF	**1.9686**	3.5156	3.9173	6.2500
Harten	1.9328	4.6875	7.1476	14.8438

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Sine entropy wave

Sine entropy wave:

$$\rho(x,0) = \begin{cases} 3.857142, & x < -4, \\ 1 + 0.2\sin(5x), & x \geq -4. \end{cases}$$

(Shu and Osher, J. Comput. Phys. 1989)

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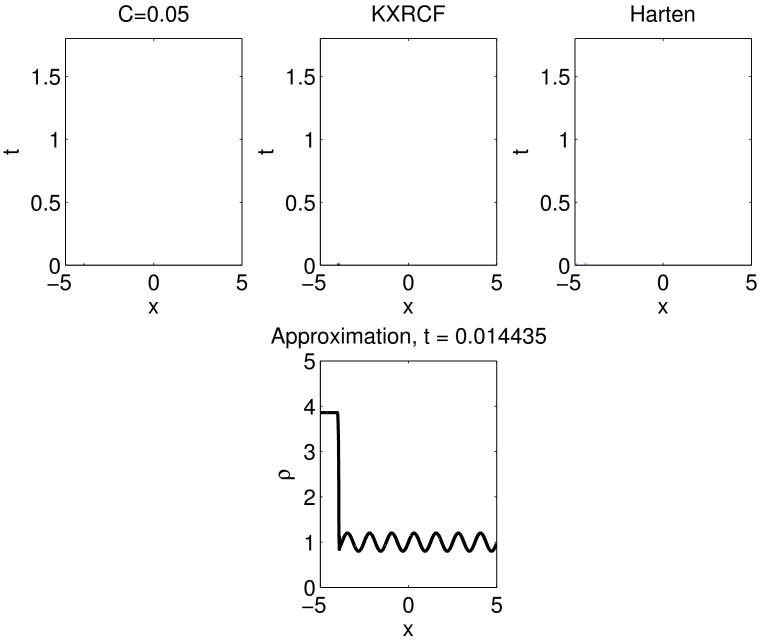

C=0.05
KXRCF
Harten
t
x
Approximation, t = 0.014435
ρ
x

Sine: percentages

Percentages of detected troubled cells, 512 elements:

	$k = 2$		$k = 3$	
	Ave	Max	Ave	Max
$C = 0.05$	1.2584	8.7891	**1.0360**	3.3203
KXRCF	**1.2059**	2.1484	2.9361	6.2500
Harten	2.4105	6.2500	5.3323	13.8672

Outline

Two-dimensional approach

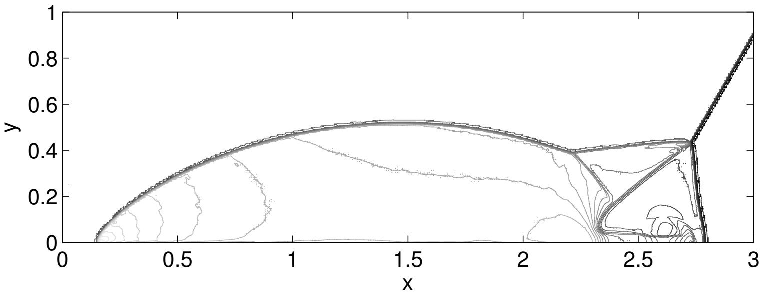
In two-dimensions, the multiwavelet expansion is:

$$S^0(x,y) + \sum_{m_x=0}^{n_x-1} \sum_{m_y=0}^{n_y-1} \left\{ D^{\alpha,\mathbf{m}}(x,y) + D^{\beta,\mathbf{m}}(x,y) + D^{\gamma,\mathbf{m}}(x,y) \right\}$$

number of elements: $2^{n_x} \times 2^{n_y}$

- α mode: multiwavelets in y-direction
- β mode: multiwavelets in x-direction
- γ mode: multiwavelets both x- and y-direction

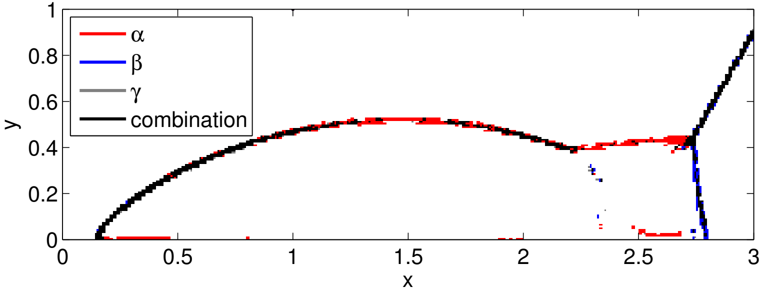
Double Mach reflection



Density contours using $C = 0.05$
$T = 0.2,\ \Delta x = \Delta y = \frac{1}{128},\ k = 3$

(Woodward and Colella, J. Comput. Phys. 1984)

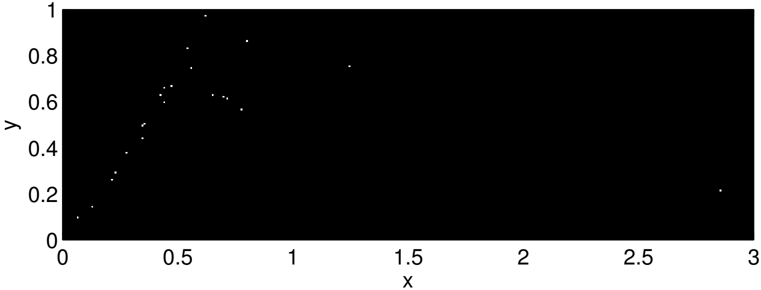
Detected troubled cells, $C = 0.05$



Detected troubled cells at $T = 0.2$, $C = 0.05$

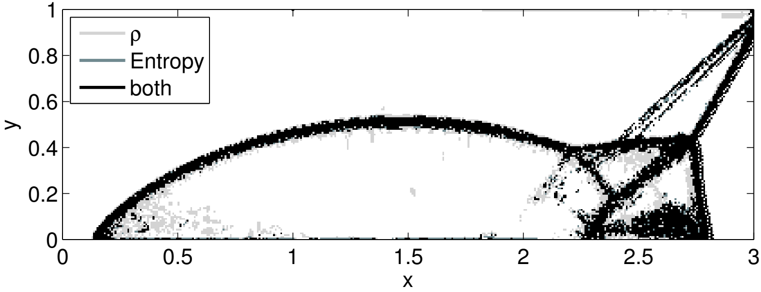
Different troubled cells are detected by modes

Detected troubled cells, moment limiter



Use the moment limiter's own switch, $T = 0.2$

Detected troubled cells, KXRCF indicator



Use the KXRCF indicator, $T = 0.2$

Computation time

Compare computation time, double Mach reflection:

- More accurate result: don't limit continuous regions
- Decrease of computation time

Computation time

k	limit everywhere	$C = 0.05$	KXRCF
1	57 min	50 min	85 min
2	490 min	214 min	335 min
3	28 hours	13 hours	36 hours

$T = 0.2$, 512×128 elements

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Outline

Conclusion

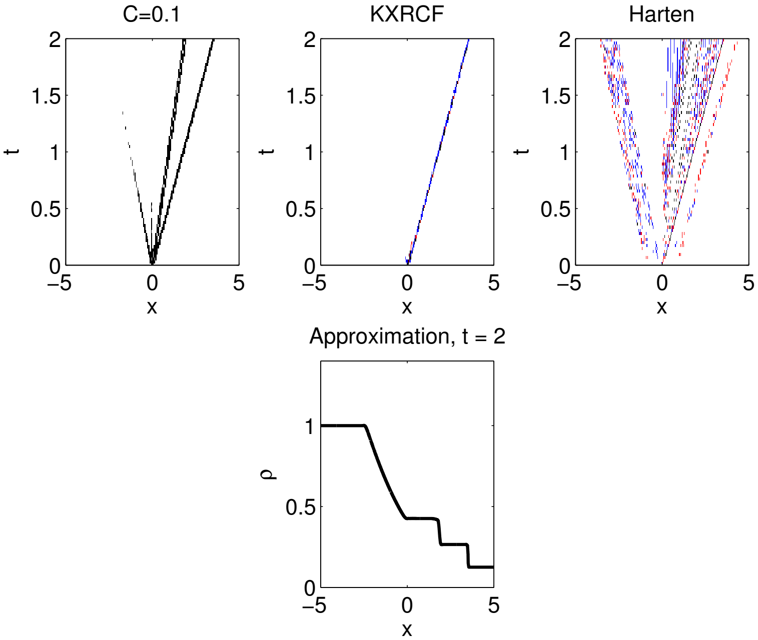
- Troubled-cell indicator is switch in limiter
- Multiwavelet decomposition: D^{n-1} detects discontinuity
- Parameter C defines strictness of detector
- Accurately detects troubled cells
- Two-dimensional detection in different modes
- Decrease of computation time

More details in JCP(270), pp 138-160

Future work:

→ How to choose parameter C

→ Applying to unstructured meshes


C=0.1
KXRCF
Harten
t
x
Approximation, t = 2
ρ
x