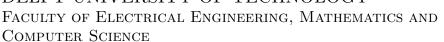
## DELFT UNIVERSITY OF TECHNOLOGY





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Examination reviewer: D. den Ouden-van der Horst

## TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS ( CTB2400 ) Thursday July 5th 2018, 13:30-16:30

**Number of questions:** This is an exam with 10 open questions, subdivided in 3 main questions.

**Answers** All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.

Tools Only a non-graphical calculator is permitted. All other tools are not permitted.

**Assessment** In total 20 points can be earned. The final not-rounded grade is given by P/20, where P is the number of points earned.

1. A method to integrate the initial value problem defined by y' = f(t, y),  $y(t_0) = y_0$ , is given by

$$w_{n+1} = w_n + (1 - \theta)\Delta t f(t_n, w_n) + \theta \Delta t f(t_{n+1}, w_{n+1}),$$

where  $\Delta t$  denotes the time–step,  $w_n$  represents the numerical solution at time  $t_n$  and  $0 \le \theta \le 1$ .

(a) The amplification factor of this method is given by

$$Q(\lambda \Delta t) = \frac{1 + (1 - \theta)\lambda \Delta t}{1 - \theta \lambda \Delta t}.$$

Derive this amplification factor for the given method.

 $(1\frac{1}{2} \text{ pt.})$ 

 $(3\frac{1}{2} \text{ pt.})$ 

(b) Show that the local truncation error of the given method is  $\mathcal{O}(\Delta t)$  in general for the test equation  $y' = \lambda y$ . Also determine for which value of  $\theta$  the method is  $\mathcal{O}(\Delta t^2)$ .

Hint:  $e^x = 1 + x + \frac{1}{2}x^2 + \mathcal{O}(x^3)$ . Hint:  $\frac{1}{1-x} = 1 + x + x^2 + \mathcal{O}(x^3)$  for |x| < 1.

(c) Take  $\theta = \frac{1}{2}$ . Given is the initial value problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -1 & -3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\tag{1}$$

with initial conditions  $x_1(0) = 1, x_2(0) = 0.$ 

Is the given method applied to this initial value problem stable for  $\Delta t = 1$ ? (3\frac{1}{2} pt.)

(d) Perform one step with the given method with  $\Delta t = 1$ ,  $\theta = \frac{1}{2}$  and  $t_0 = 0$  for the initial value problem (1) and the given initial conditions.  $(1\frac{1}{2} \text{ pt.})$ 

2. We will analyse Lagrange interpolation. For given points  $x_0, x_1, \ldots, x_n$  and their respective function values  $f(x_0), f(x_1), \ldots, f(x_n)$ , the interpolating polynomial  $L_n(x)$  is given by

$$L_n(x) = \sum_{k=0}^n f(x_k) L_{kn}(x), \quad \text{with}$$

$$L_{kn}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}.$$

(a) Determine  $\hat{L}_2(2)$  (the perturbed version of  $L_2(2)$ ) given the following measured values:

$$\begin{array}{c|ccccc}
k & x_k & f(x_k) \\
\hline
0 & 1 & 3 \\
\hline
1 & 3 & 6 \\
\hline
2 & 4 & 5
\end{array}$$
(3 pt.)

(b) Given is that we know

$$\left| f(x) - \hat{f}(x) \right| \le \epsilon,$$
$$|f'''(x)| \le \delta,$$

and

$$f(x) - L_n(x) = \frac{(x - x_0) \cdots (x - x_n)}{(n+1)!} f^{(n+1)}(\zeta(x)),$$

for  $x \in [1, 4]$ . Determine an upper bound for the error  $|f(2) - \hat{L}_2(2)|$ . (2 pt.)

- 3. We want to find a root of the function  $f(x) = -x^3 + 6x \frac{23}{8}$ .
  - (a) We choose to use the fixed point iteration  $p_{n+1} = g(p_n)$ , with  $g(x) = \frac{x^3}{6} + \frac{23}{48}$  to find a root. Show that a fixed point of g(x) is also a root of f(x). (1 pt.)
  - (b) We start the fixed point iteration at  $p_0 = 1$ . Calculate  $p_1, p_2$  and  $p_3$  exact to four decimals and sketch the fixed point iteration in a figure. (2 pt.)
  - (c) Show that the chosen fixed point iteration *converges* for all  $p_0 \in [0, 1]$ . (2 pt.)