Limiting and shock detection for discontinuous Galerkin solutions using multiwavelets

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Outline

Outline

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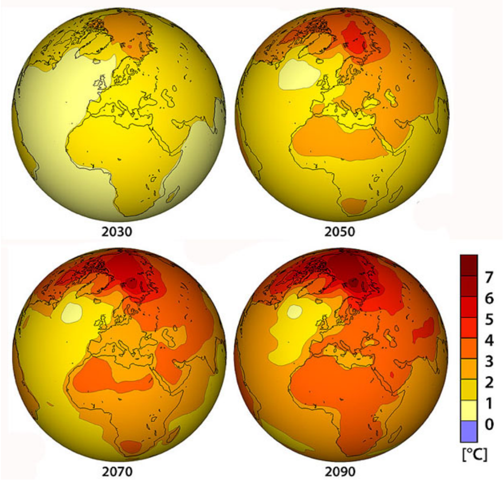
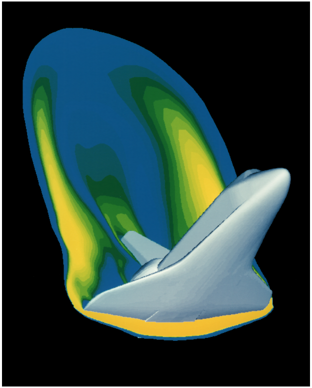
Motivatie: klimaatmodellering



Klimaatmodellering: simulatie van temperatuurverandering

Motivatie: luchtstroming



Luchtstroming rond Space Shuttle, terugkeer op aarde

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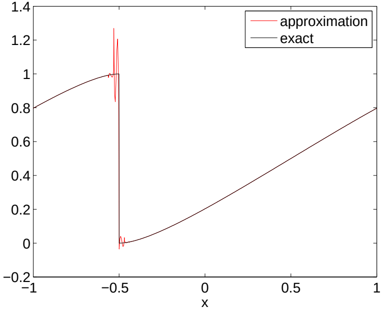
Most important result

Wanted:

Solve PDE using DG, solution contains shocks.



Bad idea:

Use multiwavelets for limiting DG solutions.

Excellent idea:

Use multiwavelets as shock detector for DG.

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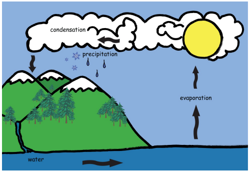
Discontinuous Galerkin, [3]

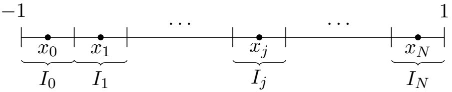
Linear advection equation on $[-1, 1]$:

$$u_t + u_x = 0, \quad x \in [-1, 1], t \geq 0,$$
$$u(x, 0) = u^0(x), \quad x \in [-1, 1],$$



u: concentration/density, periodic boundary conditions.

Exact solution: $u(x, t) = u^0(x - t)$.

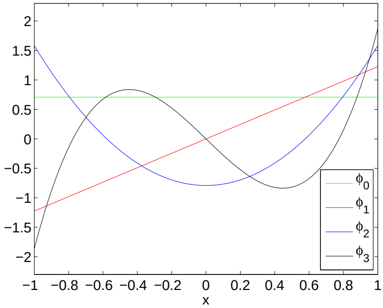


Discretize in space: $x_j = -1 + (j + \frac{1}{2})\Delta x$, $j = 0, \dots, N$.

Discontinuous Galerkin: approximations

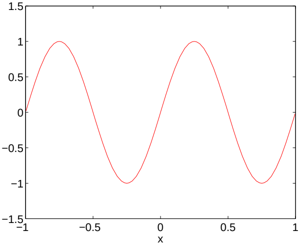
Approximate u by $u_h(x,t)$:

- linear combination of $\phi_0, \dots, \phi_k$, on each element;
- piecewise polynomial of degree k;
- choice of basis due to multiwavelets.

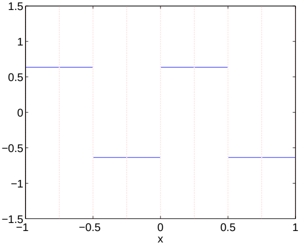


Scaled Legendre polynomials, $\phi_0, \dots, \phi_3$

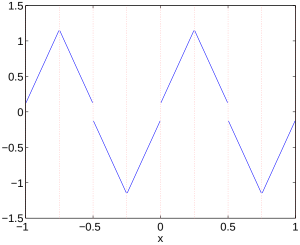
Approximation, 8 elements



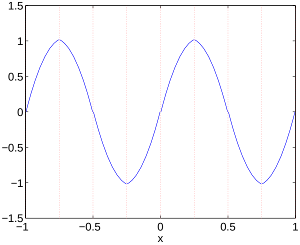
$u(x) = \sin(2\pi x)$



$k = 0$



$k = 1$



$k = 2$

Discontinuous Galerkin: weak formulation

$$u_{h,t} + u_{h,x} = 0$$

On each element I_j, $j = 0, \ldots, N$:

- Multiply $u_{h,t} + u_{h,x} = 0$ by test function, $v_h = \phi_m$, $m \in \{0, \ldots, k\}$;
- Integrate over element I_j, use partial integration.

$$\frac{\Delta x}{2} \frac{du_j^{(m)}}{dt} = \sum_{\ell=0}^{k} \alpha_{\ell m} u_{j-1}^{(\ell)} + \sum_{\ell=0}^{k} \beta_{\ell m} u_j^{(\ell)},$$

$u_j^{(m)}$: DG coefficient, element I_j, and ϕ_m.

Time stepping

$$\frac{\Delta x}{2}\frac{du_j^{(m)}}{dt} = \sum_{\ell=0}^{k} \alpha_{\ell m} u_{j-1}^{(\ell)} + \sum_{\ell=0}^{k} \beta_{\ell m} u_j^{(\ell)}$$

DG transforms PDE into ODE:

$$\frac{d}{dt}\mathbf{u}_j = L(\mathbf{u}_{j-1}, \mathbf{u}_j);$$

Time stepping: total variation diminishing RK, $\mathcal{O}((\Delta t)^3)$, [4].

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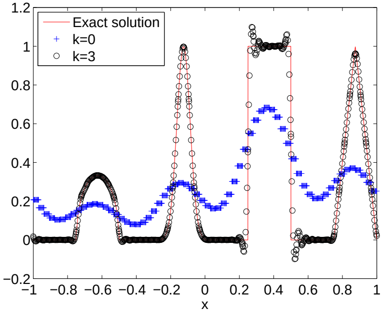
Limiters needed

Example: discontinuities in exact solution.



Approximation at $T = 0.5$, $u_t + u_x = 0$, discontinuous initial condition, $\phi_0, \ldots, \phi_k$, 64 elements

- Low order: smearing;
- High order: oscillations.

Nonlinear equations: results are worse.

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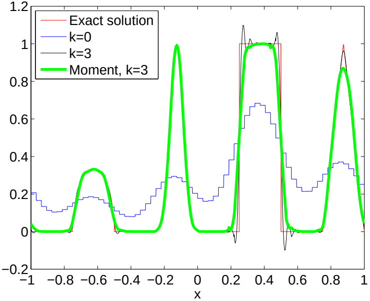
Currently used: moment limiter, [5]

Moment limiter: for every element I_j, $j = 0, \ldots, N$,

- Start at $u_j^{(k)}$: limit this coefficient;
- Limit $u_j^{(k-1)}, \ldots, u_j^{(1)}$, if solution not smooth enough.



Approximation, $u_t + u_x = 0$, $\phi_0, \ldots, \phi_k$, $T = 0.5$, 64 elements

- Problems with multidimensions or complex geometries.

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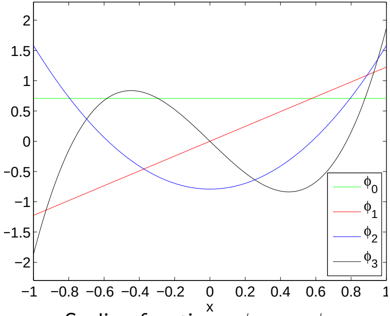
Combine DG with multiwavelets, [1, 2]

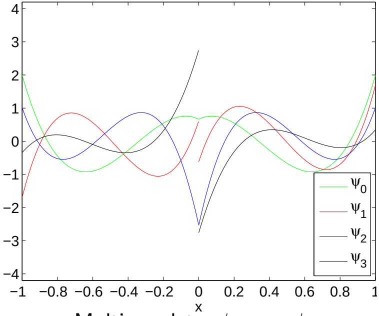
Number of elements equals 2^n;

Write $u_h(x,t) = \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi) = \underbrace{S^0(x,t)}_{\text{average}} + \sum_{m=0}^{n-1} \underbrace{D^m(x,t)}_{\text{finer details}}$,

- $S^0(x,t)$: based on $\phi_0, \ldots, \phi_3$: polynomial on $[-1,1]$;
- $D^0(x,t)$: use $\psi_0, \ldots, \psi_3$: polynomial on $[-1,0]$, $[0,1]$;

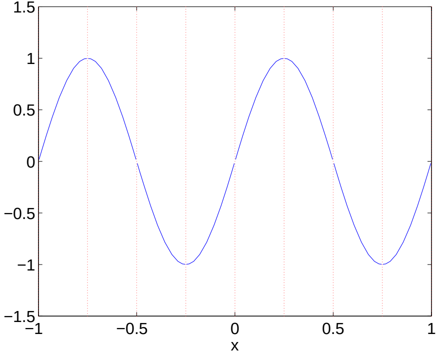


Scaling functions, $\phi_0, \ldots, \phi_3$



Multiwavelets, $\psi_0, \ldots, \psi_3$

Example: Multiwavelets and DG



$u_h(x,0)$: DG approximation of $u(x,0) = \sin(2\pi x)$, $n = 3$ ($2^n = 8$ elements), $k = 3$ (polynomial of degree 3 on each element)

Multiwavelet decomposition:

$$u_h(x) = S^0(x) + \sum_{m=0}^{n-1} D^m(x) = S^0(x) + \sum_{m=0}^{2} D^m(x).$$

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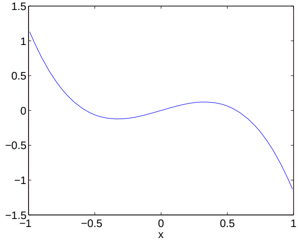
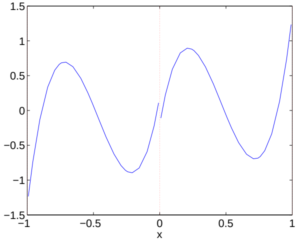
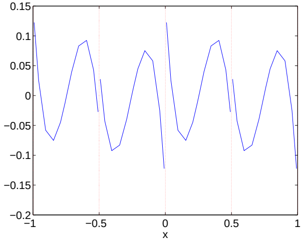
Example: multiwavelet decomposition



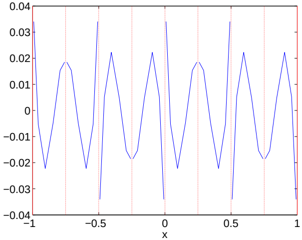
$S^0(x)$: 1 element



$D^0(x)$: 2 elements



$D^1(x)$: 4 elements



$D^2(x)$: $2^3 = 8$ elements

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Shock detection using multiwavelets

Next slides: multiwavelet decomposition on $2^6 = 64$ elements,

$$u_h(x,t) = S^0(x,t) + \sum_{m=0}^{5} D^m(x,t), \;\; n = 6, k = 3.$$

Focus: $D^m(x,t)$ for high levels m.

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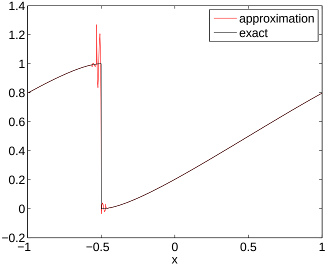
Inviscid Burgers' equation: approximation

Inviscid Burgers' equation:

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad x \in [-1, 1], t \geq 0;$$

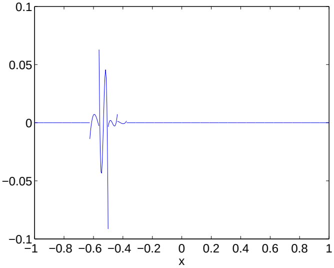
$$u^0(x) = \frac{1}{2} + \frac{1}{2}\sin(\pi x), \quad x \in [-1, 1].$$



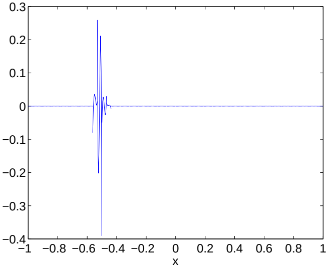
Approximation at $T = 1$, $k = 3$, 64 elements

Inviscid Burgers' equation: multiwavelets

Multiwavelet decomposition of approximation detects shock!



$D^4(x), 2^5 = 32$ elements

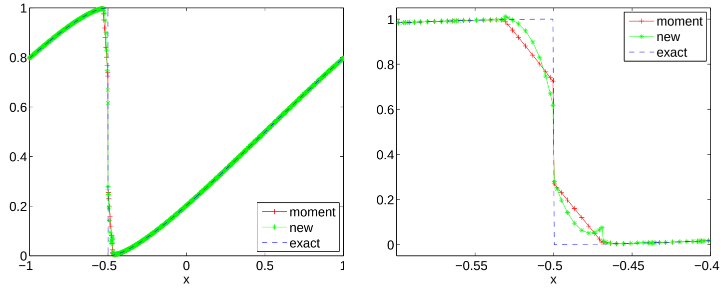


$D^5(x), 2^6 = 64$ elements

Multiwavelet contribution of approximation at $T = 1,\ n = 6,\ k = 3$

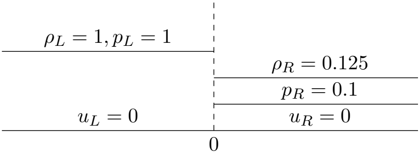
Inviscid Burgers' equation: limiter results

Moment limiter if absolute average $\bar{D}^5$ is maximal:



Approximation of solution, 64 elements, $k = 3$, $T = 1$

Sod's shock tube, [6]: equations



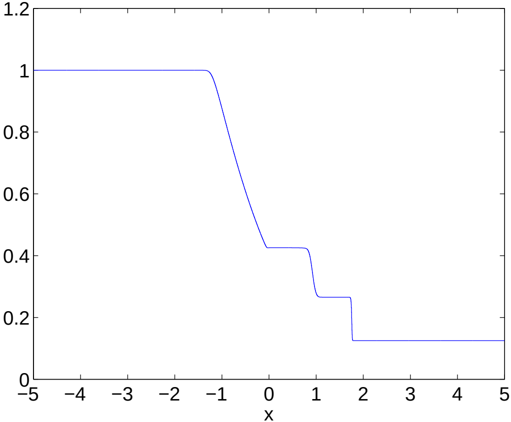
Euler equations:

$$\rho_t + (\rho u)_x = 0;$$
$$(\rho u)_t + (\rho u^2 + p)_x = 0;$$
$$E_t + ((E + p)u)_x = 0,$$

where,

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2.$$

Sod's shock tube: expected results



Exact solution for density, ρ, at $T = 1$

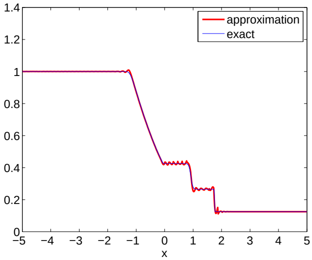
Important domains:

$[-1, 0]$: Rarefaction wave;

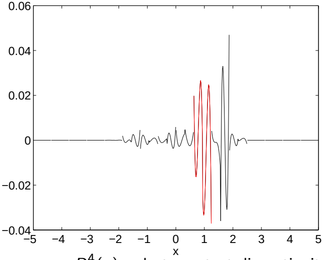
$x = 0.9$: Contact discontinuity;

$x = 1.75$: Shock.

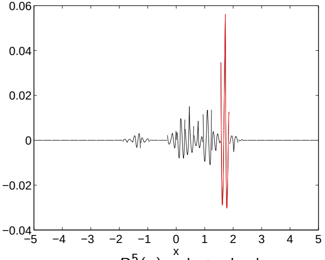
Sod's shock tube: results at $T = 1$ for density



ρ_h, $n = 6$, $k = 3$



$D^4(x)$, selects contact discontinuity



$D^5(x)$, selects shock

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Further research I: decoupled system

Different approaches Sod's shock tube:

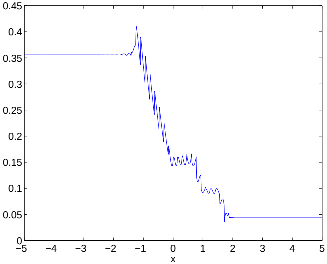
- Report: conserved variables (density, momentum, energy);
- Now: decouple Euler equations using characteristic speeds:

$$\lambda_1 = u - c, \ \lambda_2 = u, \ \lambda_3 = u + c,$$

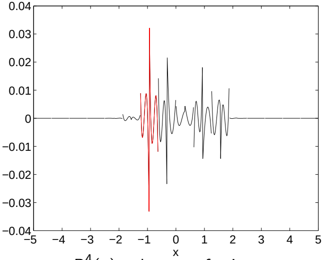
$$u: \text{ velocity of fluid}, \ c: \text{ sound speed};$$

- Next slides: multiwavelet decomposition, belonging to $\lambda_i, i = 1, 2, 3$.

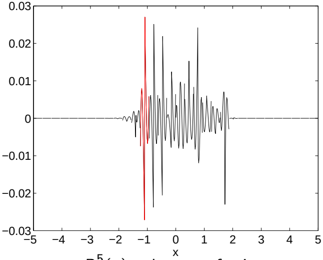
Results, speed $\lambda_1 = u - c$, at $T = 1$



Approximation, travels with speed λ_1

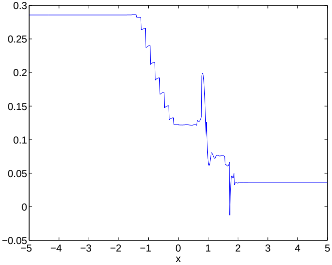


$D^4(x)$, selects rarefaction wave

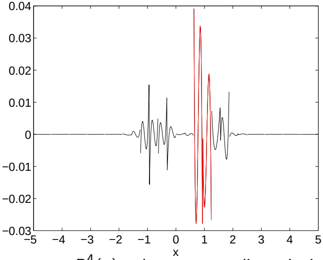


$D^5(x)$, selects rarefaction wave

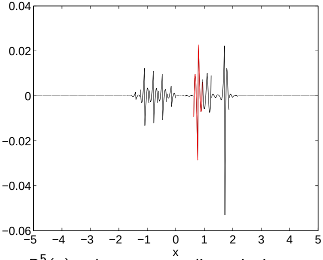
Results, speed $\lambda_2 = u$, at $T = 1$



Approximation, travels with speed λ_2



$D^4(x)$, selects contact discontinuity



$D^5(x)$, selects contact discontinuity

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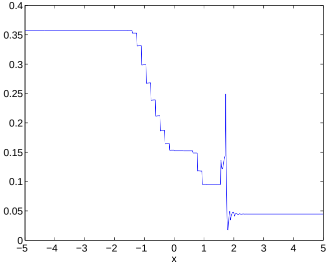
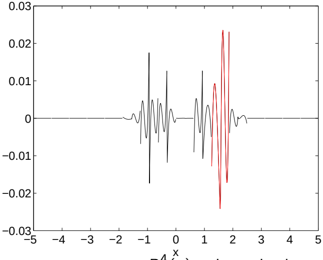
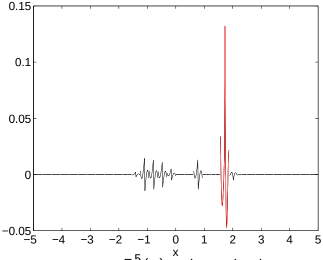
Results, speed $\lambda_3 = u + c$, at $T = 1$



Approximation, travels with speed λ_3



$D^4(x)$, selects shock



$D^5(x)$, selects shock

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Conclusion, further research II

- Multiwavelets can not be used as a limiter (13 weeks);
- Multiwavelets form an excellent shock detector (5 weeks)!
- Possible to detect different regions using characteristics.

- Reason why multiwavelets detect shocks (literature);
- Use other approaches to select regions;
- More examples in one dimension;
- Two dimensional case.

References

[1] Alpert, ..., and Vozovoi, Journal of Comp. Physics (2002);

[2] Archibald, Fann, and Shelton, Appl. Num. Math. (2011);

[3] Cockburn, Advanced Numerical Approximation of Nonlinear Hyperbolic Equations (1998);

[4] Gottlieb and Shu, Mathematics of Computation (1998);

[5] Krivodonova, Journal of Computational Physics (2007);

[6] Sod, Journal of Computational Physics (1978).