DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210)
Thursday February 1st 2018, 18:30-21:30

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical calculator may be used. All other tools are prohibited.

Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/20$, where P is the number of points earned.

1. A method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$\begin{cases} k_1 &= \Delta t\, f\left(t_n, w_n\right) \\ k_2 &= \Delta t\, f\left(t_n + \frac{1}{2}\Delta t, w_n + \frac{1}{2}k_1\right) \\ k_3 &= \Delta t\, f\left(t_n + \Delta t, w_n - k_1 + 2k_2\right) \\ w_{n+1} &= w_n + \left(\alpha k_1 + \beta k_2 + \gamma k_3\right) \end{cases} \tag{1}$$

where Δt denotes the time–step and w_n represents the numerical solution at time t_n.

(a) The *amplification factor* of this method is given by

$$Q(\lambda\Delta t) = 1 + (\alpha + \beta + \gamma)\,\lambda\Delta t + \left(\frac{\beta}{2} + \gamma\right)(\lambda\Delta t)^2 + \gamma\,(\lambda\Delta t)^3 .$$

Derive this amplification factor for the given method. ($2\frac{1}{2}$ pt.)

(b) Show that the *local truncation error* of the given method for the test equation $y' = \lambda y$ is $\mathcal{O}(\Delta t^3)$ *only* for $\alpha = \gamma = \frac{1}{6}$ and $\beta = \frac{2}{3}$. ($2\frac{1}{2}$ pt.)

(c) Given the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} + \dfrac{1}{2}y = t, \\ \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 1. \end{cases} \tag{2}$$

Show that this problem can be written as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}. \tag{3}$$

Give also the initial conditions for $x_1(0)$ and $x_2(0)$. ($1\frac{1}{2}$ pt.)

(d) Take $\alpha = \gamma = \frac{1}{6}$ and $\beta = \frac{2}{3}$.
Is the given method applied to this initial value problem stable for $\Delta t = 2$? ($1\frac{1}{2}$ pt.)

(e) Perform *one step* with the given method with $\Delta t = 2$, $t_0 = 0$, $\alpha = \gamma = \frac{1}{6}$ and $\beta = \frac{2}{3}$ for the initial value problem and the given initial conditions from (2). (2 pt.)

please turn over

2. In this exercise an estimate is determined for the velocity of a rowing boat. The measured distances of the boat from the starting line are given in the table below.

t (s)	0	10	20
$d(t)$ (m)	0	40	100

   (a) Give the first order backward difference formula and use this to determine an estimate of the velocity for $t = 20$ ($d'(20)$). (1 pt.)

   (b) We are looking for a difference formula of the first derivative of d in $2h$ of the form:

$$Q(h) = \frac{\alpha_0}{h}d(0) + \frac{\alpha_1}{h}d(h) + \frac{\alpha_2}{h}d(2h),$$

   such that

$$d'(2h) - Q(h) = O(h^2).$$

   In the remainder of this exercise we use this formula. Show that the coefficients α_0, α_1 and α_2 should satisfy the next system:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,. \end{array}$$

   (2 pt.)

   (c) The solution of this system is given by $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$. Give for these values an expression for the truncation error $d'(2h) - Q(h)$. (1 pt.)

   (d) Use $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$ in $Q(h)$ to give an estimate of the velocity at $t = 20$. (1 pt.)

3. We derive and use Newton–Raphson's Method to solve a nonlinear problem.

   (a) Given the scalar nonlinear problem:

$$\text{Find } p \in \mathbb{R} \text{ such that } f(p) = 0. \tag{4}$$

   Derive Newton–Raphson's formula, given by

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \tag{5}$$

   to solve the problem. Explain the method with a graph. (2pt.)

   (b) Given the nonlinear problem: Find $\mathbf{p} \in \mathbb{R}^m$ such that $\mathbf{f}(\mathbf{p}) = \mathbf{0}$, where $\mathbf{f}(\mathbf{x}) \in \mathbb{R}^m$. Give the Newton–Raphson's formula for this problem. (1pt.)

   (c) Perform one step of the Newton–Raphson scheme applied to the following system for p_1 and p_2:

$$\begin{cases} 18p_1 - 9p_2 + p_1^2 = 0, \\ -9p_1 + 18p_2 + p_2^2 = 9. \end{cases} \tag{6}$$

   Use $p_1 = p_2 = 0$ as the initial estimate. (2pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html