Extra exercises

Section 4.2

1. (a) Compute with the bisection-method the root of the function $f(x) = (x-3)^2 - 5$ on the interval [5,6] in 3 iterations.
   (b) Calculate the difference with the exact value.
   (c) When we calculate with 4 decimals, what's the length of the unreliability interval?

Section 4.4

1. (a) Compute the exact value of the root of the function $f(x) = 2x^2 - 5$, for $x > 0$.
   (b) Compute the root in 6 digits with the Newton-Raphson-method. Take as starting point $p_0 = 1$.
2. (a) Compute the value of the root of the function $f(x) = x^4 - 2x^2 - 6$ with your calculator or with maple.
   (b) Compute the root in 5 digits, with the Newton-Raphson-method. Take $p_0 = 2$
   (c) What is the error between the exact value of the root and the one computed with the Newton-Raphson-method?

Section 4.5

1. (a) What is the Jacobian of the following system?
   $$\begin{cases} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{cases}$$
   (b) Given the non-linear system
   $$\begin{cases} x_1^3 + 3x_1x_2 - x_2 - 2x_3^2 &= 0 \\ x_1^2 - 3x_2 + 5x_2^2 + x_3^3 &= 0 \\ x_1^2 + 2x_1 - 4x_3 + x_2^4 &= 0 \end{cases}$$
   Compute the Jacobian.
   (c) Give the Jacobian in (3,1,-1).
2. (a) Given the non-linear system
   $$\begin{aligned} x_1^2 - x_2^2 + 2x_2 &= 0 \\ 2x_1 + x_2^2 - 6 &= 0 \end{aligned}$$
   Compute the Jacobian.
   (b) Do 1 iteration with the Newton-Raphson-method, with starting vector $\begin{pmatrix} 0.6 \\ 2.2 \end{pmatrix}$, to approximate the solution of the non-linear system given above.
   (c) Compare the approximation with the exact solution $\begin{pmatrix} 0.625204094 \\ 2.179355825 \end{pmatrix}$

Answers of the extra exercises

Section 4.2

1. (a) $a_1 = 5 \qquad f(5) = (5-3)^2 - 5 = 4 - 5 = -1$
   $b_1 = 6 \qquad f(6) = (6-4)^2 - 5 = 9 - 5 = 4$
   $p_1 = \frac{6+5}{2} = 5.5 \qquad f(5.5) = (5.5-3)^2 - 5 = 6.25 - 5 = 1.25$
   $f(p_1)f(a_1) = 1.25 \cdot -1 < 0 \qquad \Rightarrow a_2 = a_1,\, b_2 = p_1$

   $a_2 = 5,\, b_2 = 5.5$
   $p_2 = \frac{5+5.5}{2} = 5.25 \qquad f(5.25) = (5.25-3)^2 - 5 = 0.0625$
   $f(p_2)f(a_2) = 0.0625 \cdot -1 < 0 \qquad \Rightarrow a_3 = a_2,\, b_3 = p_2$

   $a_3 = 5,\, b_3 = 5.25$
   $p_3 = \frac{5+5.25}{2} = 5.125 \qquad f(5.125) = (5.125-3)^2 - 5 = -0.484375$
   $f(p_3)f(a_3) = -0.484375 \cdot -1 > 0$

   (b) Exact value: $(x-3)^2 - 5 = 0$
   $(x-3)^2 = 5$
   $x - 3 = \sqrt{5} \Rightarrow x = 3 + \sqrt{5}$
   Difference: $|3 + \sqrt{5} - 5.125| = 0.111068$

   (c) The rounding error is then maximal: $\bar{\epsilon} = 0.00005$. For the derivation of this number we refer to the first chapter of the book.
   So the unreliability interval is

$$\begin{aligned} I &= \left[p - \frac{\bar{\epsilon}}{|f'(p)|}, p + \frac{\bar{\epsilon}}{|f'(p)|}\right] \\ &= \left[3 + \sqrt{5} - \frac{5 \cdot 10^{-5}}{|2(3+\sqrt{5}-3)|}, 3 + \sqrt{5} + \frac{5 \cdot 10^{-5}}{|2(3+\sqrt{5}-3)|}\right] \\ &= \left[3 + \sqrt{5} - \frac{5 \cdot 10^{-5}}{2\sqrt{5}}, 3 + \sqrt{5} + \frac{5 \cdot 10^{-5}}{2\sqrt{5}}\right] \\ &= [5.236056797, 5.236079158] \end{aligned}$$

Section 4.4

1. (a) $2x^2 - 5 = 0$
   $2x^2 = 5$
   $x^2 = \frac{5}{2} \Rightarrow x = \sqrt{\frac{5}{2}} = 1.58114$

   (b)
$$\begin{aligned} p_0 &= 1 \\ p_1 &= p_0 - \frac{f(p_0)}{f'(p_0)} = 1 - \frac{2 \cdot 1^2 - 5}{4 \cdot 1} = 1 - \frac{-3}{4} = 1.75 \\ p_2 &= p_1 - \frac{f(p_1)}{f'(p_1)} = 1.75 - \frac{2 \cdot 1.75^2 - 5}{4 \cdot 1.75} = 1.75 - \frac{-1.125}{7} = 1.58929 \\ p_3 &= p_2 - \frac{f(p_2)}{f'(p_2)} = 1.58929 - \frac{2 \cdot 1.58929^2 - 5}{4 \cdot 1.58929} = 1.58116 \\ p_4 &= p_3 - \frac{f(p_3)}{f'(p_3)} = 1.58116 - \frac{2 \cdot 1.58116^2 - 5}{4 \cdot 1.58116} = 1.58114 \end{aligned}$$

2. (a) $f(x) = 0$ if $x = 1.90939$

   (b)
$$\begin{aligned} p_0 &= 2 \\ p_1 &= 2 - \frac{2^4 - 2 \cdot 2^2 - 6}{4 \cdot 2^3 - 4 \cdot 2} = 2 - \frac{2}{24} = 1.9167 \\ p_2 &= 1.9167 - \frac{1.9167^4 - 2 \cdot 1.9167^2 - 6}{4 \cdot 1.9167^3 - 4 \cdot 1.9167} = 1.9094 \end{aligned}$$

   (c) $|x - p_4| = |1.90939 - 1.9094| = 0.00001$

Section 4.5

1. (a) $J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix}$

   (b) $f_1(x) = x_1^3 + 3x_1x_2 - x_2 - 2x_3^2$
   $f_2(x) = x_1^2 - 3x_2 + 5x_2^2 + x_3^3$
   $f_3(x) = x_1^2 + 2x_1 - 4x_3 + x_2^4$
   $J(x) = \begin{pmatrix} 3x_1^2 + 3x_2 & 3x_1 - 1 & -4x_3 \\ 2x_1 & -3 + 10x_2 & 3x_3^2 \\ 2x_1 - 2 & 4x_2^3 & -4 \end{pmatrix}$

   (c) $J(3, 1, -1) = \begin{pmatrix} 3 \cdot 3^2 + 3 \cdot 1 & 3 \cdot 3 - 1 & -4 \cdot (-1) \\ 2 \cdot 3 & -3 + 10 \cdot 1 & 3 \cdot (-1)^2 \\ 2 \cdot 3 - 2 & 4 \cdot 1^3 & -4 \end{pmatrix} = \begin{pmatrix} 30 & 8 & 4 \\ 6 & 7 & 3 \\ 4 & 4 & -4 \end{pmatrix}$

2. (a) $J(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & -2x_2 + 2 \\ 2 & 2x_2 \end{pmatrix}$

   (b) $x^{(p)} = x^{(p-1)} + y^{(p-1)}$, with $y^{(p-1)} = -J(x^{(p-1)})^{-1} \cdot F(x^{(p-1)}$.
   $F(x^{(p-1)} = \begin{pmatrix} f_1(x^{(p-1)} \\ f_2(x^{(p-1)} \end{pmatrix} = \begin{pmatrix} (x_1^{(p-1)})^2 - (x_2^{(p-1)})^2 + 2x_2^{(p-1)} \\ 2x_1^{(p-1)} + (x_2^{(p-1)})^2 - 6 \end{pmatrix}$
   $F(x^0) = \begin{pmatrix} 0.6^2 - 2.2^2 + 2 \cdot 2.2 \\ 2 \cdot 0.6 + 2.2^2 - 6 \end{pmatrix} = \begin{pmatrix} -0.08 \\ 0.04 \end{pmatrix}$
   $J(x^0) = \begin{pmatrix} 2 \cdot 0.6 & -2 \cdot 2.2 + 2 \\ 2 & 2 \cdot 2.2 \end{pmatrix} = \begin{pmatrix} 1.2 & -2.4 \\ 2 & 4.4 \end{pmatrix}$
   $J(x^0)^{-1} = \frac{1}{1.2 \cdot 4.4 + 2.4 \cdot 2} \begin{pmatrix} 4.4 & 2.4 \\ -2 & 1.2 \end{pmatrix} = \begin{pmatrix} 0.4365 & 0.2381 \\ -0.1984 & 0.1190 \end{pmatrix}$
   $y^{(0)} = -\begin{pmatrix} 0.4365 & 0.2381 \\ -0.1984 & 0.1190 \end{pmatrix} \begin{pmatrix} -0.08 \\ 0.04 \end{pmatrix} = \begin{pmatrix} -0.0254 \\ 0.0206 \end{pmatrix}$
   $x^{(1)} = x^{(0)} + y^{(0)} = \begin{pmatrix} 0.6 \\ 2.2 \end{pmatrix} + \begin{pmatrix} -0.0254 \\ 0.0206 \end{pmatrix} = \begin{pmatrix} 0.5746 \\ 2.2206 \end{pmatrix}$

   (c) $|0.625204094 - 0.5746| = 0.0506$
   $|2.179355825 - 2.2206| = 0.0412$