Extra exercises

Section 3.2

1. (a) Compute the derivative of $f(x) = x^4 + 2x^2 - x$ in x=1, using **forward differention**. Take $h = 0.1$, $h = 0.05$ and $h = 0.01$.
   (b) Compute the error (the difference between the approximation and the exact value).
2. (a) Compute the derivative of $f(x) = -2x^2+3x-1$ in x=0, using **backward differention**. Take $h = 0.1$, $h = 0.05$ and $h = 0.01$.
   (b) Compute the error.
3. (a) Compute the derivative of $f(x) = -2x^2 + 3x - 1$ in x=0, using **central differention**. Take $h = 0.1$, $h = 0.05$ and $h = 0.01$.
   (b) Compute the error.
   (c) Which method, backward or central differention, has the smallest error?
4. (a) Compute the derivative of $f(x) = x^3+5x^2-2x+3$ in x=1, using **central differention**. Take $h = 0.1$, $h = 0.05$ and $h = 0.01$.
   (b) Compute the error.

Section 3.4

1. (a) Apply the formula

$$Q(h) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

   to $f(x) = -2x^2 + 3x - 1$, with x=1 and h=0.5.
   (b) What is the meaning of this number?
   (c) Show, for this formula, that Q(h)=f'(x).
2. Given $f(x-2h)$, $f(x-h)$ and $f(x)$. Determine a formula to approximate $f'(x)$ with a minimal error.

Section 3.5

1. (a) Use the formula for linear interpolation to find an expression for the derivative when $n = 1$, $x = x_0, x_1 = x_0 - h$
   (b) Is the differention formula known?

Section 3.6

1. (a) Apply the formula

$$Q(h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

   to $f(x) = 3x^2 + 5x - 2$, with x=1, h=0.5.
   (b) What is the error?

2. Suppose we want to approximate $f''(x)$ on the border of our interval. We have given the points $x_0 = x, x_1 = x + h, x_2 = x + 2h$
   (a) What kind of differention do we need, central or one-sided?
   (b) Compute $Q(h)$ for the second derivative.
   (c) Compute the order of the error. (Hint: Use Taylorexpansions)
3. Compute, by repeatingly applying the formula for the first derivative with forward differention, the second derivative.

Section 3.7

1. (a) Compute with forward differention $f'(1)$, when $f(x) = \ln(x)$ and $h = 0.025$, using the numbers from the following table.

h	$\ln(1+h)$
0.1	0.09531
0.05	0.04879
0.025	0.02469

   (b) Give an estimate of the error with Richardsons error-estimation.
   (c) Compare this estimate of the error with the exact error.
2. (a) Compute with central differention the derivative of $f(x) = \cos(x)$, $x \in [0, \pi]$ in $x = 1$ for $h = 0.01$
   (b) Give an estimate of the error with Richardsons error-estimation.
   (c) Compare this estimate of the error with the exact error.

Answers of the extra exercises

Section 3.2

1. (a) Forward differention: $\frac{f(x+h)-f(x)}{h}$
   $f(1) = 1^4 + 2 \cdot 1^2 - 1 = 1 + 2 - 1 = 2$
   $h = 0.1$: $f'(1) \approx \frac{1.1^4+2\cdot 1.1^2-1.1-2}{0.1} = \frac{0.7841}{0.1} = 7.841$
   $h = 0.05$: $f'(1) \approx \frac{1.05^4+2\cdot 1.05^2-1.05-2}{0.05} = \frac{0.3705}{0.05} = 7.410$
   $h = 0.01$: $f'(1) \approx \frac{1.01^4+2\cdot 1.01^2-1.01-2}{0.01} = \frac{0.0708}{0.01} = 7.080$

   (b) $R_v(h) = f'(x) - \frac{f(x+h)-f(x)}{h}$
   $f'(x) = 4x^3 + 4x - 1, \Rightarrow f'(1) = 4 \cdot 1^3 + 4 \cdot 1 - 1 = 7$
   $R_v(0.1) = 7 - 7.841 = -0.841$
   $R_v(0.05) = 7 - 7.410 = -0.410$
   $R_v(0.01) = 7 - 7.080 = -0.080$

2. (a) Backward differention: $\frac{f(x)-f(x-h)}{h}$
   $f(0) = -2 \cdot 0^2 + 3 \cdot 0 - 1 = -1$
   $h = 0.1$: $f(-0.1) = -2 \cdot (-0.1)^2 + 3 \cdot (-0.1) - 1 = -1.32$
   $f'(0) \approx \frac{-1-(-1.32)}{0.1} = \frac{0.32}{0.1} = 3.2$
   $h = 0.05$: $f(-0.05) = -2 \cdot (-0.05)^2 + 3 \cdot (-0.05) - 1 = -1.155$
   $f'(0) \approx \frac{-1-(-1.155)}{0.05} = \frac{0.155}{0.05} = 3.1$
   $h = 0.01$: $f(-0.01) = -2 \cdot (-0.01)^2 + 3 \cdot (-0.01) - 1 = -1.0302$
   $f'(0) \approx \frac{-1-(-1.0302)}{0.01} = \frac{0.0302}{0.01} = 3.02$

   (b) $R_a(h) = f'(x) - \frac{f(x)-f(x-h)}{h}$
   $f'(x) = -4x + 3, \Rightarrow f'(0) = -4 \cdot 0 + 3 = 3$
   $R_a(0.1) = 3 - 3.2 = -0.2$
   $R_a(0.05) = 3 - 3.1 = -0.1$
   $R_a(0.01) = 3 - 3.02 = -0.02$

3. (a) Central differention: $\frac{f(x+h)-f(x-h)}{2h}$
   $h = 0.1$: $f(0.1) = -2 \cdot 0.1^2 + 3 \cdot 0.1 - 1 = -0.72$
   $f(-0.1) = -2 \cdot (-0.1)^2 + 3 \cdot (-0.1) - 1 = -1.32$
   $f'(0) \approx \frac{-0.72-(-1.32)}{2\cdot 0.1} = 3.00$
   $h = 0.05$: $f(0.05) = -2 \cdot 0.05^2 + 3 \cdot 0.05 - 1 = -0.855$
   $f(-0.05) = -2 \cdot (-0.05)^2 + 3 \cdot (-0.05) - 1 = -1.155$
   $f'(0) \approx \frac{-0.855-(-1.155)}{2\cdot 0.05} = 3.000$
   $h = 0.01$: $f(0.01) = -2 \cdot 0.01^2 + 3 \cdot 0.01 - 1 = -0.9702$
   $f(-0.01) = -2 \cdot (-0.01)^2 + 3 \cdot (-0.01) - 1 = -1.0302$
   $f'(0) \approx \frac{-0.9702-(-1.0302)}{2\cdot 0.01} = 3.000$

   (b) The error is exactly zero. For this problem the central differention is the best method.

4. (a) Central differention: $\frac{f(x+h)-f(x-h)}{2h}$
   $h = 0.1$: $f(1.1) = 1.1^3 + 5 \cdot 1.1^2 - 2 \cdot 1.1 + 3 = 8.181$
   $f(0.9) = 0.9^3 + 5 \cdot 0.9^2 - 2 \cdot 0.9 + 3 = 5.979$
   $f'(1) \approx \frac{8.181-5.979}{2\cdot 0.1} = 11.01$
   $h = 0.05$: $f(1.05) = 1.05^3 + 5 \cdot 1.05^2 - 2 \cdot 1.05 + 3 = 7.570125$
   $f(0.95) = 0.95^3 + 5 \cdot 0.95^2 - 2 \cdot 0.95 + 3 = 6.469875$
   $f'(1) = \frac{7.570125-6.469875}{2\cdot 0.05} = 11.0025$

$h = 0.01$: $f(1.01) = 1.01^3 + 5 \cdot 1.01^2 - 2 \cdot 1.01 + 3 = 7.110801$
$f(0.99) = 0.99^3 + 5 \cdot 0.99^2 - 2 \cdot 0.99 + 3 = 6.890799$
$f'(1) = \frac{7.110801 - 6.890799}{2 \cdot 0.01} = 11.0001$

(b) $R_2(h) = f'(x) - \frac{f(x+h) - f(x-h)}{2h}$
$f'(x) = 3x^2 + 10x - 2, \Rightarrow f'(1) = 3 \cdot 1^2 + 10 \cdot 1 - 2 = 3 + 10 - 2 = 11$
$R_2(0.1) = 11 - 11.01 = -0.01$
$R_2(0.05) = 11 - 11.0025 = -0.0025$
$R_2(0.01) = 11 - 11.0001 = -0.0001$

Section 3.4

1. (a) $Q(h) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$
$f(1) = -2 \cdot 1^2 + 3 \cdot 1 - 1 = 0$
$f(1 + 0.5) = f(1.5) = -2 \cdot 1.5^2 + 3 \cdot 1.5 - 1 = -1$
$f(1 + 2 \cdot 0.5) = f(2) = -2 \cdot 2^2 + 3 \cdot 2 - 1 = -3$
$Q(0.5) = \frac{-3 \cdot 0 + 4 \cdot (-1) - (-3)}{2 \cdot 0.5} = \frac{0 - 4 + 3}{1} = -1$

(b) This is the derivative of the function in the point $x = 1$.

(c) $f'(x) = -4x + 3$

$$
\begin{aligned}
Q(h) &= \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} \\
&= \frac{-3(-2x^2+3x-1) + 4(-2(x+h)^2 + 3(x+h) - 1) - (-2(x+2h)^2 + 3(x+2h) - 1)}{2h} \\
&= \frac{6x^2 - 9x + 3 - 8(x^2 + 2xh + h^2) + 12x + 12h - 4 + 2(x^2 + 4xh + 4h^2) - 3x - 6h + 1}{2h} \\
&= \frac{6x^2 - 9x + 3 - 8x^2 - 16xh - 8h^2 + 12x + 12h - 4 + 2x^2 + 8xh + 8h^2 - 3x - 6h + 1}{2h} \\
&= \frac{-8xh + 6h}{2h} \\
&= -4x + 3
\end{aligned}
$$

2. Choose $x_{-2} = x - 2h$, $x_{-1} = x - h$, $x_0 = x$ and $Q(h) = \frac{\alpha_{-2} f(x-2h) + \alpha_{-1} f(x-h) + \alpha_0 f(x)}{h}$

Taylorexpansion:
$f(x) = f(x)$
$f(x-h) = f(x) + (x - h - x)f'(x) + \frac{(x-h-x)^2}{2!} f''(x) + \mathcal{O}(h^3)$
$= f(x) - hf'(x) + \frac{h^2}{2} f''(x) + (O)(h^3)$
$f(x-2h) = f(x) + (x - 2h - x)f'(x) + \frac{(x-2h-x)^2}{2!} f''(x) + \mathcal{O}(h^3)$
$= f(x) - 2hf'(x) + \frac{4h^2}{2} f''(x) + (O)(h^3)$

This gives the following conditions:
$f(x)$: $\frac{\alpha_{-2}}{h} + \frac{\alpha_{-1}}{h} + \frac{\alpha_0}{h} = 0$
$f'(x)$: $-2\alpha_{-2} + -\alpha_{-1} = 1$ $\Rightarrow \alpha_{-1} = -2\alpha_{-2} - 1$
$f''(x)$: $2h\alpha_{-2} + \frac{1}{2}h\alpha_{-1} = 0$ $\Rightarrow 2h\alpha_{-2} + \frac{1}{2}h(-2\alpha_{-2} - 1) = 0$
$\Rightarrow 2h\alpha_{-2} - h\alpha_{-2} - \frac{1}{2}h = 0$
$\Rightarrow h\alpha_{-2} - \frac{1}{2}h = 0$
$\Rightarrow h(\alpha_{-2} - \frac{1}{2}) = 0$
$\Rightarrow \alpha_{-2} = \frac{1}{2} \Rightarrow \alpha_{-1} = -2 \cdot \frac{1}{2} - 1 = -2$

$$
Q(h) = \frac{\frac{1}{2}f(x-2h) - 2f(x-h) + \frac{3}{2}f(x)}{h} = \frac{f(x-2h) - 4f(x-h) + 3f(x)}{2h}
$$

Section 3.5

1. (a) The formula becomes:
$f'(x_0) = f(x_0)L'_{01}(x_0) + f(x_1)L'_{11}(x_0) + (x_0 - x_1))\frac{1}{2}f''(\xi)$

From 2.3 follows:
$L_{01}(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-x_1}{x_0-(x_0-h)} = \frac{x-x_1}{h}$
$L'_{01}(x) = \frac{1}{h}$
$L_{11}(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-x_0}{x_0-h-x_0} = \frac{x-x_0}{-h}$
$L'_{11}(x) = -\frac{1}{h}$

$f'(x_0) = \frac{1}{h}f(x_0) - \frac{1}{h}f(x_1) + (x_0 - (x_0 - h)) \cdot \frac{1}{2}f''(\xi)$

(b) $f'(x_0) = \frac{1}{h}f(x_0) - \frac{1}{h}f(x_1) + (x_0 - (x_0 - h)) \cdot \frac{1}{2}f''(\xi)$
$= \frac{1}{h}f(x_0) - \frac{1}{h}f(x_1) + \frac{h}{2}f''(\xi)$
$f'(x) = \frac{1}{h}f(x) - \frac{1}{h}f(x-h) + \frac{h}{2}f''(\xi)$
$= \frac{f(x)-f(x-h)}{h} + \frac{h}{2}f''(\xi)$
This is equal to backward differention.

Section 3.6

1. (a) $Q(h) = \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$
$f(1+0.5) = f(1.5) = 3 \cdot 1.5^2 + 5 \cdot 1.5 - 2 = 12.25$
$f(1) = 3 \cdot 1^2 + 5 \cdot 1 - 2 = 6$
$f(1-0.5) = f(0.5) = 3 \cdot 0.5^2 + 5 \cdot 0.5 - 2 = 1.25$
$Q(0.5) = \frac{12.25-2\cdot6+1.25}{0.5^2} = \frac{1.5}{0.25} = 6$

(b) $f'(x) = 6x + 5$
$f''(x) = 6$
$f''(1) = 6$
The error is $f''(1) - Q(0.5)$=6-6=0

2. (a) One-sided differention

(b) $Q(h) = \frac{\alpha_0 f(x)+\alpha_1 f(x+h)+\alpha_2 f(x+2h)}{h^2}$

Taylorexpansion:

$$\begin{aligned} f(x) &= f(x) \\ f(x+h) &= f(x) + (x+h-x)f'(x) + \frac{(x+h-x)^2}{2!}f''(x) + \mathcal{O}(h^3) \\ &= = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + (O)(h^3) \\ f(x+2h) &= f(x) + (x+2h-x)f'(x) + \frac{(x+2h-x)^2}{2!}f''(x) + \mathcal{O}(h^3) \\ &= f(x) + 2hf'(x) + 2h^2f''(x) + (O)(h^3) \end{aligned}$$

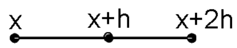
This gives the following conditions:
$f(x)$: $\frac{\alpha_0}{h^2} + \frac{\alpha_1}{h^2} + \frac{\alpha_2}{h^2} = 0$
$f'(x)$: $\frac{\alpha_1}{h} + \frac{2\alpha_2}{h} = 0$ $\Rightarrow \alpha_1 = -2\alpha_2$
$f''(x)$: $\frac{1}{2}\alpha_1 + 2\alpha_2 = 1$ $\Rightarrow \frac{1}{2}(-2\alpha_2) + 2\alpha_2 = 1$
$\Rightarrow -\alpha_2 + 2\alpha_2 = \alpha_2 = 1$
$\Rightarrow \alpha_1 = -2 \cdot 1 = -2$
$\Rightarrow \alpha_0 - 2 + 1 = \alpha_0 - 1 = 0 \Rightarrow \alpha_0 = 1$

$$Q(h) = \frac{f(x) - 2f(x+h) + f(x+2h)}{h^2}$$

(c) $f''(x) - Q(h) = f''(x) - \frac{f(x)-2f(x+h)+f(x+2h)}{h^2}$
$= f'' - \frac{f-2f-2hf'-2\cdot\frac{1}{2}h^2f''+\mathcal{O}(h^3)+f+2hf'+2h^2f''+\mathcal{O}(h^3)}{h^2}$

$= f'' - \frac{h^2 f'' + \mathcal{O}(h^3)}{h^2}$
$= f'' - f'' + \mathcal{O}(h) = \mathcal{O}(h)$
The order of the error is 1.

3. Forward differention gives as approximation for $f''(x)$: $\frac{f'(x+h)-f'(x)}{h}$
Again forward differention gives:
$\frac{1}{h}\left(\frac{f(x+2h)-f(x+h)}{h} - \frac{f(x+h)-f(x)}{h}\right)$
$= \frac{f(x+2h)-2f(x+h)+f(x)}{h^2}$
(Check your answer with the last exercise.)



Section 3.7

1. (a) $N(0.025) = \frac{f(1.025)-f(1)}{h} = \frac{0.02469-0}{0.025} = 0.9876$
   (b) $N(0.05) = \frac{f(1.05)-f(1)}{0.05} = \frac{0.04879-0}{0.05} = 0.9758$
   $N(0.1) = \frac{f(1.1)-f(1)}{0.1} = \frac{0.09531-0}{0.1} = 0.9531$

   $\frac{N(0.05)-N(0.1)}{N(0.025)-N(0.05)} = \frac{0.9758-0.9531}{0.9876-0.9758} = \frac{0.0227}{0.0118} = 1.9237 \Rightarrow \alpha \approx 1$

   $N(0.025) - N(0.05) = 0.0118 = K(2\cdot 0.025)^1(1-(\frac{1}{2})^1) = 0.025K \Rightarrow K = 0.4720$

   So, the error$=f'(1) - N(0.025) = 0.025^1 \cdot K = 0.0118$
   (c) $f'(x) = \frac{1}{x} \Rightarrow f'(1) = 1/1 = 1$
   Exact error $= f'(1) - N(0.025) = 1 - 0.9876 = 0.0124$
   Difference $= |0.0124 - 0.0118| = 0.0006$

2. (a) Central differention: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$
   $f(1.01) = \cos(1.01) = 0.531860$
   $f(0.99) = \cos(0.99) = 0.548689$

   $f'(1) \approx \frac{0.531860-0.548689}{2\cdot 0.01} = -0.841456$
   (b) $N(0.01) = -0.841456$ (from a)
   $N(0.02) = \frac{f(1.02)-f(0.98)}{2\cdot 0.02} = -0.841414$
   $N(0.04) = \frac{f(1.04)-f(0.96)}{2\cdot 0.04} = -0.841246$

   $\frac{N(0.02)-N(0.04)}{N(0.01)-N(0.02)} = \frac{-0.841414-(-0.841246)}{-0.841456-(-0.841414)} = 3.999701 \approx 2^2$
   So $\alpha = 2$

   $N(0.01) - N(0.02) = -0.0000420725 = K(2\cdot 0.01)^2(1-(\frac{1}{2})^2) = 0.0003K \Rightarrow K = \frac{-0.0000420725}{0.0003} = 0.140241$

   So, the error is $f'(1) - N(0.01) = 0.01^2 \cdot K = 1.402417 \cdot 10^{-5}$
   (c) $f'(x) = -\sin(x) \Rightarrow f'(1) = -\sin(1) = -0.841471$
   Exact error $= f'(1) - N(0.01) = -0.841471 + 0.841457 = 1.402441 \cdot 10^{-5}$ Difference $= \left|1.402441 \cdot 10^{-5} - 1.402417 \cdot 10^{-5}\right| = 2.4123 \cdot 10^{-10}$