DELFT UNIVERSITY OF TECHNOLOGY

FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)

Thursday January 30 2014, 18:30-21:30

1. In this exercise we use the Trapezoidal rule to approximate the solution of the following initial value problem $y' = f(t, y)$ with $y(t_0) = y_0$. This method is given by:

$$w_{n+1} = w_n + \frac{h}{2}\left(f(t_n, w_n) + f(t_{n+1}, w_{n+1})\right) \tag{1}$$

   (a) Show that the amplification factor of the Trapezoidal rule is given by

$$Q(h\lambda) = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}.$$

   (2 pt.)

   (b) Give the order (+ proof) of the local truncation error of the Trapezoidal rule for the test equation. *Hint:* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$, $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$. (3 pt.)

   (c) Show that for a general complex valued $\lambda = \mu + i\nu$ the method is stable for all step size $h > 0$ if $\mu \leq 0$. (2 pt.)

   (d) Do one step with the Trapezoidal rule for the following initial value problem

$$y' = -(1 + 2t)y + t, \text{ met } y(0) = 1,$$

   and step size $h = \frac{1}{2}$. (1.5 pt.)

   (e) Make for this problem (given in part d) a comparison of the Trapezoidal rule and the Euler Forward method. Which method do you prefer (+ motivation)? (1.5 pt.)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. We consider the following boundary value problem

$$\begin{cases} -\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y^2 = x^2 + 1, \quad x \in (0,1), \\ \\ y(0) = 0, \qquad y(1) = 1. \end{cases} \tag{2}$$

a Show that $y(x) = x$ is a solution to the above boundary value problem. (1pt.)

The boundary value problem is nonlinear, and hence we consider a Fixed Point algorithm and the Newton–Raphson method.

b Let $y^{(k)}$ be the approximation of the solution to boundary value problem (2) after k iterations of the following Fixed Point method:

$$\text{Iterate: } \begin{cases} -\dfrac{d^2y^{(k+1)}}{dx^2} + \dfrac{dy^{(k+1)}}{dx} + y^{(k+1)}y^{(k)} = x^2 + 1, \quad x \in (0,1), \\ y^{(k+1)}(0) = 0, \qquad y^{(k+1)}(1) = 1. \end{cases} \tag{3}$$

Derive a discretization for the above boundary value problem for $y^{(k+1)}$ using n unknowns and a stepsize $h = \frac{1}{n+1}$. Thus, make a discretization such that $y^{(k+1)}(x_1), ..., y^{(k+1)}(x_n)$ can be computed. Assume $y^{(k)}$ is given. Prove that the error is of order $\mathcal{O}(h^2)$ (It is sufficient to consider the local truncation error). (3pt.)

c Let the initial guess $y^{(0)}$ be given by $y^{(0)}(x) = 0, x \in [0,1]$, use $h = \frac{1}{3}$. Compute the numerical solution of the first Fixed Point iteration. (2pt.)

Subsequently, we derive and use Newton–Raphson's Method to solve a nonlinear problem.

d Given the scalar nonlinear problem:

$$\text{Find } y \in \mathbb{R} \text{ such that } f(y) = 0. \tag{4}$$

Derive Newton–Raphson's formula, given by

$$y^{(k+1)} = y^{(k)} - \frac{f(y^{(k)})}{f'(y^{(k)})}, \tag{5}$$

to solve the problem. (2pt.)

e Perform one step of the Newton–Raphson scheme applied to the following system for w_1 and w_2:

$$\begin{cases} 18w_1 - 9w_2 + w_1^2 = 0, \\ -9w_1 + 18w_2 + w_2^2 = 9. \end{cases} \tag{6}$$

Use $w_1 = w_2 = 0$ as the initial estimate. (2pt.)