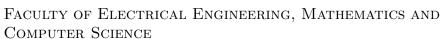
DELFT UNIVERSITY OF TECHNOLOGY





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Examination reviewer: C. Vuik

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU/Minor AESB2210) Thursday April 19th 2018, 18:30-21:30

Number of questions: This is an exam with 10 open questions, subdivided in 3 main questions.

Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical calculator is permitted. All other tools are not permitted.

Assessment In total 20 points can be earned. The final not-rounded grade is given by P/20, where P is the number of points earned.

1. A method to integrate the initial value problem defined by y' = f(t, y), $y(t_0) = y_0$, is given by

$$\begin{cases} k_1 &= \Delta t f(t_n, w_n) \\ k_2 &= \Delta t f(t_n + \Delta t, w_n + k_1) \\ w_{n+1} &= w_n + \frac{1}{2} (k_1 + k_2) \end{cases}$$

where Δt denotes the time-step and w_n represents the numerical solution at time t_n .

- (a) Show that the *local truncation error* of the given method is $\mathcal{O}(\Delta t^2)$. (3 pt.)
- (b) The amplification factor of this method is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t + \frac{1}{2} (\lambda \Delta t)^{2}.$$

Derive this amplification factor for the given method.

(2 pt.)

(c) Given is the initial value problem

$$\underline{y}' = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} -2 & 0 & 0 \\ 2 & -3 & 0 \\ 0 & 3 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A\underline{y} + \underline{f}, \tag{1}$$

with the initial conditions $y_1(0) = y_2(0) = y_3(0) = 0$.

For which value of Δt is the given time-integration method stable for this initial-value problem?

 $(3\frac{1}{2} \text{ pt.})$

(d) Perform one step with the given method with $\Delta t = \frac{1}{2}$ and $t_0 = 0$ for the initial-value problem (1) and the given initial conditions. $(1\frac{1}{2} \text{ pt.})$

2. Consider the following boundary value problem:

$$\begin{cases}
-\frac{d^2y(x)}{dx^2} + y(x) = 2e^x, & x \in (0,1), \\
y(0) = 2, & y'(1) = 0.
\end{cases}$$
(2)

We apply the finite difference method to approximate the solution to the above boundary value problem. Let the gridnodes be given by $x_j = jh$, with h as stepsize. Let $x_n = nh = 1$.

- (a) Give a finite differences scheme (including a derivation) for which the local truncation error is of order $O(h^2)$. (2pt.)
- (b) Use a virtual gridnode for the boundary condition at x = 1 and give a *finite differences scheme* (including a derivation) for which the local truncation error is of order $O(h^2)$. (1pt.)
- (c) Give a linear system $A\mathbf{w} = \mathbf{b}$ with symmetric matrix A which can be obtained from applying the finite difference discretisation with three (after processing the virtual gridnode) unknowns (h = 1/3). (2pt.)
- 3. We approximate the integral $\int_a^b f(x)dx$ by a numerical method.
 - (a) Now we derive a new integration method. Let $P_1(x)$ be the Taylor polynomial of degree one of f(x) around the point b. Show that the *integration method* I_P based on $P_1(x)$ for $\int_a^b f(x)dx$ is given by

$$(b-a)f(b) - \frac{1}{2}(a-b)^2 f'(b),$$

and give an upper bound for the corresponding truncation error

$$\int_{a}^{b} f(x)dx - I_{P}.$$

(2pt.)

- (b) Derive the composite rule I(h) for the new integration method I_P . Let the gridnodes be given by $x_j = a + jh$, with h as stepsize and $x_n = a + nh = b$. Approximate the integral for $f(x) = x^3$, a = 0 and b = 1 with this method using $h = \frac{1}{2}$ and determine the difference with the exact answer. (1.5pt.)
- (c) Which method do you *prefer*: the method given in 3b or the composite Trapezoidal rule. Give arguments that support your preference. Hint: you may use that the truncation error of the composite Trapezoidal rule is bounded by: $\frac{(b-a)h^2}{12}\max_{\xi\in[a,b]}|f''(\xi)|$. (1.5pt.)