Multiwavelet troubled-cell indicator for discontinuity detection of discontinuous Galerkin schemes

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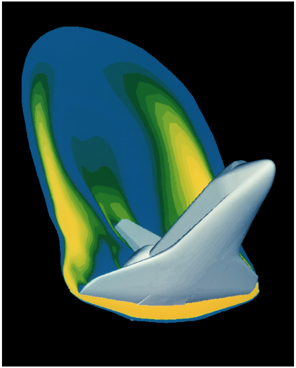
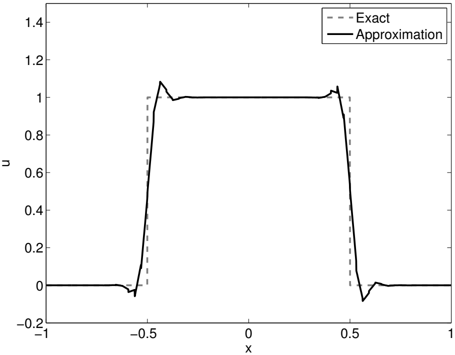
Motivation





Flow around Space Shuttle

Solution linear advection equation

Outline

Outline

Discontinuous Galerkin

Hyperbolic partial differential equation:

$$u_t + f(u)_x = 0; \quad x \in [-1,1],\ t \geq 0.$$

- DG approximation: for $x \in I_j$, write,

$$u_h(x) = \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j), \quad \xi_j = \frac{2}{\Delta x}(x - x_j)$$

- approximation space: orthonormal Legendre polynomials

$$\int_{-1}^{1} \phi_\ell(x)\phi_m(x)dx = \delta_{\ell m}$$

- k: highest polynomial degree of the approximation

Outline

Limiters

Limiter:

- Helps to control spurious oscillations
- Reduces polynomial order in nonsmooth regions
- May flatten local extrema (diffusive property)

Troubled-cell indicator:

- Helps to limit at discontinuities only

Troubled-cell indicators

Examples of troubled-cell indicators for DG:

- minmod-based TVB limiter
  (Cockburn and Shu, Math. Comput. 1989)
- KXRCF indicator
  (Krivodonova et al., Appl. Numer. Math. 2004)
- Harten's subcell resolution
  (Qiu and Shu, SIAM J. Sci. Comput. 2005)

These indicators use local information (neighbouring cells)
Multiwavelet approach: global and local information

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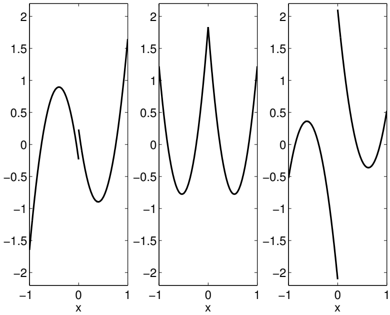
Multiwavelets

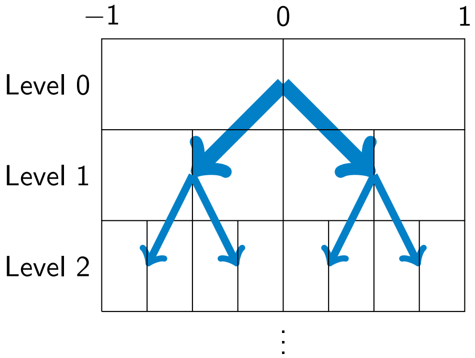
Multiwavelets (Alpert, SIAM J. Math. Anal. 1993):

- specific set of piecewise polynomials
- based on orthonormal Legendre polynomials
- possible to decompose function into several levels



Basis spans piecewise polynomials on $[-1,0] \cup [0,1]$, degree ≤ 2

Multiwavelet decomposition: next levels



Dilate and translate basis functions

Relation between DG and multiwavelets (2^n elements):

$$u_h(x) = \sum_{j=0}^{2^n-1} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) = S^0(x) + \sum_{m=0}^{n-1} D^m(x)$$

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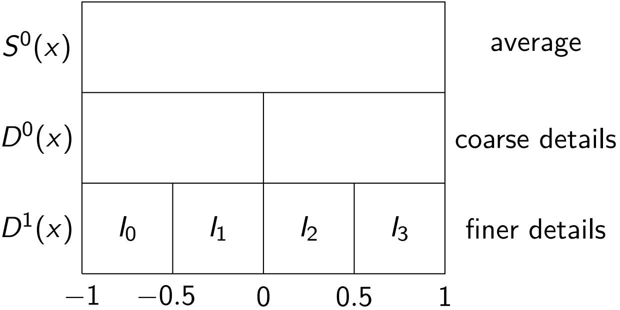
Multiwavelet decomposition

Example uses $n = 2$: 4 elements on $[-1, 1]$

$$u_h(x) = \sum_{j=0}^{3} \sum_{\ell=0}^{k} u_j^{(\ell)} \phi_\ell(\xi_j) = S^0(x) + \sum_{m=0}^{n-1} D^m(x), \; n-1=1$$

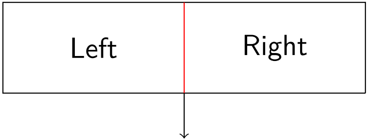


Regions where multiwavelet contributions are continuous

Both $D^1(x)$ and $u_h(x)$: continuous on $I_0, \ldots, I_3$

Highest level

- D^{n-1} constructed using $\mathbf{d}^{n-1} = (d_0^{n-1} \cdots d_k^{n-1})^\top$
- Jump between cells: $\Delta \mathbf{u} = ([u_h]^{(0)} \cdots [u_h]^{(k)})^\top$



$$\mathbf{d}^{n-1}, \Delta \mathbf{u}$$

$$\mathbf{d}^{n-1} = A \Delta \mathbf{u},$$

where

$$A(\ell + 1, r + 1) = 2^{-\frac{n-1}{2}} \frac{2^{(-n+1)r}}{r!} \int_0^1 x^r \psi_\ell(x) dx.$$

Highest level

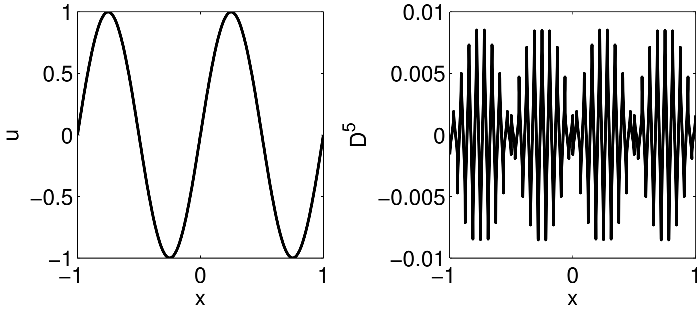
This means that D^{n-1}:

- Measures jumps in approximation (derivatives) at element boundaries;
- Can be used for detection of discontinuities (in derivatives).

Continuous example

Most details are visible in $D^{n-1}(x)$

Example: use $n = 6$: 2^6 elements

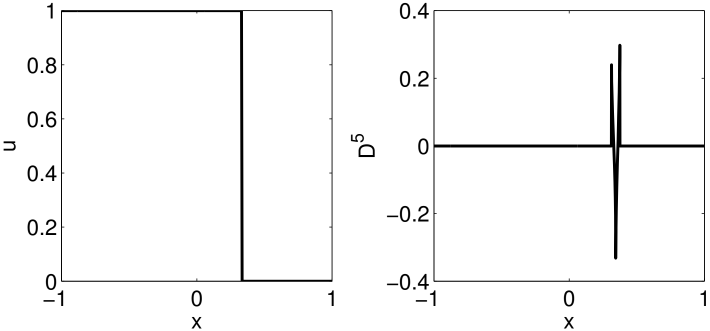


Multiwavelet approximation $D^5(x)$ of $\sin(2\pi x)$

Discontinuous example

Most details are visible in $D^{n-1}(x)$

Example: use $n = 6$: 2^6 elements



Multiwavelet approximation $D^5(x)$ of square wave

Outline

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Multiwavelet troubled-cell indicator

- Troubled cells: focus on highest level $D^{n-1}(x)$
- Compute absolute average $\bar{D}_j^{n-1}$ on element I_j
- Element I_j is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max\left\{\bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1\right\}, C \in [0, 1]$$

Choice of C

I_j is troubled cell if,

$$\bar{D}_j^{n-1} \geq C \cdot \max\left\{\bar{D}_i^{n-1}, i = 0, \ldots, 2^n - 1\right\}, C \in [0, 1]$$

Parameter C: defines strictness of indicator,

- $C = 0$: every element is detected
- $C = 0.2$: select largest 80% of averages
- $C = 0.8$: select largest 20% of averages

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Multiwavelet troubled-cell indicator

Applications: Euler equations

- Local detector: shock in different locations
  (Zaide and Roe, 20th AIAA CFD Conf. 2011)

  Our indicator: combines local and global nature

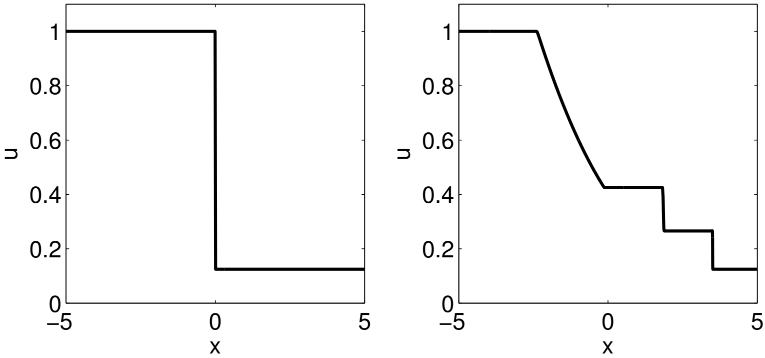
- Limiter: mechanism to control limited regions
  Now: troubled-cell indicator as switch

- Moment limiter (Krivodonova, J. Comput. Phys. 2007)

  Only a choice, other limiters possible

Outline

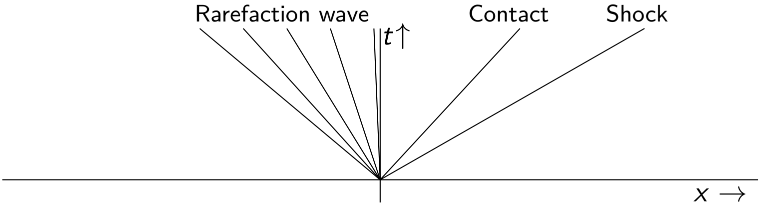
Sod's shock tube (J. Comput. Phys. 1978)



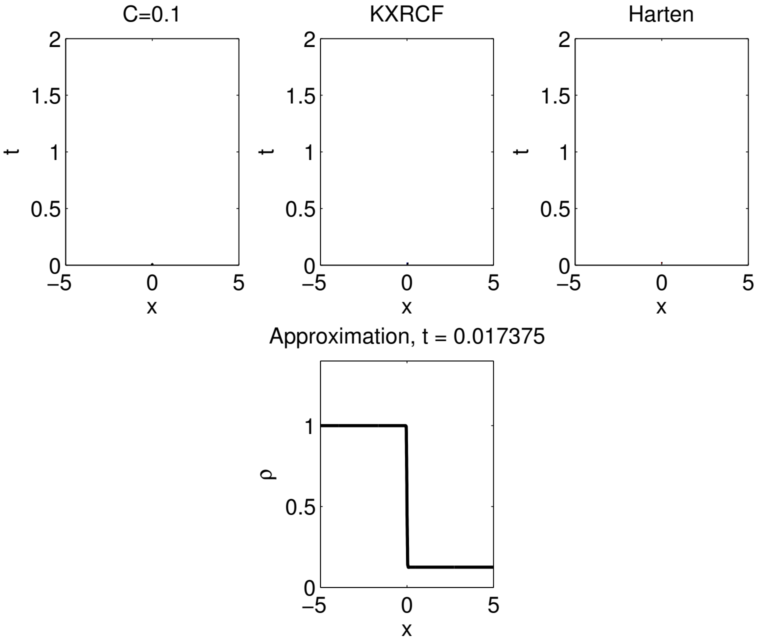
Density in Sod's shock tube at $T = 0$ (left) and $T = 2$ (right)

Sod: time history

Results: focus on detected troubled cells



Time history of troubled cells


C=0.1
KXRCF
Harten
t
x
Approximation, t = 0.017375
ρ
x

Sine entropy wave

Sine entropy wave:

$$\rho(x,0)=\begin{cases}3.857142, & x<-4,\\ 1+0.2\sin(5x), & x\geq -4.\end{cases}$$

(Shu and Osher, J. Comput. Phys. 1989)

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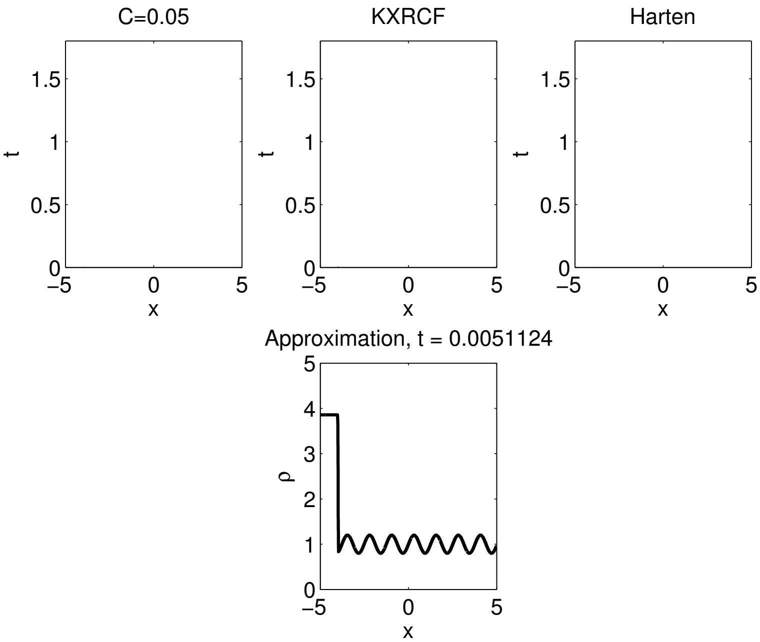


C=0.05
KXRCF
Harten
1.5
1
0.5
0
t
–5
0
5
x
Approximation, t = 0.0051124
ρ

Outline

Two-dimensional approach

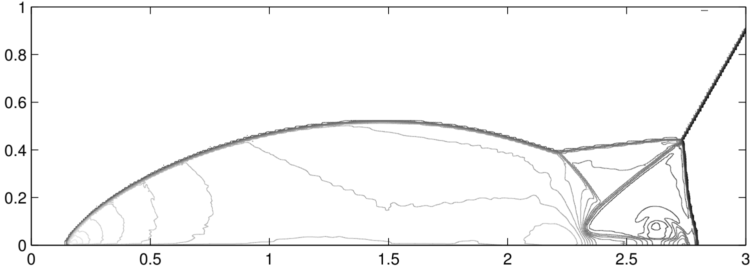
In two-dimensions, the multiwavelet expansion is:

$$S^0(x,y) + \sum_{m_x=0}^{n_x-1} \sum_{m_y=0}^{n_y-1} \left\{ D^{\alpha,\mathbf{m}}(x,y) + D^{\beta,\mathbf{m}}(x,y) + D^{\gamma,\mathbf{m}}(x,y) \right\}$$

number of elements: $2^{n_x} \times 2^{n_y}$

- α mode: multiwavelets in y-direction
- β mode: multiwavelets in x-direction
- γ mode: multiwavelets both x- and y-direction

Double Mach reflection

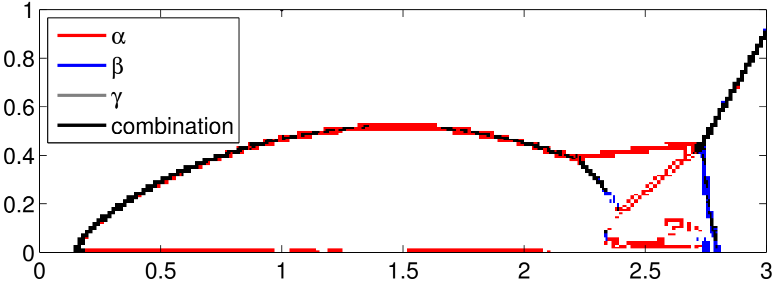


Density contours using $C = 0.05$

$T = 0.2$, $\Delta x = \Delta y = \frac{1}{128}$, $k = 1$

(Woodward and Colella, J. Comput. Phys. 1984)

Detected troubled cells



Detected troubled cells at $T = 0.2,\ C = 0.05$

Different troubled cells are detected by modes

Computation time

Compare computation time, double Mach reflection:

- More accurate result: don't limit continuous regions
- Decrease of computation time

Number of elements	limit everywhere	$C = 0.05$
512×128	57	50
1024×256	493	441

Computation time in minutes, $T = 0.2$, $k = 1$

Outline

Conclusion

- Troubled-cell indicator is switch in limiter
- Multiwavelet decomposition: D^{n-1} detects discontinuity
- Parameter C defines strictness of detector
- More accurate than existing detectors
- Two-dimensional detection in different modes
- Decrease of computation time

More details in JCP(270), pp 138-160

Future work:

→ How to choose parameter C

→ Applying to unstructured meshes