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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday January 2014, 18:30-21:30

1. (a) The amplification factor can be derived as follows. Consider the test equation $y' = \lambda y$. Application of the trapezoidal rule to this equation gives:

$$w_{j+1} = w_j + \frac{h}{2}(\lambda w_j + \lambda w_{j+1}) \tag{1}$$

Rearranging of w_{j+1} and w_j in (1) yields

$$\left(1 - \frac{h}{2}\lambda\right) w_{j+1} = \left(1 + \frac{h}{2}\lambda\right) w_j.$$

It now follows that

$$w_{j+1} = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda} w_j,$$

and thus

$$Q(h\lambda) = \frac{1 + \frac{h}{2}\lambda}{1 - \frac{h}{2}\lambda}.$$

(b) The definition of the local truncation error is

$$\tau_{j+1} = \frac{y_{j+1} - Q(h\lambda)y_j}{h}.$$

The exact solution of the test equation is given by

$$y_{j+1} = e^{h\lambda} y_j.$$

Combination of these results shows that the local truncation error of the test equation is determined by the difference between the exponential function and the amplification factor $Q(h\lambda)$

$$\tau_{j+1} = \frac{e^{h\lambda} - Q(h\lambda)}{h} y_j. \tag{2}$$

The difference between the exponential function and amplification factor can be computed as follows. The Taylor series of $e^{h\lambda}$ with known point 0 is:

$$e^{h\lambda} = 1 + \lambda h + \frac{(\lambda h)^2}{2} + \mathcal{O}(h^3). \tag{3}$$

The Taylor series of $\frac{1}{1-\frac{h}{2}\lambda}$ with known point 0 is:

$$\frac{1}{1-\frac{h}{2}\lambda} = 1 + \frac{1}{2}h\lambda + \frac{1}{4}h^2\lambda^2 + \mathcal{O}(h^3). \tag{4}$$

With (4) it follows that $\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda}$ is equal to

$$\frac{1+\frac{h}{2}\lambda}{1-\frac{h}{2}\lambda} = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \mathcal{O}(h^3). \tag{5}$$

In order to determine $e^{h\lambda} - Q(h\lambda)$, we subtract (5) from (3). Now it follows that

$$e^{h\lambda} - Q(h\lambda) = \mathcal{O}(h^3). \tag{6}$$

The local truncation error can be found by substituting (6) into (2), which leads to

$$\tau_{j+1} = \mathcal{O}(h^2).$$

(c) The trapezoidal rule is stable if

$$\frac{|1+\frac{h}{2}\lambda|}{|1-\frac{h}{2}\lambda|} \leq 1.$$

Using the complex valued $\lambda = \mu + i\nu$ it appears that the condition is equal to:

$$\frac{|1+\frac{h}{2}(\mu+i\nu)|}{|1-\frac{h}{2}(\mu+i\nu)|} \leq 1$$

This is equivalent with

$$\frac{\sqrt{(1+\frac{h}{2}\mu)^2 + (\frac{h}{2}\nu)^2}}{\sqrt{(1-\frac{h}{2}\mu)^2 + (\frac{h}{2}\nu)^2}} \leq 1$$

Since $\mu \leq 0$ it easily follows that

$$\sqrt{(1+\frac{h}{2}\mu)^2 + (\frac{h}{2}\nu)^2} \leq \sqrt{(1-\frac{h}{2}\mu)^2 + (\frac{h}{2}\nu)^2}$$

which implies that

$$\frac{|1+\frac{h}{2}\lambda|}{|1-\frac{h}{2}\lambda|} \leq 1.$$

and the method is stable.

(d) Application of the trapezoidal rule to

$$y' = -(1+2t)y + t, \text{ with } y(0) = 1,$$

and step size $h = \frac{1}{2}$ gives:

$$w_1 = w_0 + \frac{h}{2}[-w_0 + 0 - 2w_1 + \frac{1}{2}].$$

Using the initial value $w_0 = y(0) = 1$ and step size $h = \frac{1}{2}$ gives:

$$w_1 = 1 + \frac{1}{4}[-1 - 2w_1 + \frac{1}{2}].$$

This leads to

$$1\frac{1}{2}w_1 = \frac{7}{8}, \text{ so } w_1 = \frac{7}{12}.$$

(e) For the comparison we use the following items: accuracy, stability, and amount of work. Below we make the comparison:

- Accuracy: since the error of Euler Forward is $O(h)$ and that of the trapezoidal rule is $O(h^2)$, the error is smaller for the trapezoidal rule.
- Stability: since the value of $-(1+2t)$ is always negative the trapezoidal rule is stable for all step sizes, whereas for Euler Forward the step size should satisfy the inequality $h \leq \frac{2}{1+2t}$.
- Amount of work: since the differential equation is linear the amount of work for the implicit trapezoidal rule is comparable to the work of the explicit Euler Forward method.

From the above comparisons we conclude that for this problem the trapezoidal rule is preferred.

2. a Consider $y(x) = x$, then $y'(x) = 1$ and $y''(x) = 0$. Hence

$$-y''(x) + y'(x) + y^2(x) = 0 + 1 + x^2 = x^2 + 1, \tag{7}$$

and hence $y(x) = x$ satisfies the differential equation. Since $y(x) = x$ gives $y(0) = 0$ and $y(1) = 1$, we see that $y(x) = x$ also satisfies the boundary conditions. Hence $y(x) = x$ is a solution to the boundary value problem.

b On the gridnodes $x_j = jh$, we approximate all derivatives by difference formulae, and compute truncation errors by expanding Taylor Series around x_j, to obtain for the first–order derivative:

$$\frac{dy^{(k+1)}}{dx}(x_j) - \frac{y_{j+1}^{(k+1)} - y_{j-1}^{(k+1)}}{2h} =$$

$$\frac{dy^{(k+1)}}{dx}(x_j) - \frac{y_j^{(k+1)} + h\frac{dy^{(k+1)}}{dx}(x_j) + \frac{h^2}{2}\frac{d^2y^{(k+1)}}{dx^2}(x_j) + \frac{h^3}{6}\frac{d^3y^{(k+1)}}{dx^3}(x_j) + \mathcal{O}(h^4)}{2h} \tag{8}$$

$$+ \frac{y_j^{(k+1)} - h\frac{dy^{(k+1)}}{dx}(x_j) + \frac{h^2}{2}\frac{d^2y^{(k+1)}}{dx^2}(x_j) - \frac{h^3}{6}\frac{d^3y^{(k+1)}}{dx^3}(x_j) + \mathcal{O}(h^4)}{2h} = \mathcal{O}(h^2).$$

For the second–order derivative, we get

$$\frac{d^2y^{(k+1)}}{dx^2}(x_j) - \frac{y_{j+1}^{(k+1)}-2y_j^{(k+1)}+y_{j-1}^{(k+1)}}{h^2} = \frac{d^2y^{(k+1)}}{dx^2}(x_j)$$

$$-\frac{y_j^{(k+1)}+h\frac{dy^{(k+1)}}{dx}(x_j)+\frac{h^2}{2}\frac{d^2y^{(k+1)}}{dx^2}(x_j)+\frac{h^3}{6}\frac{d^3y^{(k+1)}}{dx^3}(x_j)+\mathcal{O}(h^4)-2y_j^{(k+1)}}{h^2} \quad (9)$$

$$-\frac{y_j^{(k+1)}-h\frac{dy^{(k+1)}}{dx}(x_j)+\frac{h^2}{2}\frac{d^2y^{(k+1)}}{dx^2}(x_j)-\frac{h^3}{6}\frac{d^3y^{(k+1)}}{dx^3}(x_j)+\mathcal{O}(h^4)}{h^2} = \mathcal{O}(h^2).$$

Herewith, we get

$$-\frac{d^2y^{(k+1)}}{dx^2}(x_i) + \frac{dy^{(k+1)}}{dx}(x_j) + y^{(k)}(x_j)y^{(k+1)}(x_j)-$$
$$(-\frac{y_{j+1}^{(k+1)}-2y_j^{(k+1)}+y_{j-1}^{(k+1)}}{h^2} + \frac{y_{j+1}^{(k+1)}-y_{j-1}^{(k+1)}}{2h} + y_j^{(k)}y_j^{(k+1)}) = \mathcal{O}(h^2), \quad (10)$$

and using the above given difference formulae, we get a local truncation error of $\mathcal{O}(h^2)$. Regarding the discretization at point $x_j = jh$, we have

$$-\frac{w_{j+1}^{(k+1)} - 2w_j^{(k+1)} + w_{j-1}^{(k+1)}}{h^2} + \frac{w_{j+1}^{(k+1)} - w_{j-1}^{(k+1)}}{2h} + w_j^{(k)}w_j^{(k+1)} = f(x_j) = x_j^2+1 = (jh)^2+1. \quad (11)$$

for $j \in \{1,\ldots,n\}$. Here $w_j^{(k)} \approx y_j^{(k)}$ represents the approximation under neglecting the truncation error. Substitution of the boundary conditions, $w_0 = 0$, $w_{n+1} = 1$, gives

$$j = 1: -\frac{w_2^{(k+1)} - 2w_1^{(k+1)}}{h^2} + \frac{w_2^{(k+1)}}{2h} + w_1^{(k+1)}w_1^{(k)} = f(x_1) = h^2+1, \quad (12)$$

and

$$j = n: -\frac{-2w_n^{(k+1)} + w_{n-1}^{(k+1)}}{h^2} - \frac{w_{n-1}^{(k+1)}}{2h} + w_n^{(k)}w_n^{(k+1)} = (1-h)^2+1+\frac{1}{h^2}-\frac{1}{2h}. \quad (13)$$

c We consider $h = \frac{1}{3}$, which means that $n = 2$ (note that $(n+1)h) = 1$), and that, using the iteration procedure given in the assignment, we get the following 2×2–system

$$\begin{cases} 18w_1^{(k+1)} - 9w_2^{(k+1)} + \frac{3}{2}w_2^{(k+1)} + w_1^{(k+1)}w_1^{(k)} = \frac{1}{9} + 1 = \frac{10}{9}, \\ 18w_2^{(k+1)} - 9w_1^{(k+1)} - \frac{3}{2}w_1^{(k+1)} + w_2^{(k+1)}w_2^{(k)} = \frac{4}{9} + 1 + 9 - \frac{3}{2} = \frac{161}{18}. \end{cases} \quad (14)$$

Substituting $w_1^{(0)} = w_2^{(0)} = 0$, gives

$$\begin{cases} 18w_1^{(1)} - 7.5w_2^{(1)} = \frac{10}{9}, \\ -10.5w_1^{(1)} + 18w_2^{(1)} = \frac{161}{18}. \end{cases} \quad (15)$$

Solution of this system gives $w_1^{(1)} \approx 0.35508$ and $w_2^{(1)} \approx 0.70404$.

d Newton–Raphson's Method is an iterative method to find $y \in \mathbb{R}$ such that $f(y) = 0$. One constructs a sequence of successive approximations $\{y^{(k)}\}$. Given the k–th estimate, then $y^{(k+1)}$ is obtained through linearizing around $y^{(k)}$ and by finding $y^{(k+1)}$ by determining the point where the linearization (tangent) equals zero. Linearization of $f(y)$ around $y^{(k)}$ gives (upon neglecting the error)

$$f(y) \approx f(y^{(k)}) + f'(y^{(k)})(y - y^{(k)}) =: L(y; y^{(k)}), \tag{16}$$

for any y provided the second derivative of $f(y)$ is bounded and where $L(y; y^{(k)})$ denotes the tangent (linearization) of $f(y)$ at point $(y_k, f(y^{(k)}))$. Then the next point is found upon setting $L(y^{(k+1)}; y^{(k)}) = 0$:

$$f(y^{(k)}) + f'(y^{(k)})(y^{(k+1)} - y^{(k)}) = 0. \tag{17}$$

The above equation is solved for $y^{(k+1)}$, and gives

$$y^{(k+1)} = y^{(k)} - \frac{f(y^{(k)})}{f'(y^{(k)})}, \tag{18}$$

which is the famous Newton–Raphson formula for root–finding.

Remark 1 *One can also give a graphical derivation, or alternatively using a derivation from first principles using* $L(y; y^{(k)}) = a_0 + a_1 y$*, where* a_0 *and* a_1 *need to be determined using* $L(y^{(k)}; y^k) = f(y^{(k)})$*,* $L'(y^{(k)}; y^{(k)}) = f'(y^{(k)})$ *and* $L(y^{(k+1)}, y^{(k)}) = 0$*.*

e First, we rewrite the system into the form

$$\begin{aligned} f_1(w_1, w_2) &= 0, \\ f_2(w_1, w_2) &= 0, \end{aligned} \tag{19}$$

by setting

$$\begin{aligned} f_1(w_1, w_2) &:= 18w_1 - 9w_2 + (w_1)^2, \\ f_2(w_1, w_2) &:= -9w_1 + 18w_2 + (w_2)^2 - 9. \end{aligned} \tag{20}$$

We denote the Jacobi–matrix by $J(w_1, w_2)$. At the first step we compute

$$\underline{w}^{(1)} = \underline{w}^{(0)} - J(\underline{w}^{(0)})^{-1} F(\underline{w}^{(0)}), \tag{21}$$

where $\underline{w} = [w_1 \; w_2]^T$. Note that

$$J(\underline{w}^{(0)}) = \begin{pmatrix} 18 + 2w_1^{(0)} & -9 \\ -9 & 18 + 2w_2^{(0)} \end{pmatrix}. \tag{22}$$

Using $w_1^{(0)} = w_2^{(0)} = 0$ we obtain:

$$J(\underline{w}^{(0)}) = \begin{pmatrix} 18 & -9 \\ -9 & 18 \end{pmatrix}. \tag{23}$$

This implies that

$$J(\underline{w}^{(0)})^{-1} = \frac{1}{18^2 - 81} \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix}. \tag{24}$$

Furthermore

$$F(\underline{w}^{(0)}) = \begin{pmatrix} 0 \\ -9 \end{pmatrix}, \tag{25}$$

so

$$\underline{w}^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{18^2 - 81} \begin{pmatrix} 18 & 9 \\ 9 & 18 \end{pmatrix} \begin{pmatrix} 0 \\ -9 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}. \tag{26}$$