DELFT UNIVERSITY OF TECHNOLOGY

FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400 WI3097 TU)
Thursday June 30th 2016, 18:30-21:30

1. (a) The **fourth-order Runge-Kutta method** (RK_4) for the differential equation $y' = f(t, y)$ is given by the following formulas:

$$\begin{aligned} k_1 &= \Delta t f(t_n, w_n) \\ k_2 &= \Delta t f(t_n + \tfrac{1}{2}\Delta t, w_n + \tfrac{1}{2}k_1) \\ k_3 &= \Delta t f(t_n + \tfrac{1}{2}\Delta t, w_n + \tfrac{1}{2}k_2) \\ k_4 &= \Delta t f(t_n + \Delta t, w_n + k_3) \end{aligned}$$

$$w_{n+1} = w_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Derive the amplification factor $Q(\lambda\Delta t)$ of RK_4 by applying the method to the *homogeneous test equation* $y' = \lambda y$. (2 pt.)

(b) Use the fact that $y(t_{n+1}) = e^{\lambda\Delta t}y(t_n)$ holds for the exact solution of $y' = \lambda y$ to show that RK_4 solves the *homogeneous test equation* with a local truncation error of $O(\Delta t^4)$. (Hint: $e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + ...$) (2 pt.)

In the following you may assume without proof that the last result also holds for systems and that, as a consequence, the global error of RK_4 applied to (1) is $O(\Delta t^4)$.

We consider the following **second-order initial-value problem**:

$$y'' + py' + qy = \sin t, \quad y(0) = y_0, \; y'(0) = y_0'. \qquad (1)$$

(c) Write (1) as a system of first order differential equations of type

$$\mathbf{y}' = \mathbf{A}\mathbf{y} + \mathbf{g}(t)$$

Give $\mathbf{A}$ and $\mathbf{g}$, and determine the eigenvalues of $\mathbf{A}$, for arbitrary p and q. (2 pt.)

(d) Let us consider p=1000 and q=250001. Give an approximate stability condition for this case. Hint: use the drawing of the RK_4-stability region on page 3. (2 pt.)

(e) Suppose you have the choice between the Trapezoidal rule and RK_4 to integrate the problem given in (d). Motivate your choice as good as possible, taking into account the stability condition and the order of magnitude of the global error. (2 pt.)

please turn over

2. We are interested in the **numerical integration** of $\int_0^1 y(x)dx$ with $y(x) = x^2$.

   (a) Give the *Rectangle Rule* I^R, the corresponding *composed integration rule* $I^R(h)$ and compute the approximate integral $\int_0^1 y(x)dx$ with $h = 1/3$. (1 pt.)

   (b) Repeat part (a) for the *Trapezoidal Rule* (I^T and $I^T(h)$) with $h = 1/3$. (1 pt.)

   (c) If one approximates $\int_0^1 y(x)dx$, the magnitude of the error of the *composed integration rules* (ε_R and ε_T for the Rectangle and Trapezoidal Rule, respectively) is bounded by

$$\varepsilon_R \leq \frac{h}{2} \max_{x\in[0,1]} |y'(x)|, \qquad \varepsilon_T \leq \frac{h^2}{12} \max_{x\in[0,1]} |y''(x)|. \tag{2}$$

   Which method do you recommend if the number of integration points is large? Give a proper motivation. (2 pt.)

3. We derive and use **Newton-Raphson's method** to solve a nonlinear problem.

   (a) Given is the *scalar* nonlinear problem:

$$\text{Find } p \in \mathbb{R} \text{ such that } f(p) = 0. \tag{3}$$

   Derive Newton–Raphson's formula, given by

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \text{ for } n \geq 1 \tag{4}$$

   with initial guess p_0 to solve the above problem. (2 pt.)

   (b) To prove convergence of Newton-Raphson's method it can be reformulated as fixed-point method for which a rigorous convergence proof exists:

$$g(x) = x - \frac{f(x)}{f'(x)}. \tag{5}$$

   One of the prerequisites of the convergence proof is that $g'(x)$ exists and satisfies

$$|g'(x)| \leq k < 1. \tag{6}$$

   Show that Newton-Raphson's method applied to $f(x) = \sin(x)$ converges to the root $p = 0$ for any initial guess $p_0 \in (-\sqrt{2}/2, \sqrt{2}/2)$. (1 pt.)

   (c) Formulate Newton-Raphson's method for the *general* nonlinear problem:

$$\text{Find } \mathbf{p} \in \mathbb{R}^m \text{ such that } \mathbf{f}(\mathbf{p}) = \mathbf{0}. \tag{7}$$

   (1 pt.)

   (d) Perform **one** step of Newton-Raphson's method applied to the following nonlinear problem for w_1 and w_2:

$$\begin{cases} 18w_1 - 9w_2 + w_1^2 = 0, \\ -9w_1 + 18w_2 + w_2^2 = 9. \end{cases} \tag{8}$$

   Use $w_1 = w_2 = 0$ as the initial estimate. (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html

List of Figures

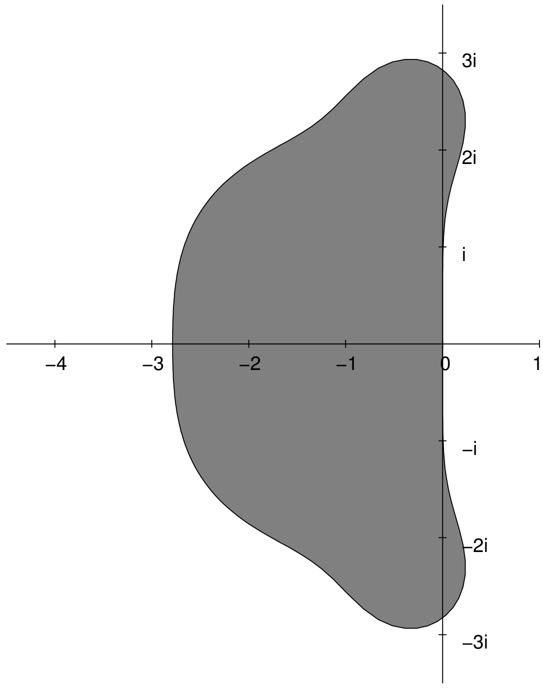

Figure 1: Stability region for RK_4

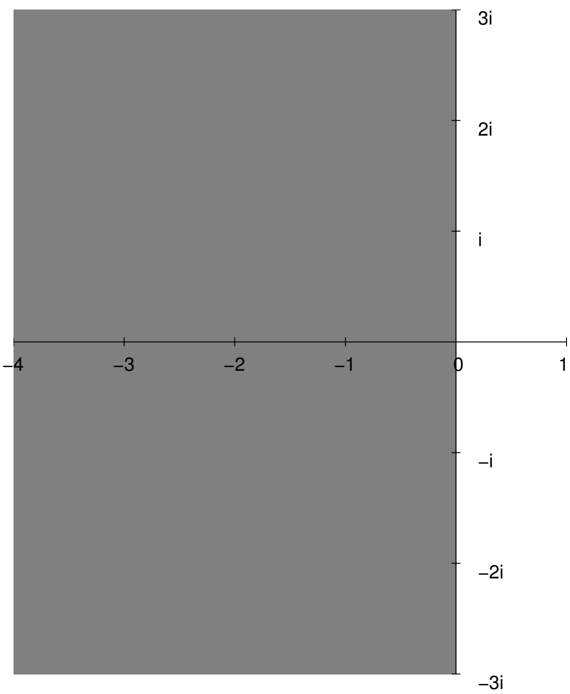

Figure 2: Stability region for the Trapezoidal rule