DELFT UNIVERSITY OF TECHNOLOGY
FACULTY OF ELECTRICAL ENGINEERING, MATHEMATICS AND COMPUTER SCIENCE

TU Delft

Examiner responsible: C. Vuik
Examination reviewer: D. den Ouden-van der Horst

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (CTB2400)
Tuesday August 14th 2018, 13:30-16:30

PLEASE WRITE DOWN THE CORRECT COURSE CODE UNDER WHICH YOU WANT YOUR RESULT TO BE RECORDED WITHIN OSIRIS ON YOUR ANSWER SHEETS.

Number of questions: This is an exam with 11 open questions, subdivided in 3 main questions.
Answers All answers require arguments and/or shown calculation steps. Answers without arguments or calculation steps will not give points.
Tools Only a non-graphical calculator is permitted. All other tools are not permitted.
Assessment In total 20 points can be earned. The final not-rounded grade is given by $P/2$, where P is the number of points earned.

1. The Forward Euler method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$w_{n+1} = w_n + \Delta t f(t_n, w_n),$$

where Δt denotes the time–step and w_n represents the numerical solution at time t_n.

(a) Show that the *local truncation error* of the given method is $\mathcal{O}(\Delta t)$ in general. *Remark:* Do not use the test equation. (3 pt.)

(b) The *amplification factor* of this method is given by

$$Q(\lambda \Delta t) = 1 + \lambda \Delta t.$$

Derive this amplification factor for the given method. (1 pt.)

(c) Given is the initial value problem

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} -3 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (1)$$

with initial conditions $x_1(0) = 1, x_2(0) = 1$.
For which values of $\Delta t \geq 0$ is the given method applied to this initial value problem *stable*? ($3\frac{1}{2}$ pt.)

(d) Perform *two steps* with the given method with $\Delta t = 1/5$ and $t_0 = 0$ for the initial value problem (1) and the given initial conditions. ($2\frac{1}{2}$ pt.)

please turn over

2. We consider the one-dimensional convection–diffusion equation with Dirichlet boundary conditions:

$$\begin{cases} -u'' + u' = 1, & 0 < x < 1, \\ u(0) = 0, & u(1) = 0, \end{cases} \tag{2}$$

where $u = u(x)$, $u' = \frac{du}{dx}$ and $u'' = \frac{d^2u}{dx^2}$. The solution is given by $u(x) = x - \frac{1-e^x}{1-e}$. In this exercise we try to approximate this solution with a numerical method.

(a) Give a (physical or mathematical) *motivation* why oscillatory numerical solutions to (2) should be considered unreliable. (1 pt.)

(b) We solve the boundary value problem (2) using finite differences, upon setting $x_j = j\Delta x$, $(n+1)\Delta x = 1$, where Δx denotes the uniform stepsize. After discretization we obtain the following formulas:

$$\begin{aligned} -\frac{w_2 - 2w_1}{(\Delta x)^2} + \frac{w_2}{2\Delta x} &= 1 \\ -\frac{w_{j+1} - 2w_j + w_{j-1}}{(\Delta x)^2} + \frac{w_{j+1} - w_{j-1}}{2\Delta x} &= 1, \quad \text{for } j \in \{2, \ldots, n-1\}, \\ \frac{2w_n - w_{n-1}}{(\Delta x)^2} - \frac{w_{n-1}}{2\Delta x} &= 1. \end{aligned}$$

Give (with proof) the *order* of the *local truncation error* of this scheme. (3 pt.)

(c) Choose $\Delta x = 1/4$ and derive the *system of equations* $A\mathbf{w} = \mathbf{b}$ with $\mathbf{w} = [w_1, \ldots, w_n]^T$. (1 pt.)

3. To approximate $\int_a^b f(x)\,dx$ the Trapezoidal rule $\frac{b-a}{2}(f(a) + f(b))$ can be used.

(a) Give the *linear Lagrange interpolatory polynomial* $p_1(x)$ with nodes a and b and *derive* the Trapezoidal rule by the use of $p_1(x)$. ($1\frac{1}{2}$ pt.)

(b) The error for linear interpolation over nodes a and b is given by

$$f(x) - p_1(x) = \frac{1}{2}(x-a)(x-b)f''(\xi(x)), \text{ for some } \xi(x) \in (a, b).$$

Derive that an *upper bound* of the truncation error of the Trapezoidal rule applied to the interval $[a, b]$ is given by

$$\frac{1}{12}(b-a)^3 \max_{x \in [a,b]} |f''(x)|,$$

given that the second-order derivative of f is continuous over $[a, b]$. ($1\frac{1}{2}$ pt.)

(c) *Approximate* $\int_0^1 x^2\,dx$ with the composite Trapezoidal rule using $h = \frac{1}{4}$. (1 pt.)

(d) Determine the absolute value of the *truncation error* of the answer given in (c). (1 pt.)

For the answers of this test we refer to:
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html