Multiwavelets and outlier detection for troubled-cell indication

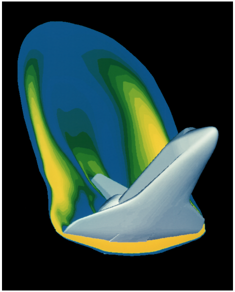
Thea Vuik
Delft University of Technology

Collaboration with Jennifer Ryan, University of East Anglia

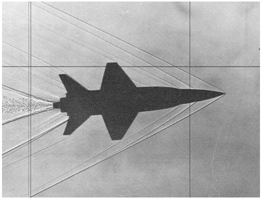
ICOSAHOM 2016
June 28, 2016

TU Delft

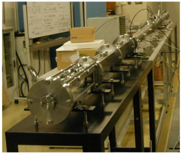
Motivation



Flow around Space Shuttle



Shock wave

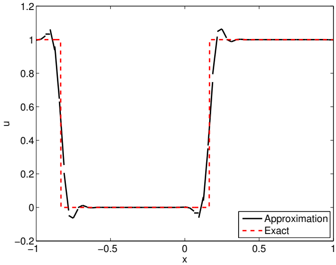


Shock tube

http://www.wikiwand.com
http://projectthunderstruck.org
http://akagi.nuae.nagoya-u.ac.jp

Motivation: hyperbolic PDEs

Discontinuities $\rightarrow$ spurious oscillations!



Remove wiggles by:

- Limiting
- Filtering
- Adding artificial viscosity

Motivation

Limiters:

- Advantage: approximation no longer oscillatory
- Disadvantage: limits smooth extrema, too diffusive

Troubled-cell indicator: detects discontinuous regions

Multiwavelet troubled-cell indicator & outlier detection

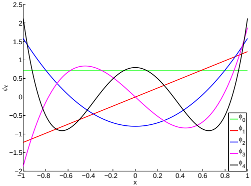
Outline

1. Discontinuous Galerkin method & multiresolution analysis
2. Relation multiwavelets and DG
3. Multiwavelet troubled-cell indicator
4. Outlier detection for parameter choice
5. Results

Outline

Discontinuous Galerkin method

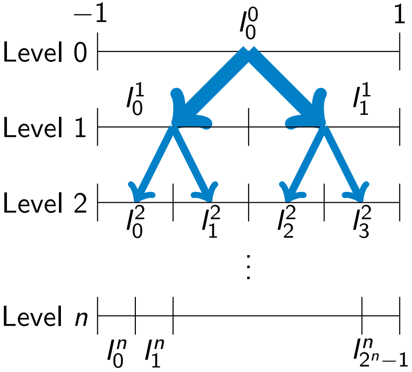
- Discretize in space: 2^n elements
- Approximation space: $V_h = \{f \in \mathbb{P}^k(I_j), j = 0, \ldots, 2^n - 1\}$
- Basis of scaled Legendre polynomials $\{\phi_0, \phi_1, \ldots, \phi_k\}$



- Local Lax-Friedrichs flux
- Time integration: 3rd-order SSPRK method

TUDelft

Multiresolution idea



$$V_0^{k+1} = \{f : f \in \mathbb{P}^k(I_0^0)\}$$

$$V_1^{k+1} = \{f : f \in \mathbb{P}^k(I_j^1)\}$$

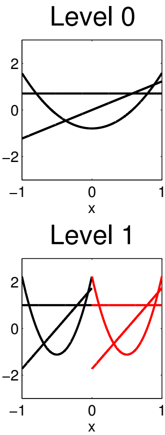
$$V_2^{k+1} = \{f : f \in \mathbb{P}^k(I_j^2)\}$$

$$\vdots$$

$$V_n^{k+1} = \{f : f \in \mathbb{P}^k(I_j^n)\}$$

(Alpert, SIAM J. Math. Anal. 1993)

Scaling functions



$k = 2$

- Basis multiresolution space: start from level 0
- Orthonormal Legendre polynomials $\phi_0, \ldots, \phi_k$
- Basis functions higher levels: dilation and translation

(Archibald et al., Appl. Num. Math. 2011)

Multiwavelets

Multiwavelet space W_m^{k+1}:
Orthogonal complement of V_m^{k+1} in V_{m+1}^{k+1}:

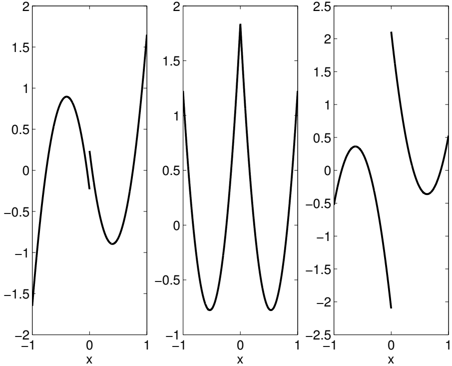
$$V_m^{k+1} \oplus W_m^{k+1} = V_{m+1}^{k+1}, \quad W_m^{k+1} \perp V_m^{k+1}, \quad W_m^{k+1} \subset V_{m+1}^{k+1}$$

V_n^{k+1} can be split into $n+1$ orthogonal subspaces:

$$\begin{aligned} V_n^{k+1} &= V_{n-1}^{k+1} \oplus W_{n-1}^{k+1} = V_{n-2}^{k+1} \oplus W_{n-2}^{k+1} \oplus W_{n-1}^{k+1} \\ &= V_0^{k+1} \oplus W_0^{k+1} \oplus W_1^{k+1} \oplus \cdots \oplus W_{n-1}^{k+1} \end{aligned}$$

(Alpert, SIAM J. Math. Anal. 1993)

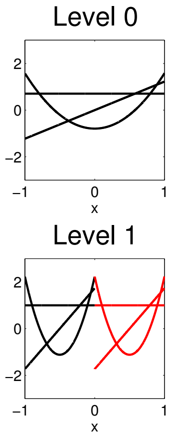
Multiwavelets, $k = 2$



$$\psi_0(x) = \frac{1}{3}\sqrt{\frac{1}{2}}(1 - 24x + 30x^2)$$

$$\psi_1(x) = \frac{1}{2}\sqrt{\frac{3}{2}}(3 - 16x + 15x^2)$$

$$\psi_2(x) = \frac{1}{3}\sqrt{\frac{5}{2}}(4 - 15x + 12x^2)$$



Scaling functions, $k = 2$

TU Delft

Multiwavelets on higher levels

Multiwavelets on higher levels: dilation and translation

$$\psi_{\ell j}^{m}(x) = \sqrt{\frac{2}{\Delta x^m}}\psi_\ell\left(\frac{2}{\Delta x^m}(x - x_j^m)\right),$$

Δx^m is mesh width on level m

$\ell = 0, \ldots, k, \; j = 0, \ldots, 2^m - 1$

Outline

TUDelft

Multiwavelets and DG

V_n^{k+1} (multiresolution scheme) equals V_h (DG scheme)!

$$V_h = V_n^{k+1} = V_0^{k+1} \oplus W_0^{k+1} \oplus W_1^{k+1} \oplus \cdots \oplus W_{n-1}^{k+1}$$

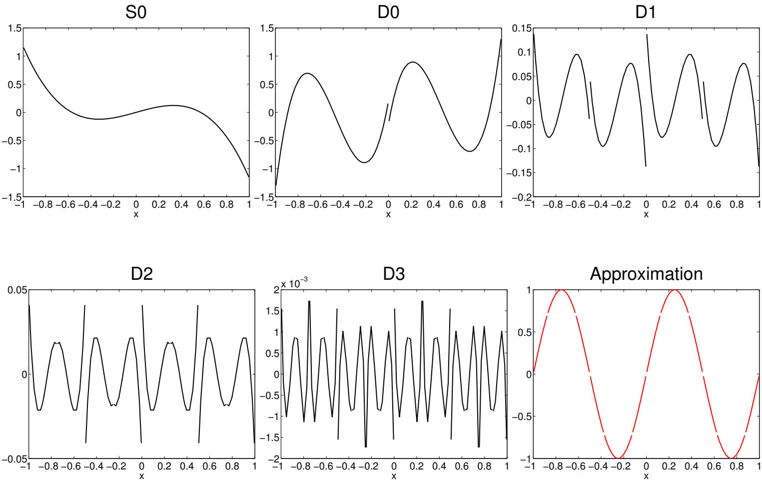
This means that:

$$\begin{aligned} u_h(x) &= \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) \\ &= \underbrace{\sum_{\ell=0}^{k} s_{\ell 0}^{0} \phi_\ell(x)}_{\substack{\text{global average} \\ \in V_0^{k+1}}} + \sum_{m=0}^{n-1} \underbrace{\sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^{m} \psi_{\ell j}^{m}(x)}_{\substack{\text{finer details} \\ \in W_m^{k+1}}} \end{aligned}$$

Coefficients efficiently computed by decomposition method

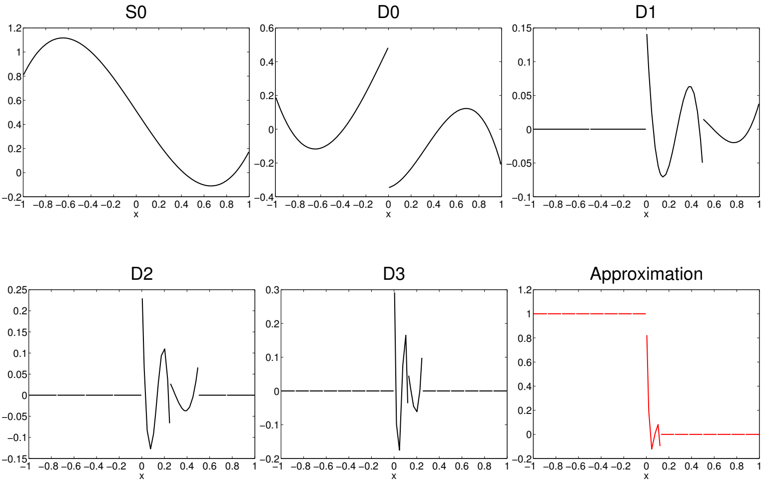
Sine, $k = 3, n = 4$

Projection on DG basis, multiwavelet decomposition



Square wave, $k = 3, n = 4$

Projection on DG basis, multiwavelet decomposition



Applications multiwavelets and DG

- Multiwavelet DG method

  (Archibald, Fann, and Shelton, APNUM 2011)

- Thresholding: cancelation property for decay rate

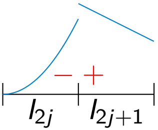
  (Gerhard and Müller, CAM 2014)

- Sparse-grid representation

  (Wang, Tang, Guo, and Cheng, JCP 2016)

Jumps in DG approximations

$$u_h(x) = \sum_{\ell=0}^{k} s_{\ell 0}^0 \phi_\ell(x) + \sum_{m=0}^{n-1} \sum_{j=0}^{2^m-1} \sum_{\ell=0}^{k} d_{\ell j}^m \psi_{\ell j}^m(x)$$

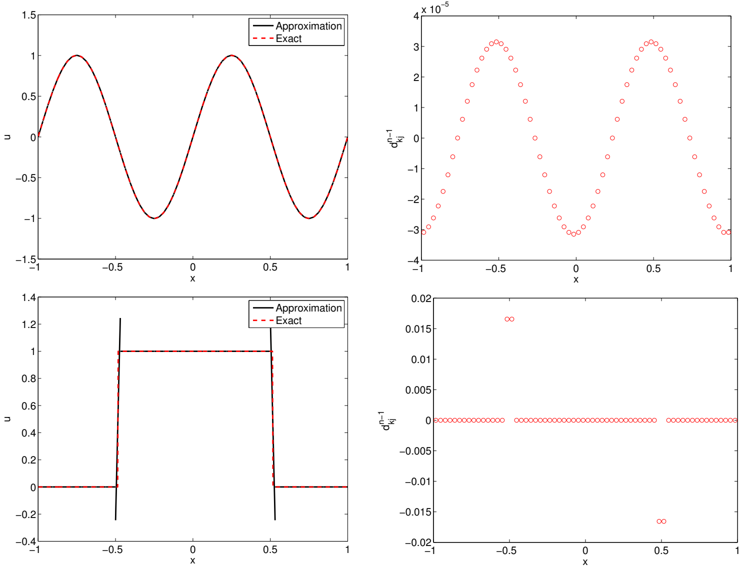


Coefficient $d_{\ell j}^{n-1}$: measures jump in (derivatives) approximation

$u_h^{(m)}$: mth derivative of u_h

$$d_{\ell j}^{n-1} = \sum_{m=0}^{k} c_{m\ell}^n \left(u_h^{(m)}(x_{2j+1/2}^+) - u_h^{(m)}(x_{2j+1/2}^-) \right)$$

(Vuik and Ryan, Proc. ICOSAHOM 2014)



Approximation

d_{kj}^{n-1}

Outline

Multiwavelet troubled-cell indicator

Detect an element as being troubled if

$$|d_{kj}^{n-1}| > C \cdot \max\{|d_{kj}^{n-1}|,\ j = 0, \dots, 2^n - 1\}$$

C prescribes strictness of indicator:

- $C = 0$: all elements are detected
- $C = 1$: no elements are detected

(Vuik and Ryan, JCP 2014)

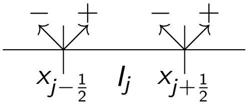
Overview troubled-cell indicators

- Multiwavelet troubled-cell indicator: d_{kj}^{n-1}
- KXRCF indicator: jump across inflow edges

$$\hat{\mathcal{I}}_j = \frac{\left|\int_{\partial I_j^-} (u_h|_{I_j} - u_h|_{I_{n_j}})ds\right|}{h^{\frac{k+1}{2}}|\partial I_j^-|||u_h|_{I_j}||}$$

- Minmod-based TVB indicator:

$$\begin{aligned} \tilde{u}_j &= \quad u^-_{j+\frac{1}{2}} - \bar{u}_j, \\ \tilde{\tilde{u}}_j &= -u^+_{j-\frac{1}{2}} + \bar{u}_j, \end{aligned}$$



where $\bar{u}_j$ is the average.

(Krivodonova et al., APNUM 2004)
(Cockburn and Shu, Math. Comp. 1989)

Parameter choice

$$|d_{kj}^{n-1}| > C \cdot \max\{|d_{kj}^{n-1}|\}, \quad \hat{\mathcal{I}}_j > 1, \quad |\tilde{u}_j|, |\tilde{\tilde{u}}_j| > M\Delta x^2$$

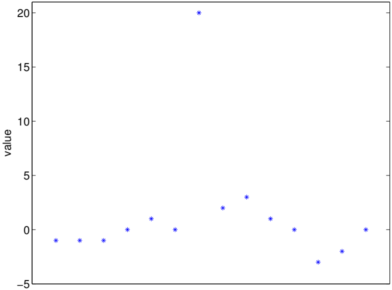
Troubled-cell indication methods rely on parameters

How should we choose the parameters?

Outline

TUDelft

Indication vector



- Troubled-cell indication vector: $\mathbf{d} = (d_0, \ldots, d_N)^\top$
- Detect sudden changes compared to neighboring values
- No problem-dependent parameters

Outlier-detection algorithm

1. Sort $\mathbf{d}$ to obtain $\mathbf{d}^s = (d_0^s, d_1^s, \ldots, d_N^s)$
2. Compute quartiles $Q_1,\ Q_2,\ Q_3$ of $\mathbf{d}$
3. Detect outliers outside interval

$$[Q_1 - 3(Q_3 - Q_1), Q_3 + 3(Q_3 - Q_1)]$$

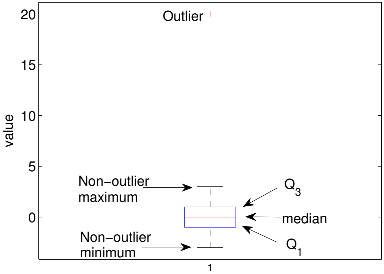
Normal distribution: 0.0002% identified as outlier

(Tukey, 1977)

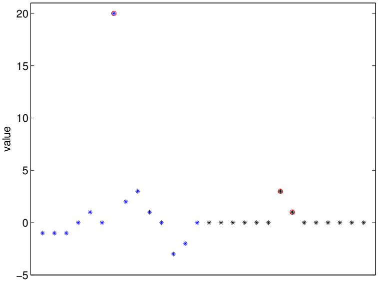
Boxplot

$$\mathbf{d} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 20 \\ 2 \\ 3 \\ 1 \\ 0 \\ -3 \\ -2 \\ 0 \end{pmatrix}, \quad \mathbf{d}^s = \begin{pmatrix} -3 \\ -2 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 2 \\ 3 \\ 20 \end{pmatrix}$$

- 25th and 75th percentiles:
  $Q_1 = -1, \quad Q_3 = 1$
- Lower bound:
  $Q_1 - 3(Q_3 - Q_1) = -7$
- Upper bound:
  $Q_3 + 3(Q_3 - Q_1) = 7$



Local information



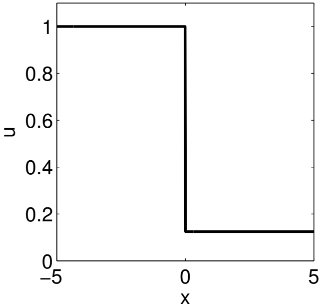
- Divide global vector in locals
- Apply boxplot approach for each local vector
- Ignore 'outliers' near split boundaries
- Local vectors: size 16

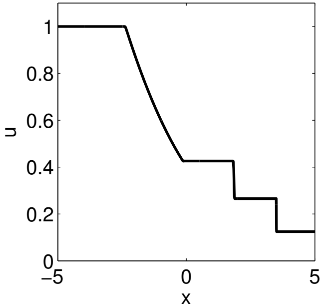
(Vuik and Ryan, SISC 2016)

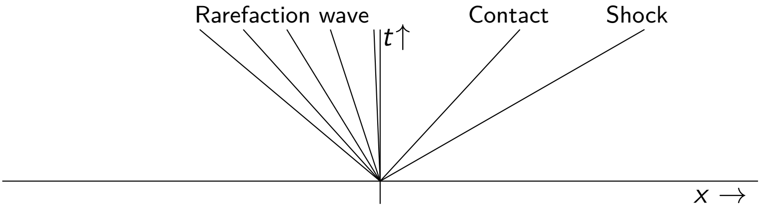
Outline

TU Delft

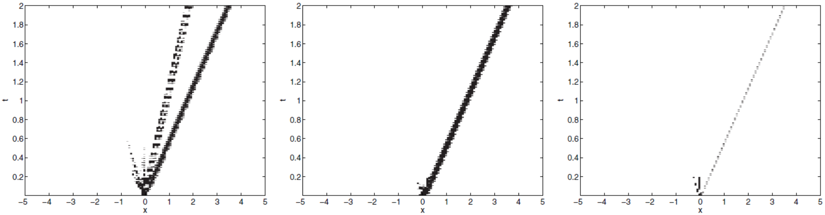
Sod's shock tube



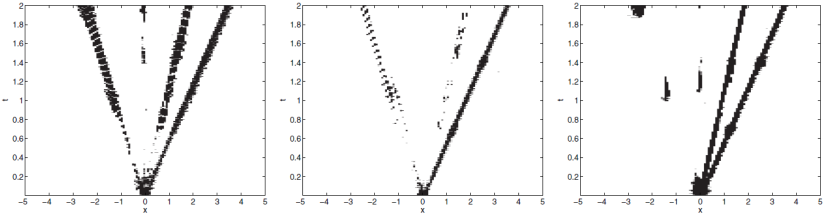




Sod, $k = 2$



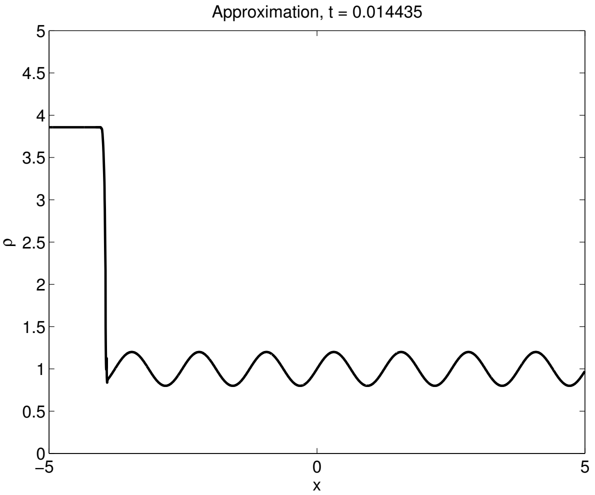
(a) Original, $C = 0.1$ (b) Original, KXRCF (c) Original, $M = 10$



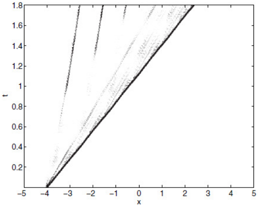
(d) Outlier, multiwavelets (e) Outlier, KXRCF value (f) Outlier, minmod-based TVB

(Vuik and Ryan, SISC 2016)

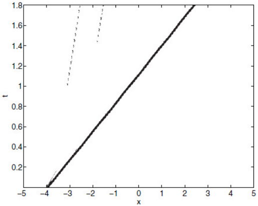
Sine-entropy wave



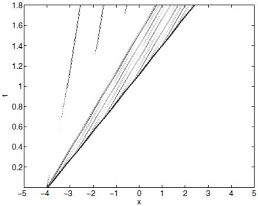
Sine-entropy wave, $k = 2$



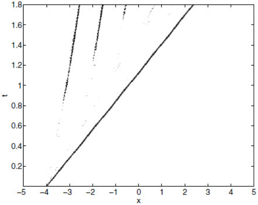
(a) Original, $C = 0.01$



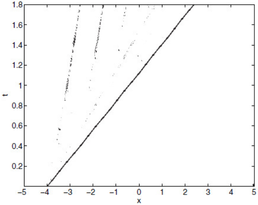
(b) Original, KXRCF



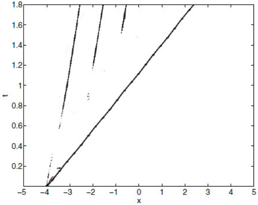
(c) Original, $M = 100$



(d) Outlier, multiwavelets



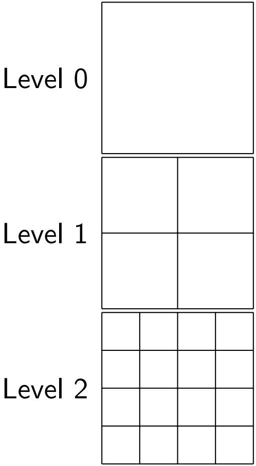
(e) Outlier, KXRCF value



(f) Outlier, minmod-TVB

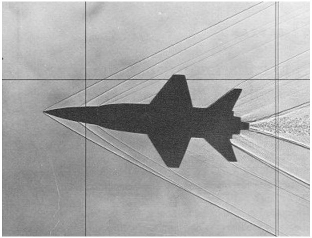
Two dimensions: rectangular mesh (tensor)

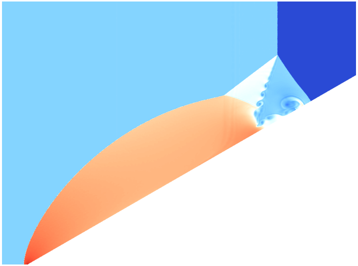
$$u_h(x,y) = S^{\mathbf{n}-1}(x,y) + D^{\alpha,\mathbf{n}-1}(x,y) + D^{\beta,\mathbf{n}-1}(x,y) + D^{\gamma,\mathbf{n}-1}(x,y)$$



- Scaling functions: $\phi_{\ell_x}(x)\phi_{\ell_y}(y)$
- Multiwavelets:
  - α mode: $\phi_{\ell_x}(x)\psi_{\ell_y}(y)$
  - β mode: $\psi_{\ell_x}(x)\phi_{\ell_y}(y)$
  - γ mode: $\psi_{\ell_x}(x)\psi_{\ell_y}(y)$

Double Mach reflection



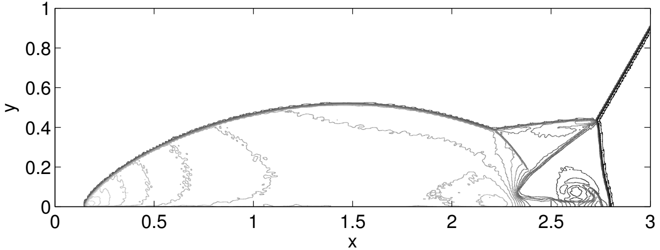


http://projectthunderstruck.org

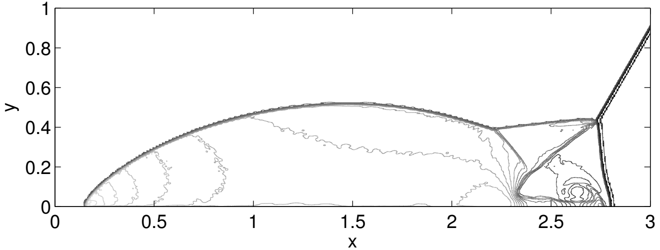
http://www.math.sciences.univ-nantes.fr

TU Delft

Double Mach reflection: contour plots



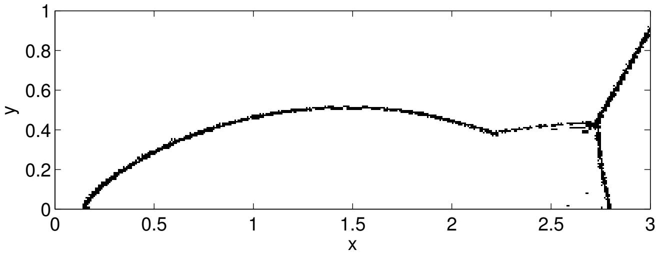
Original
$C = 0.05$



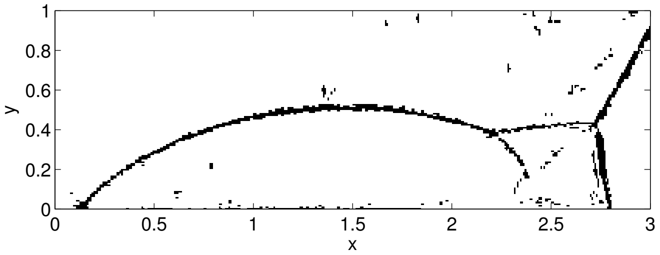
Outlier

TUDelft

Double Mach reflection: troubled cells



Original
$C = 0.05$



Outlier

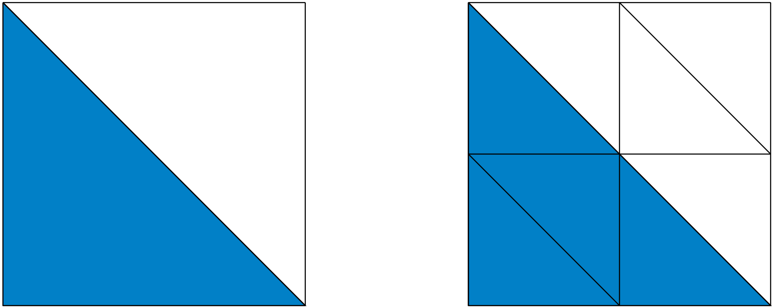
Conclusion and future research

- Exact relation DG approximation and multiwavelet coefficients
- Multiwavelet coefficients for troubled-cell indication
- Originally: problem-dependent parameter
- Outlier-detection technique using boxplots
- Problem-dependent parameters no longer needed!

(Vuik and Ryan, JCP 2014, ICOSAHOM 2014, SISC 2016)

- Non-uniform Cartesian meshes
- Triangular meshes

Two dimensions: triangular mesh



- No tensor product, but genuinely two dimensional!
- Multiwavelets: theory of Yu et al. (1997)
  - Based on Alpert's algorithm
  - Efficient coefficient computation still possible
  - Relation with DG coefficients