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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday February 2nd 2017, 18:30-21:30

1. In this exercise we use the Trapezoidal rule to approximate the solution of the following **initial value problem** $y' = f(t, y)$ with $y(t_0) = y_0$. This method is given by:

$$w_{n+1} = w_n + \frac{\Delta t}{2}\left(f(t_n, w_n) + f(t_{n+1}, w_{n+1})\right) \tag{1}$$

   (a) Show that the amplification factor of the Trapezoidal rule is given by

$$Q(\lambda\Delta t) = \frac{1 + \frac{\lambda\Delta t}{2}}{1 - \frac{\lambda\Delta t}{2}}.$$

   (2 pt.)

   (b) Give the order (+ proof) of the local truncation error of the Trapezoidal rule for the test equation.
   Hint:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$$

   (3 pt.)

   (c) Show that for a general complex valued $\lambda = \mu + i\nu$ the method is stable for all step size $\Delta t > 0$ if $\mu \leq 0$. (2 pt.)

   (d) Do one step with the Trapezoidal rule for the following initial value problem

$$y' = -(1 + 2t)y + t, \text{ met } y(0) = 1,$$

   and step size $\Delta t = \frac{1}{2}$. (1.5 pt.)

   (e) Make for this problem (given in part d) a comparison of the Trapezoidal rule and the Euler Forward method. Which method do you prefer (+ motivation)? (1.5 pt.)

2. We consider the following **boundary value problem** (differential equation with boundary conditions):

$$\begin{cases} -y'' + xy = x^3 - 2, x \in (0, 1) \\ y'(0) = 0, \quad y(1) = 1. \end{cases} \tag{2}$$

   (a) Let Δx be the step size. Give a discretization with an error of $\mathcal{O}(\Delta x^2)$ (+ proof) such that $x_n = 1$. Use a virtual grid node near $x = 0$. (3 pt.)

   (b) Use a step size of $\Delta x = 1/3$ to derive the system of equations. Take care of the boundary conditions. The derived system must be of 3×3 (three unknowns and three equations). (2 pt.)

please turn over

3. Consider a **hypothetical computer**, which operates with floating point (decimal) numbers. This computer has the following properties:

   - Each real number is represented as a floating point number with four digits after the comma;
   - The floating point representation is obtained by *rounding*.

   Hence, as an illustration: $fl(5/7) = fl(0.714285714\ldots) = 0.7143 \cdot 10^0$.

   In the exercise we consider the following given numbers

$$x = 2/3 = 0.666666666\ldots$$
$$y = 1999/3000 = 0.666333333\ldots$$

   (a) Determine $x + y$, $x - y$, $fl(fl(x) + fl(y))$ and $fl(fl(x) - fl(y))$, with the given values for x and y as exact results and computer representations of the outcomes. (1.5 pt.)

   (b) Calculate the relative error for $x+y$ and $x-y$ due to rounding in the computations by our hypothetical computer. (1.5 pt.)

   (c) Motivate why the relative error is much higher for $x - y$ than it is for $x + y$ if $x \approx y$, assuming that $x, y > 0$. (2 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html