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Part I: European-Style Products

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Efficient Pricing of Asian Options under Lévy Processes based on Fourier Cosine Expansions *Part I: European-Style Products*

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Abstract

We propose an efficient pricing method for arithmetic, and geometric, Asian options under Lévy processes, based on Fourier cosine expansions and Clenshaw–Curtis quadrature. The pricing method is developed for both European–style and American–style Asian options, and for discretely and continuously monitored versions. In the present paper we focus on European–style Asian options; American-style options are treated in an accompanying part II of this paper. The exponential convergence rate of Fourier cosine expansions and Clenshaw–Curtis quadrature reduces the CPU time of the method to milli-seconds for geometric Asian options and a few seconds for arithmetic Asian options. The method's accuracy is illustrated by a detailed error analysis, and by various numerical examples.

Keywords: Arithmetic Asian options, Lévy processes, Fourier cosine expansions, Clenshaw–Curtis quadrature, exponential convergence.

AMS MSC: 65C30, 60H35, 65T50

1 Introduction

Asian options, introduced in 1987, belong to the class of path–dependent options. Their payoff is typically based on a geometric or arithmetic average of underlying asset prices at monitoring dates before maturity. The number of monitoring dates can be finite (*discretely–monitored*) or infinite (*continuously–monitored*). Volatility inherent in an asset is reduced due to the averaging feature, leading to cheaper options compared to plain vanilla option equivalents.

For geometric Asian options a closed–form solution under the Black–Scholes model has been presented in [15]. Other Lévy asset models have been studied

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in [12], resulting in an efficient valuation method based on the Fast Fourier Transform.

For arithmetic Asian options the prices have to be approximated numerically. Monte Carlo methods have been applied for this task, for example in [15]. An efficient PDE method for arithmetic Asian options which works particularly well for short maturities has been presented in [18].

Advanced pricing methods for options on the arithmetic average are based on a recursive integration procedure, in which the probability density function of the log-return of the sum of asset prices is approximated, see [6, 2, 16, 12]. In [6, 2] an FFT and inverse FFT have been incorporated in the procedure to approximate the governing densities. The study in [6] was focused on log–normally distributed underlying processes and required a fine grid to approximate the probability density function. This method is extended to more general densities in [2], where the size of the grid was reduced by re–centering the probability densities at each monitoring step, resulting in reduced CPU time. In [12] the FFT was used to approximate the density of the increment between consecutive monitoring dates, in combination with a series of recursive quadrature rules. The total computational complexity in [12] was $O(Mn^2)$, where M is the number of monitoring dates and n the number of points used in the quadrature. A recent contribution was presented in [7], where discretely sampled Asian options were priced via backward price convolutions.

Another pricing approach can be found in [14], where the governing densities were computed by a special Laplace inversion, for guaranteed return rate products, that can be seen as generalized discretely sampled Asian options. Alternatively, upper and lower bounds of the Asian option prices have been determined, for example in [16], for Lévy asset processes.

In this paper, Asian options are priced with the help Fourier cosine expansions. We name the resulting method, the *ASCOS* method (Asian cosine method), as it is related to the COS method from [10, 11]. The COS method recovers the transitional density function in the risk–neutral formula in terms of the conditional characteristic function, by a Fourier cosine expansion. The characteristic function of a Lévy process is typically available in closed form. In our pricing method, the Fourier cosine expansions are not only used in the risk–neutral pricing formula, but also to recursively recover the *characteristic function* of the log-return of the sum of asset prices. Moreover, the Clenshaw–Curtis quadrature rule is used in the pricing procedure.

Exponential convergence has been proved and observed for plain vanilla European and Bermudan options in [10, 11]. We will perform an extensive error analysis here to confirm exponential convergence also for Asian options.

We present, in section 2, a technique to price geometric Asian options under Lévy processes (discretely and continuously monitored), which is highly efficient. The pricing algorithm for arithmetic Asian options is presented in section 3. A detailed error analysis is given in section 4 and numerical results are presented in section 5. We compare our results to those presented in [12].

The ASCOS pricing method can be seen as an efficient alternative to the

FFT and convolution methods in [6, 12, 2, 16]. The method remains robust as the number of monitoring dates, M, increases for arithmetic Asian options. The ASCOS method is extended to pricing American–style Asian options, in an accompanying *Part II* of the present paper. Key is here that instead of recovering the density function (like in [6, 12, 2, 16]), the characteristic function is recovered, which enables us to also price American–style Asian options.

In our paper, we focus on a fixed–strike Asian options. The extension to floating–strike Asian options follows directly from the symmetry between floating–strike and fixed–strike Asian options, as explained in [13] and [9].

2 ASCOS method for European-style geometric Asian options

The ASCOS pricing technique for geometric and arithmetic Asian options is described in sections 2 and 3, respectively. In our method, the characteristic function of the geometric or arithmetic mean value of the underlying is recovered, which is then used to calculate the Asian option value by Fourier cosine expansions. For geometric Asian options, the characteristic function of the geometric mean can be calculated directly, as we will see below.

2.1 Introduction to the COS method

We depart from the risk-neutral option valuation formula (discounted expected payoff approach) for plain vanilla European options:

$$v(x, t_0) = e^{-r\Delta t} \int_{-\infty}^{\infty} v(y, T) f(y|x) dy, \tag{1}$$

where $v(x, t_0)$ is the present option value, r the interest rate, $\Delta t = T - t_0$ and x, y can be any monotone functions of the underlying asset at initial time t_0 and the expiration date T, respectively. Payoff function $v(y, T)$ is known, but the transitional density function, $f(y|x)$, typically is not. Based on (1), we approximate the transitional density function on a truncated domain $[a, b]$, by a truncated Fourier cosine series expansion, with N terms, based on its conditional characteristic function (see [10]), as follows:

$$f(y|x) \approx \frac{2}{b-a} \sum_{k=0}^{N-1}{}' Re \left(\phi(\frac{k\pi}{b-a}; x) \exp{(-i\frac{ak\pi}{b-a})} \right) \cos{(k\pi \frac{y-a}{b-a})}, \tag{2}$$

with $\phi(u; x)$ the conditional characteristic function of $f(y|x)$, a, b determine the integration interval and Re means taking the real part of the argument. The prime at the sum symbol indicates that the first term in the expansion is multiplied by one-half. The appropriate size of the integration interval can be determined with the help of the cumulants [10] [1].

[1]So that $| \int_{\mathbb{R}} f(y|x)dy - \int_a^b f(y|x)dy| < TOL$.

Replacing $f(y|x)$ by its approximation (2) in (1) and interchanging integration and summation gives the COS formula for the computation of plain vanilla European options:

$$\hat{v}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi \frac{a}{b-a}}\right) V_k, \tag{3}$$

where $\hat{v}(x, t_0)$ indicates the approximate option value, and

$$V_k = \frac{2}{b-a} \int_a^b v(y, T) \cos\left(k\pi \frac{y-a}{b-a}\right) dy,$$

are the Fourier cosine coefficients of $v(y, T)$, available in closed form for several payoff functions.

With integration interval $[a, b]$ chosen sufficiently wide, it was found that the series truncation error dominates the overall error. For transitional density functions $f(y|x) \in C^\infty([a, b] \subset \mathbb{R})$, the method converges exponentially; otherwise convergence is algebraically [10, 11].

2.2 European-style geometric Asian options

The payoff function of a geometric Asian options with M monitoring dates and a fixed strike reads:

$$v(S, T) \equiv g(S) = \begin{cases} \max((\prod_{j=0}^{M} S_j)^{\frac{1}{M+1}} - K, 0), & \text{for a call,} \\ \max(K - (\prod_{j=0}^{M} S_j)^{\frac{1}{M+1}}, 0). & \text{for a put.} \end{cases}$$

Here S, K, $g(S)$ denote the stock price, the strike price and the payoff function, respectively; $M = 1, 2, \cdots$.

For geometric Asian options, the characteristic function of the geometric mean can be calculated directly. The underlying process is transformed to the logarithm domain and we use the notation:

$$y \quad := \quad \log((\prod_{j=0}^{M} S_j)^{\frac{1}{M+1}}) = \frac{1}{M+1} \sum_{j=0}^{M} \log(S_j) =: \frac{1}{M+1} \sum_{j=0}^{M} x_j. \tag{4}$$

In order to use the Fourier cosine expansion, we need to determine the conditional characteristic function of y given x_0. From the definition of characteristic function we have:

$$\begin{aligned}
\phi(u;x_0) &= \mathbf{E}[\exp(iuy)|\mathcal{F}_0] = \mathbf{E}[\exp(iu\frac{1}{M+1}\sum_{j=0}^{M} x_j)|\mathcal{F}_0] \\
&= \mathbf{E}[\exp(iu(x_0 + \frac{1}{M+1}(M(x_1-x_0) + (M-1)(x_2-x_1) \\
&+ (M-2)(x_3-x_2) + \ldots + 2(x_{M-1}-x_{M-2}) + (x_M - x_{M-1}))|\mathcal{F}_0))] \\
&= e^{iux_0}\mathbf{E}\left[\exp\left(i(u\frac{M}{M+1})(x_1-x_0)\right)|\mathcal{F}_0\right]\cdot\mathbf{E}\left[\exp\left(i(u\frac{M-1}{M+1})(x_2-x_1)\right)|\mathcal{F}_0\right]\cdot \\
&\cdots\mathbf{E}\left[\exp\left(i(u\frac{1}{M+1})(x_M-x_{M-1})\right)|\mathcal{F}_0\right]. \qquad (5)
\end{aligned}$$

The last step is due to the fact that Lévy processes have independent increments. A Lévy process also has stationary increments, which implies that the increments $x_1 - x_0,\ x_2 - x_1,\ \cdots,\ x_M - x_{M-1}$ are identically distributed, and they are all independent of x_0. Denoting the (identical) characteristic functions of these increments by $\varphi(u,t)$, and substitution of $\varphi(u,t)$ into (5) gives the characteristic function of y given x_0:

$$\phi(u;x_0) = e^{iux_0}\cdot\prod_{j=1}^{M}\varphi\left(u\frac{M+1-j}{M+1}, \frac{T-t_0}{M}\right). \qquad (6)$$

Substitution of characteristic function (6) into (3) results in the *ASCOS pricing formula for European-style geometric Asian options*, with the underlying asset modeled by a Lévy process:

$$v(x_0,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}' Re\left(\phi(\frac{k\pi}{b-a};x_0)e^{-ik\pi\frac{a}{b-a}}\right)V_k, \qquad (7)$$

where

$$V_k = \begin{cases} \frac{2}{b-a}(\chi_k(\log(K),b) - K\psi_k(\log(K),b)), & \text{for a call,} \\ \frac{2}{b-a}(K\psi_k(a,\log(K)) - \chi_k(a,\log(K))), & \text{for a put,} \end{cases}$$

with

$$\begin{aligned}
\chi_k(x_1,x_2) &:= \int_{x_1}^{x_2} e^y\cos\left(k\pi\frac{y-a}{b-a}\right)dy, \\
\psi_k(x_1,x_2) &:= \int_{x_1}^{x_2} \cos\left(k\pi\frac{y-a}{b-a}\right)dy, \qquad (8)
\end{aligned}$$

which are known analytically.

The computational complexity to get the characteristic function for each $u = k\pi/b - a,\ k = 0,\cdots,N-1$ is linear in M, so that $O(MN)$ computations are required. The complexity of the work in (7) is linear in N, so that the total computational complexity of the method is $O(MN)$.

For geometric Asian options there is no error in deriving the characteristic function by (5) and (6). The only errors made are due to the COS formula (7). Detailed error analysis of the COS method for European options can be found in [10]. The ASCOS pricing method for geometric Asian options under Lévy processes is thus expected to have an exponential convergence rate in the number of cosine terms, for all density functions that satisfy $f(y|x) \in C^{\infty}([a,b] \subset \mathbb{R})$.

Remark 2.1 (Black–Scholes Model)**.** *Under the Black-Scholes model, we have that*

$$y := \frac{1}{M+1}\sum_{j=0}^{M} x_j = x_0 + \frac{1}{M+1}\left(M(x_1 - x_0) + (M-1)(x_2 - x_1) + \cdots + (x_M - x_{M-1})\right)$$

is normally distributed:

$$y \sim \mathcal{N}\left(\frac{M}{2}(\mu - 0.5\sigma^2)\Delta t, \frac{M(2M+1)}{6(M+1)}\sigma^2 \Delta t\right). \tag{9}$$

The characteristic function can be derived directly and the resulting computational complexity is therefore only $O(N)$. Note however that a closed form solution is available in this case.

3 ASCOS method for arithmetic Asian options

For arithmetic Asian options, the characteristic function of the arithmetic mean will be derived recursively by Fourier cosine expansions and Clenshaw–Curtis quadrature. The Fourier cosine expansion is used each time step (i.e. at each monitoring date), whereas the Clenshaw–Curtis quadrature rule is used once at the beginning of the computation. In subsection 2.2 the characteristic function of the geometric average was recovered, which was explicitly a function of $x_0 = \log(S_0)$, as $x_0 + ... + x_T$ is a function of x_0, so that the characteristic function took the form $\phi(u; x_0)$. In the present section, we recover the characteristic function of the sum of Lévy asset price increments which is independent of x_0. Therefore we write the characteristic function here in the form $\phi(u)$, rather than $\phi(u; x_0)$.

The payoff function of an arithmetic Asian options reads:

$$v(S,T) \equiv g(S) = \begin{cases} \max(\frac{1}{M+1}\sum_{j=0}^{M} S_j - K, 0), & \text{for a call,} \\ \max(K - \frac{1}{M+1}\sum_{j=0}^{M} S_j, 0), & \text{for a put.} \end{cases} \tag{10}$$

In this section we denote by n_q the number of terms in the Clenshaw–Curtis quadrature (q stands for quadrature).

We first explain the recursion procedure to recover the characteristic function of the arithmetic mean value of the underlying. We denote by:

$$R_j := \log\left(\frac{S_j}{S_{j-1}}\right), \quad j = 1, \cdots, M. \tag{11}$$

For Lévy processes, the increments $R_j,\ j = 1, \cdots, M$ are identically and independently distributed, so that $R_j \stackrel{d}{=} R$. Then $\forall u, j$, we can write $\phi_{R_j}(u) = \phi_R(u)$. Characteristic function $\phi_R(u)$ is known in closed form for different Lévy processes.

We introduce a stochastic process, Y_j, where $Y_1 = R_M$ and for $j = 2, \cdots, M$, we have

$$Y_j := R_{M+1-j} + \log(1 + \exp(Y_{j-1})). \tag{12}$$

We denote by $Z_j := \log(1 + \exp(Y_j)), \forall j$, so that (12) is rewritten as

$$Y_j := R_{M+1-j} + Z_{j-1}. \tag{13}$$

In this setting Y_j admits the form

$$Y_j = \log\left(\frac{S_{M-j+1}}{S_{M-j}} + \frac{S_{M-j+2}}{S_{M-j}} + \cdots + \frac{S_M}{S_{M-j}}\right),$$

and we have that

$$\frac{1}{M+1}\sum_{j=0}^{M} S_j = \frac{(1 + \exp(Y_M))S_0}{M+1}. \tag{14}$$

Convolution scheme (13) has already been used in [6, 2, 16], for example, in combination with other numerical methods, to recover the probability density function of Y_M. Here, however, we will recover the *characteristic function of* Y_M instead, by a forward recursion procedure, which is then used in turn to recover the density of the European-style arithmetic mean of the underlying process in the risk–neutral formula (15). The arithmetic Asian option value is now defined as:

$$v(x_0, t_0) = e^{-r\Delta t}\int_{-\infty}^{\infty} v(y, T) f_{Y_M}(y)dy. \tag{15}$$

By (14), $v(y, T)$ in (15) is of the following form:

$$v(y, T) = \begin{cases} \left(\dfrac{S_0(1 + \exp(y))}{M+1} - K\right)^+, & \text{for a call,} \\ \left(K - \dfrac{S_0(1 + \exp(y))}{M+1}\right)^+, & \text{for a put.} \end{cases}$$

3.1 Recovery of characteristic function

To recover the characteristic function of Y_M, i.e. $\phi_{Y_M}(u)$, we start with Y_1, for which the characteristic function reads:

$$\phi_{Y_1}(u) = \phi_R(u). \tag{16}$$

Then, at time step t_j, $j = 2, \cdots, M$, $\phi_{Y_j}(u)$ can be recovered in terms of $\phi_{Y_{j-1}}(u)$. This is done by application of (13) and the fact that Lévy processes have independent increments. This implies that $\forall j$, R_{M+1-j} and Z_{j-1} are independent, which gives us:

$$\phi_{Y_j}(u) = \phi_{R_{M+1-j}}(u)\phi_{Z_{j-1}}(u) = \phi_R(u)\phi_{Z_{j-1}}(u). \tag{17}$$

From the definition of characteristic function, we have

$$\phi_{Z_{j-1}}(u) = \mathbf{E}[e^{iu\log(1+\exp(Y_{j-1}))}] = \int_{-\infty}^{\infty}(e^x+1)^{iu} f_{Y_{j-1}}(x)dx. \tag{18}$$

To apply the Fourier cosine series expansion to *approximate* the characteristic function, we first truncate the integration range

$$\hat{\phi}_{Z_{j-1}}(u) = \int_a^b (e^x+1)^{iu} f_{Y_{j-1}}(x)dx. \tag{19}$$

If we define the following error

$$\epsilon_T(X) := \int_{\mathbf{R}\backslash[a,b]} f_X(x)dx,$$

then, as $\forall j, u \in \mathbf{R}$,

$$|(e^x+1)^{iu}| = |\cos(u\log(1+e^x)) + i\sin(u\log(1+e^x)))| = 1, \tag{20}$$

the error in (19) can be bounded by:

$$|\int_{\mathbf{R}\backslash[a,b]} (e^x+1)^{iu} f_{Y_{j-1}}(x)dx| \le \int_{\mathbf{R}\backslash[a,b]} f_{Y_{j-1}}(x)dx = \epsilon_T(Y_{j-1}). \tag{21}$$

We apply the Fourier cosine expansion to approximate $f_{Y_{j-1}}(x)$, giving:

$$\begin{aligned}\hat{\phi}_{Z_{j-1}}(u) &= \frac{2}{b-a}{\sum_{l=0}^{N-1}}' Re\left(\hat{\phi}_{Y_{j-1}}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right) \\ &\cdot \int_a^b (e^x+1)^{iu}\cos\left((x-a)\frac{l\pi}{b-a}\right)dx,\end{aligned} \tag{22}$$

where $\hat{\phi}_{Y_{j-1}}$ is an approximation of $\phi_{Y_{j-1}}$.

In this way, $\hat{\phi}_{Z_{j-1}}$ is recovered in terms of $\hat{\phi}_{Y_{j-1}}$. Application of (17) gives us an approximation $\hat{\phi}_{Y_j}(u)$ for any u. Equation (22) can be written in matrix–vector form:

$$\Phi_{j-1} = \mathcal{M}A_{j-1}, \tag{23}$$

using:

$$\begin{aligned}\Phi_{j-1} &= (\Phi_{j-1}(k))_{k=0}^{N-1}, \quad \Phi_{j-1}(k) = \hat{\phi}_{Z_{j-1}}(u_k), \\ u_k &= \frac{k\pi}{b-a}, \ k = 0, \cdots, N-1, \\ \mathcal{M} &= (\mathcal{M}(k,l))_{k,l=0}^{N-1}, \quad \mathcal{M}(k,l) = \int_a^b (e^x+1)^{iu_k}\cos((x-a)u_l)dx, \\ A_j &= \frac{2}{b-a}(A_j(l))_{l=0}^{N-1}, \quad A_j(l) = Re(\hat{\phi}_{Y_{j-1}}(u_l)\exp(-iau_l)).\end{aligned}$$

By the recursion procedure in (17) and (23), the characteristic function, $\phi_{Y_M}(u)$, can be approximated by $\hat{\phi}_{Y_M}(u)$ efficiently. Application of (3) in (15) finally gives us the European-style arithmetic Asian option value:

$$\hat{v}(x,t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\hat{\phi}_{Y_M}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}}\right) V_k, \tag{24}$$

in which

$$V_k = \begin{cases} \frac{2}{b-a}\left(\frac{S_0}{M+1}\chi_k(x^*,b) + (\frac{S_0}{M+1} - K)\psi_k(x^*,b)\right), & \text{for a call,} \\ \frac{2}{b-a}\left((K - \frac{S_0}{M+1})\psi(a,x^*) - \frac{S_0}{M+1}\chi(a,x^*)\right), & \text{for a put.} \end{cases} \tag{25}$$

Functions $\chi_k(x_1,x_2)$ and $\psi_k(x_1,x_2)$ are as in (8), and $x^* = \log(\frac{K(M+1)}{S_0} - 1)$.

3.2 Clenshaw–Curtis quadrature

We discuss the efficient computation of matrix $\mathcal{M}$ in (23). An important feature is that matrix $\mathcal{M}$ remains constant for all time steps $t_j, j = 1, \cdots, M-1$, so that we need to calculate it only once. Its elements are given by:

$$\mathcal{M}(k,l) = \int_a^b (e^x+1)^{iu_k}\cos((x-a)u_l)dx, \quad k,l = 0,\cdots N-1, \tag{26}$$

which can be rewritten in terms of incomplete Beta functions (see appendix A). Here, (26) is approximated numerically by the Clenshaw–Curtis quadrature rule, which is based on an expansion of the integrand in terms of Chebyshev polynomials (as proposed in [8]; more information can be found in [4]).

To use the Clenshaw–Curtis rule for (26), we first change the integration interval from $[a,b]$ to $[-1,1]$

$$\int_a^b (e^x+1)^{iu_k}\cos((x-a)u_l)dx =$$
$$\int_{-1}^1 \frac{b-a}{2}\left(\exp(\frac{b-a}{2}x + \frac{a+b}{2}) + 1\right)^{iu_k} \cos((\frac{b-a}{2}x + \frac{a+b}{2} - a)u_l)dx.$$

The integral can then be approximated as follows

$$\int_a^b (e^x+1)^{iu_k}\cos((x-a)u_l)dx \approx (D^T d)^T y =: w^T y, \tag{27}$$

where D is an $(n_q/2+1)\times(n_q/2+1)$ matrix, whose elements read

$$D(k,n) = \frac{2}{n_q}\cos\left(\frac{(n-1)(k-1)\pi}{n_q/2}\right)\cdot\begin{cases} 1/2, & \text{if } n = \{1, n_q/2+1\}, \\ 1, & \text{otherwise.}\end{cases} \tag{28}$$

The vector d and the elements y_n in $y = \{y_n\}_{n=0}^{n_q/2}$ are defined as:

$$
\begin{aligned}
d &:= \left(1, \frac{2}{(1-4)}, \frac{2}{(1-16)}, \cdots, \frac{2}{(1-(n_q-2)^2)}, \frac{1}{(1-n_q^2)}\right)^T, \\
y_n &:= f(\cos(\frac{n\pi}{n_q})) + f(-\cos(\frac{n\pi}{n_q})),
\end{aligned} \tag{29}
$$

where in our case

$$
f(x) = \frac{b-a}{2}(\exp(\frac{b-a}{2}x + \frac{a+b}{2}) + 1)^{iu_k}\cos((\frac{b-a}{2}x + \frac{a+b}{2} - a)u_l).
$$

For all (k,l), the vector $w = D^T d$ remains the same, so that it needs to be computed only once, for all (k,l). Because $D^T d$ is a type I discrete cosine transform, the computational complexity is $O(n_q \log_2 n_q)$. Elements y_n must be calculated for each pair (k,l), with complexity $O(n_q)$ and the computational complexity, for all (k,l), is therefore $O(n_q N^2)$. When using the Clenshaw–Curtis quadrature rule to compute matrix $\mathcal{M}$ (only once, used for all time steps), the total computational complexity is thus $O(n_q \log_2 n_q) + O(n_q N^2)$.

Furthermore, at each time step t_j, we need $O(N^2)$ computations for the matrix–vector multiplication (23) and $O(N)$ computations to obtain $\hat{\phi}_{Y_j}$ by equation (16) or (17). The computational complexity for this task is thus $O(MN^2)$.

The overall computational complexity of our method for arithmetic Asian option is then $O(n_q \log_2 n_q) + O(n_q N^2) + O(MN^2)$. The number N^2 is in practice much larger than $\log_2 n_q$. The overall complexity is then of order $O((n_q + M)N^2)$.

We will show in the section on error analysis for arithmetic Asian options that for most of the Lévy processes, the Fourier cosine expansion exhibits an exponential convergence rate with respect to N. For the integrand in (26) the Clenshaw–Curtis quadrature converges exponentially with respect to n_q. Therefore, the ASCOS pricing method is an efficient alternative to the method proposed in [12], which requires $O(M\bar{N}^2)$ computations ($\bar{N}$ being the number of points used in the quadrature in [12]), with $\bar{N} > n_q$, and $\bar{N} > N$, for the same level of accuracy. Our pricing method is especially advantageous when the number of monitoring dates, M, increases. The method is summarized below.

ASCOS ALGORITHM: Pricing European-style arithmetic Asian options.

Initialization

- Use Clenshaw–Curtis quadrature (27) to compute $\mathcal{M} = (\mathcal{M}(k,l))$, $k, l = 0, \cdots, N-1$, with $\mathcal{M}$ in (23), (26).
- Compute $\phi_R(u_k), k = 0, \cdots, N-1$.
- Set $\phi_{Y_1}(u_k) = \phi_R(u_k)$.

Main Loop to Recover $\hat{\phi}_{Y_M}$: For $j = 2$ to M,

- Compute the vector Φ_{j-1} with elements $\hat{\phi}_{Z_{j-1}}(u_k)$, $k = 0, \cdots, N-1$ using (23).
- Recover $\hat{\phi}_{Y_j}(u_k)$, $k = 0, \cdots, N-1$ using (17).

Final step:

- Compute $\hat{v}(x_0, t_0)$ by inserting $\hat{\phi}_{Y_M}(u_k)$, $k = 0, \cdots, N-1$ into (24).

3.3 Extensions

In a series of remarks, we now discuss some other generalizations of the ASCOS method. The American-style Asian options generalization will be discussed in a separate part II of this paper.

Remark 3.1 (Continuously–monitored Asian options)**.** *The option values of continuously–monitored arithmetic Asian options, with payoff*

$$v(S,T) = g(S) = \begin{cases} (\frac{1}{T}\int_0^T S(t)dt - K)^+, & \text{for a call,} \\ (K - \frac{1}{T}\int_0^T S(t)dt)^+, & \text{for a put,} \end{cases}$$

can be obtained from discretely–monitored arithmetic Asian option prices by a four-point Richardson extrapolation.

Let $\hat{v}(M)$ denote the computed value of a discretely–monitored Asian option with M monitoring dates. The continuously–monitored Asian option value, denoted by $\hat{v}_\infty$, is approximated by a four-point Richardson extrapolation scheme, as follows:

$$\hat{v}_\infty(d) = \frac{1}{21}(64\hat{v}(2^{d+3}) - 56\hat{v}(2^{d+2}) + 14\hat{v}(2^{d+1}) - \hat{v}(2^d)). \tag{30}$$

The same technique can be applied for continuously monitored geometric Asian options.

Remark 3.2 (Asian options on the harmonic average)**.** *Asian options with a payoff based on the harmonic average, i.e. on $M/(\sum_{j=1}^{M} 1/S_j)$, can be priced in a similar fashion as explained above by the ASCOS method. First, we recover the characteristic function of a variable $y = \log(\sum_{j=1}^{m} S_0/S_j)$ recursively; then* we insert the approximation into the COS pricing formula. We define $\bar{R}_j = \log(S_{j-1}/S_j)$. Starting with $Y_1 = \log(\bar{R}_M)$, we find, $\forall j, u$:

$$\phi_{\bar{R}_j}(u) = \mathbf{E}[e^{iu\log(\frac{S_{j-1}}{S_j})}] = \mathbf{E}[e^{i(-u)\log(\frac{S_j}{S_{j-1}})}] = \phi_{R_j}(-u), \tag{31}$$

with ϕ_{R_j} available in closed form for Lévy processes. For this reason, $\phi_{Y_1}(u)$ is also known analytically.

For $j = 2, \cdots, M$ we then define $Y_j := \bar{R}_{M+1-j} + Z_{j-1}$, where $Z_j := \log(1 + \exp(Y_j))$. In this setting we have $Y_M \equiv \log(\sum_{j=1}^{m} S_0/S_j)$.

Again, $\bar{R}_{M+1-j}$ and Z_{j-1} are independent at each time step, due to the properties of Lévy processes. Therefore

$$\phi_{Y_j}(u) = \phi_{\bar{R}_{M+1-j}}(u)\phi_{Z_{j-1}}(u), \ \forall u,$$

where $\phi_{\bar{R}_{M+1-j}}(u)$ is known analytically from (31) and $\phi_{Z_{j-1}}(u)$ can be recovered, as $\hat{\phi}_{Z_{j-1}}(u)$ from $\hat{\phi}_{Y_{j-1}}(u)$ by Fourier cosine expansions and Clenshaw–Curtis quadrature, as in (22). We thus approximate the characteristic function of Y_M and the fixed strike Asian option value is then given by:

$$\hat{v}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\hat{\phi}_{Y_M}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}}\right) V_k,$$

in which

$$V_k = \begin{cases} \frac{2}{b-a}(MS_0\bar{\chi}_k(x^*, b) - K\psi_k(x^*, b)), & \text{for a call,} \\ \frac{2}{b-a}(K\psi(a, x^*) - MS_0\bar{\chi}(a, x^*)), & \text{for a put,} \end{cases}$$

where $x^* = \log(MS_0/K)$, $\bar{\chi}(x_1, x_2) := \int_{x_1}^{x_2} e^{-y}\cos(k\pi\frac{y-a}{b-a})dy$, and $\psi_k(x_1, x_2)$ is defined in (8).

Finally, the symmetry between floating and fixed–strike Asian options also holds for Asian options on the harmonic average, so that floating strike options can be valued as well.

Remark 3.3 (A special case: the forward contract)**.** *A forward contract, as often encountered in commodity markets, may be defined by the payoff:*

$$g(S) = \frac{1}{M+1}\sum_{j=0}^{M} S_j - K. \tag{32}$$

The contract value then reads

$$\begin{aligned} v(x_0, t_0) &= e^{-r\Delta t}\mathbf{E}\left[\frac{1}{M+1}\sum_{j=0}^{M} S_j - K\right] \\ &= e^{-r\Delta t}\left(\frac{S_0}{M+1}\mathbf{E}[e^{Y_M}] + (\frac{S_0}{M+1} - K)\right), \end{aligned} \tag{33}$$

where the last step follows from (14). The expected value of $\exp(Y_M)$ *can be obtained by a forward recursion procedure. At each monitoring date,* t_j*, we have from (13) that*

$$\mathbf{E}[e^{Y_j}] = \mathbf{E}[e^{R_{M+1-j}}(1+e^{Y_{j-1}})]. \tag{34}$$

For Lévy processes R_{M+1-j} *and* $(1+\exp(Y_{j-1}))$ *are independent and* $R_j \overset{d}{=} R$, $\forall j$*, so that equation (34) reads:*

$$\mathbf{E}[e^{Y_j}] = \mathbf{E}[e^R](1+\mathbf{E}[e^{Y_{j-1}}]),\ \forall j, \tag{35}$$

with $\mathbf{E}[e^{Y_1}] \equiv \mathbf{E}[e^R]$*. The value of* $\mathbf{E}[e^R]$ *reads*

$$\mathbf{E}[e^R] \;=\; \int_{-\infty}^{\infty} e^y f_R(y)dy = {\sum_{k=0}^{N-1}}' Re\left(\phi_R(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}}\right)\chi_k(a,b), \tag{36}$$

where function $\chi_k(x_1,x_2)$ *is defined in (8) and* ϕ_R *is the characteristic function of* R*, which is available for various Lévy processes.*

The $\mathbf{E}[e^R]$*-term needs to be calculated only once, with* $O(N)$ *complexity. In the recursion procedure to get the forward value, we use (35)* $M-1$ *times and (33) once. Therefore, the total computational complexity is* $O(N)+O(M)$*, and exponential convergence is expected for probability density functions belonging to* $C^\infty[a,b]$*.*

4 Error analysis for arithmetic Asian options

Here we give an error analysis of the ASCOS method for arithmetic Asian options. We first discuss, in general terms, three types of error occurring, i.e., the truncation error, ϵ_T, the error of the Fourier cosine expansion, ϵ_F, and the error from the use of the Clenshaw–Curtis quadrature, ϵ_Q.

The truncation error was defined as

$$\epsilon_T(Y_j) := \int_{\mathbf{R}\setminus[a,b]} f_{Y_j}(y)dy,\ \ j=1,\cdots,M, \tag{37}$$

and it decreases as interval $[a,b]$ increases. In other words, for a sufficiently large integration range $[a,b]$, this part of the error won't dominate the overall error of the arithmetic Asian option price.

Regarding the error of the Fourier cosine expansions, we know from [10] that for $f(y|x) \in \mathbf{C}^\infty[a,b]$, it can be bounded by

$$|\epsilon_F(N,[a,b])| \le P^*(N)\exp(-(N-1)\nu),$$

with $\nu > 0$ a constant and a term $P^*(N)$ which varies less than exponentially with respect to N.

When the probability density function has a discontinuous derivative, the error can be bounded by

$$|\epsilon_F(N,[a,b])| \le \frac{\bar{P}^*(N)}{(N-1)^{\beta-1}},$$

where $\bar{P}^*(N)$ is a constant and $\beta \geq 1$.

Error ϵ_F decays thus either exponentially with respect to N, if the density function $f(y|x) \in \mathbf{C}^\infty[a,b]$, or algebraically.

Let us now have a look at the error from the Clenshaw–Curtis quadrature, which we use to approximate

$$I := \int_a^b (e^x + 1)^{iu_k} \cos((x-a)u_l)dx, \tag{38}$$

by $\hat{I} := w^T y$ in (27). In other words, $\epsilon_q = I - \hat{I}$.

According to [17, 19], the Clenshaw–Curtis quadrature rule exhibits an error which can be bounded by $O((2n_q)^{-k}/k)$, for a k–times differentiable integrand. When k is bounded, we have algebraic convergence; otherwise the error converges exponentially with respect to n_q, see also [3]. The integrand in (38) belongs to $C^\infty[a,b]$, as all derivatives are continuous on any interval $[a,b]$, confirming that, for the integrand in (38), we will have exponential convergence with respect to n_q.

4.1 Error propagation in the characteristic functions

The following lemma is used in the error analysis.

Lemma 4.1. *For any random variable, X, and any $u \in \mathbf{R}$, the characteristic function can be bounded by $|\phi_X(u)| \leq 1$.*

Proof. For any X and u, the characteristic function $\phi_X(u)$ is defined by:

$$\phi_X(u) = \mathbf{E}[e^{iuX}] = \int_{-\infty}^{\infty} e^{iux} f(x)dx.$$

We have

$$|\phi_X(u)| \leq \int_{-\infty}^{\infty} |e^{iux}| f(x)dx,$$

and thus:

$$|\phi_X(u)| \leq \int_{-\infty}^{\infty} f(x)dx = 1.$$

□

Now we start with the error analysis, and denote by $\epsilon(\hat{\phi}_{Y_m}(u))$ and $\epsilon(\hat{\phi}_{Z_m}(u))$, $m = 1, \cdots, M$, the errors in $\hat{\phi}_{Y_m}(u)$ and $\hat{\phi}_{Z_m}(u)$, respectively. From (24) the error in the arithmetic Asian option price, denoted by ϵ, is given by

$$\begin{aligned}
\epsilon &= e^{-r\Delta t}\int_{-\infty}^{\infty} v(y,T) f_{Y_M}(y)dy - e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re\left(\hat{\phi}_{Y_M}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}}\right)V_k \\
&= e^{-r\Delta t}\int_{-\infty}^{\infty} v(y,T) f_{Y_M}(y)dy - e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re\left(\phi_{Y_M}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}}\right)V_k \\
&+ e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re\left((\phi_{Y_M}(\frac{k\pi}{b-a}) - \hat{\phi}_{Y_M}(\frac{k\pi}{b-a}))e^{-ik\pi\frac{a}{b-a}}\right)V_k \\
&= \epsilon_{\cos} + e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re\left(\epsilon(\hat{\phi}_{Y_M}(\frac{k\pi}{b-a}))e^{-ik\pi\frac{a}{b-a}}\right)V_k,
\end{aligned}$$

where V_k is known analytically and $\epsilon_{\cos}$ is the error resulting from the use of the COS pricing method. From [10] we know that for a sufficiently large truncation range $[a,b]$, we have $\epsilon_{\cos} = O(\epsilon_F)$ and thus

$$\epsilon = O(\epsilon_F) + e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re\left(\epsilon(\hat{\phi}_{Y_M}(\frac{k\pi}{b-a}))e^{-ik\pi\frac{a}{b-a}}\right)V_k. \quad (39)$$

The remaining part of the error (39) which we need to estimate is $\epsilon(\hat{\phi}_{Y_M}(u))$. This is done by mathematical induction. We first estimate the error in $\hat{\phi}_{Y_1}(u)$ and $\hat{\phi}_{Y_2}(u)$ and then use an induction step to bound the error in $\hat{\phi}_{Y_M}(u)$.

Characteristic function $\phi_{Y_1}(u)$ is known analytically from (16), so that $\epsilon(\hat{\phi}_{Y_1}(u)) = 0,\ \forall u$.

The error in $\hat{\phi}_{Z_1}(u)$ consists of three parts. The first part is the error due to the truncation of the integration range as in (19). The second part is due to the approximation of $f_{Y_1}(x)$ by the Fourier cosine expansion in (22). The third part is due to the use of the Clenshaw–Curtis quadrature rule to approximate the integral in (22). Summing up, we have:

$$\begin{aligned}
\epsilon(\hat{\phi}_{Z_1}(u)) &= \int_{-\infty}^{\infty}(e^x+1)^{iu} f_{Y_1}(x)dx - \int_a^b (e^x+1)^{iu} f_{Y_1}(x)dx \\
&+ \int_a^b (e^x+1)^{iu} f_{Y_1}(x)dx - \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_1}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right) I \\
&+ \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_1}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)(I-\hat{I}) \qquad (40) \\
&= \int_{\mathbf{R}\backslash[\mathbf{a},\mathbf{b}]}(e^x+1)^{iu} f_{Y_1}(x)dx + \epsilon_F + \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_1}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)\epsilon_q.
\end{aligned}$$

The lemma below gives an upper bound for the local error.

Lemma 4.2. *We define by*

$$\begin{aligned} e_j : &= \int_{\mathbf{R}\setminus[\mathbf{a},\mathbf{b}]} (e^x+1)^{iu} f_{Y_j}(x)dx + \epsilon_F \\ &+ \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_j}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)\epsilon_q, \end{aligned} \tag{41}$$

then, with integration range $[a,b]$ *sufficiently wide, we have*

$$|e_j| \le \bar{P}(N,n_q)(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|),\ \forall j,$$

where $\bar{P}(N,n_q) > 0$ *varies less than* ϵ_F *and* ϵ_q*, with respect to* N, n_q*.*

Proof. Application of (21) gives us that, $\forall j, u \in \mathbf{R}$,

$$|\int_{\mathbf{R}\setminus[\mathbf{a},\mathbf{b}]} (e^x+1)^{iu} f_{Y_j}(x)dx| \le \epsilon_T(Y_j), \tag{42}$$

with $\epsilon_T(Y_j)$ defined in (37). Substitution into (41), results in

$$|e_j| \le |\epsilon_T(Y_j)| + |\epsilon_F| + \frac{2}{b-a}\sum_{l=0}^{N-1}{}' |Re\left(\phi_{Y_j}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)||\epsilon_q|.$$

From Lemma 4.1, it follows that, $\forall j, l$, $|\phi_{Y_j}(l\pi/b-a)| \le 1$, and

$$|\exp(-ia\frac{l\pi}{b-a})| = |\cos\left(-a\frac{l\pi}{b-a}\right) + i\sin\left(-a\frac{l\pi}{b-a}\right)| = 1,\ \forall l,$$

so that $|Re\left(\phi_{Y_j}(l\pi/(b-a))\exp(-ial\pi/(b-a))\right)| \le 1,\ \forall j, l.$

For $[a,b]$ sufficiently wide, ϵ_F dominates the expression $\epsilon_F + \epsilon_T$, so that we find, $\forall j$:

$$|e_j| \le \bar{P}(N,n_q)\left(|\epsilon_F| + \frac{2}{b-a}\sum_{l=0}^{N-1}{}' |\epsilon_q|\right) = \bar{P}(N,n_q)\left(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|\right), \tag{43}$$

where $\bar{P}(N,n_q) > 0$ varies less than ϵ_F and ϵ_q with respect to N, n_q. □

Using the notation:

$$\epsilon_L := |\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|, \tag{44}$$

we can write $|e_j| \le \bar{P}(N,n_q)\epsilon_L,\ \forall j$. Application of Lemma 4.2 and (44) to (40) gives

$$|\epsilon(\hat{\phi}_{Z_1}(u))| = |e_1| \le \bar{P}(N,n_q)\epsilon_L.$$

We continue with the error in $\hat{\phi}_{Y_2}(u)$. From (17) we have that

$$\epsilon(\hat{\phi}_{Y_2}(u)) \;=\; \epsilon(\hat{\phi}_{Z_1}(u))\phi_R(u) = e_1\phi_R(u) = e_1\phi_{Y_1}(u),\ \forall u. \tag{45}$$

Applying Lemma 4.1 and Lemma 4.2 to (45) results in

$$|\epsilon(\hat{\phi}_{Y_2}(u))| = |e_1||\phi_{Y_1}(u)| \leq |e_1| \leq \bar{P}(N, n_q)\epsilon_L. \tag{46}$$

Next, we arrive at the induction step, described in the lemma below.

We use the common notation $\epsilon = O(g(a_1, \cdots, a_n))$ to indicate that a $Q > 0$ exists, so that $|\epsilon| = Q|g(a_1, \cdots, a_N)|$ with Q constant or varying less than function $g(\cdot)$ with respect to parameters $a_1, \cdots, a_N$.

Lemma 4.3. *For $m = 3, \cdots, M$, assuming that*

$$\epsilon(\hat{\phi}_{Y_{m-1}}(u)) = \bar{P}(N, n_q) \sum_{j=1}^{(m-1)-1} \phi_{Y_j}(u)e_{(m-1)-j},\ \forall u, \tag{47}$$

where $\bar{P}(N, n_q)$ is a term which varies less than exponentially with respect to N and n_q, then

$$\epsilon(\hat{\phi}_{Y_m}(u)) = O(\sum_{j=1}^{m-1} \phi_{Y_j}(u)e_{m-j}),\ \forall u, \tag{48}$$

and thus

$$|\epsilon(\hat{\phi}_{Y_m}(u))| = O(m-1)\epsilon_L. \tag{49}$$

Proof. We find that for $m = 3, \cdots, M$, and $\forall u$:

$$
\begin{aligned}
& \epsilon(\hat{\phi}_{Z_{m-1}}(u)) \\
= & \int_{-\infty}^{\infty} (e^x+1)^{iu} f_{Y_{m-1}}(x)dx - \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\hat{\phi}_{Y_{m-1}}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)\hat{I} \\
= & \int_{-\infty}^{\infty} (e^x+1)^{iu} f_{Y_{m-1}}(x)dx - \int_a^b (e^x+1)^{iu} f_{Y_{m-1}}(x)dx \\
+ & \int_a^b (e^x+1)^{iu} f_{Y_{m-1}}(x)dx - \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_{m-1}}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right) I \\
+ & \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_{m-1}}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)(I-\hat{I}) \\
+ & \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left((\phi_{Y_{m-1}}(\frac{l\pi}{b-a}) - \hat{\phi}_{Y_{m-1}}(\frac{l\pi}{b-a}))\exp(-ia\frac{l\pi}{b-a})\right)\hat{I} \\
= & \int_{\mathbf{R}\setminus[a,b]} (e^x+1)^{iu} f_{Y_{m-1}}(x)dx + \epsilon_F + \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_{m-1}}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)\epsilon_q \\
+ & \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\epsilon(\phi_{Y_{m-1}}(\frac{l\pi}{b-a}))\exp(-ia\frac{l\pi}{b-a})\right)\hat{I} \\
= & e_{m-1} + \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\epsilon(\phi_{Y_{m-1}}(\frac{l\pi}{b-a}))\exp(-ia\frac{l\pi}{b-a})\right)\hat{I}. \qquad (50)
\end{aligned}
$$

Substitution of (47) into (50) gives

$$
\begin{aligned}
& \epsilon(\hat{\phi}_{Z_{m-1}}(u)) \\
= & e_{m-1} + \bar{P}(N, n_q)\sum_{j=1}^{(m-1)-1} \frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_j}(\frac{l\pi}{b-a})e_{(m-1)-j}\exp(-ia\frac{l\pi}{b-a})\right)\hat{I} \\
= & e_{m-1} + \bar{P}(N, n_q)\sum_{j=1}^{(m-1)-1} e_{(m-1)-j}(\frac{2}{b-a}\sum_{l=0}^{N-1}{}' Re\left(\phi_{Y_j}(\frac{l\pi}{b-a})\exp(-ia\frac{l\pi}{b-a})\right)\hat{I}) \\
= & e_{m-1} + \bar{P}(N, n_q)\sum_{j=1}^{(m-1)-1} e_{(m-1)-j}\hat{\phi}_{Z_j}(u).
\end{aligned}
$$

The error in $\hat{\phi}_{Y_m}(u)$, $\forall u$, is found as

$$\begin{aligned}
\epsilon(\hat{\phi}_{Y_m}(u)) &= \phi_R(u)\epsilon(\hat{\phi}_{Z_{m-1}}(u)) \\
&= \phi_R(u)e_{m-1} + \bar{P}(N,n_q)\sum_{j=1}^{(m-1)-1} e_{(m-1)-j}\phi_R(u)\hat{\phi}_{Z_j}(u) \\
&= \phi_R(u)e_{m-1} + \bar{P}(N,n_q)\sum_{j=1}^{(m-1)-1} e_{(m-1)-j}\hat{\phi}_{Y_{j+1}}(u) \\
&= \phi_{Y_1}(u)e_{m-1} + \bar{P}(N,n_q)\sum_{j=2}^{m-1} e_{m-j}\hat{\phi}_{Y_j}(u) \\
&= O(\sum_{j=1}^{m-1}\phi_{Y_j}(u)e_{m-j}) + O(e_k e_l), \quad k,l \in 1,\cdots,m-1.
\end{aligned}$$

From Lemma 4.2 we see that $|e_j| = O(|\epsilon_F| + |\epsilon_q|)$, $\forall j$, if N and n_q increase simultaneously. Error ϵ_F decays exponentially with respect to N and ϵ_q decays exponentially with respect to n_q, so that e_j decays exponentially and the quadratic term, $e_k e_l$, converges to zero faster than e_j. We thus have that

$$\epsilon(\hat{\phi}_{Y_m}(u)) = O(\sum_{j=1}^{m-1}\phi_{Y_j}(u)e_{m-j}),$$

and application of Lemmas 4.1 and 4.2 gives, $\forall u \in \mathbf{R}$,

$$|\sum_{j=1}^{m-1}\phi_{Y_j}(u)e_{m-j}| \leq \sum_{j=1}^{m-1}|\phi_{Y_j}(u)||e_{m-j}| \leq \bar{P}(N,n_q)(m-1)\epsilon_L,$$

where $\bar{P}(N,n_q)$ varies less than ϵ_F and ϵ_q with respect to N, n_q, respectively. So

$$|\epsilon(\hat{\phi}_{Y_m}(u))| = O((m-1)\epsilon_L), \tag{51}$$

which concludes the proof. □

As a result of the lemma above, we have, $\forall u$,

$$\epsilon(\hat{\phi}_{Y_M}(u)) = O(\sum_{j=1}^{M-1}\phi_{Y_j}(u)e_{m-j}), \tag{52}$$

and

$$|\epsilon(\hat{\phi}_{Y_M}(u))| = O((M-1)\epsilon_L). \tag{53}$$

Remark 4.1 (Error of $\hat{\phi}_{Y_M}$). *Application of (53) and (44) results in*

$$|\epsilon(\hat{\phi}_{Y_M}(u))| = O((M-1)(|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|)), \ \forall u.$$

When the number of monitoring dates, M, increases, larger values of N and n_q are necessary to reach a specified level of accuracy.

Moreover, when a large value of N is necessary for accuracy, we should also increase n_q to control the error. When N and n_q both increase, the expression $|N\epsilon_q|$ converges exponentially to zero[2] *and we have that*

$$|\epsilon(\hat{\phi}_{Y_M}(u))| = O((M-1)(|\epsilon_F| + |\epsilon_q|)), \ \forall u.$$

4.2 Error in the option price

We now focus on the error in the arithmetic Asian option price. After application of (52) in (39) the error reads

$$\epsilon = O(\epsilon_F) + O(\sum_{j=1}^{M-1} e_{m-j} \exp(-r\Delta t) \sum_{k=0}^{N-1}{}' Re(\phi_{Y_j}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}})V_k). \quad (54)$$

When replacing $e^{-r\Delta t}V_k$ (V_k defined in (25)) by the following term:

$$e^{-r\Delta t_j}W_k^j := e^{-r\Delta t_j} \begin{cases} \frac{2}{b-a}(\frac{S_0}{j+1}\chi_k(x^*, b) + (\frac{S_0}{j+1} - K)\psi_k(x^*, b)), & \text{for a call,} \\ \frac{2}{b-a}((K - \frac{S_0}{j+1})\psi(a, x^*) - \frac{S_0}{j+1}\chi(a, x^*)), & \text{for a put,} \end{cases} \quad (55)$$

with $\Delta t_j := j\Delta t/M$, the expression

$$\sum_{j=1}^{M-1} e_{m-j} \exp(-r\Delta t) \sum_{k=0}^{N-1}{}' Re(\phi_{Y_j}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}})V_k, \ \forall j, k,$$

remains of the same order, regarding N and n_q.

The error in (54) therefore satisfies

$$\epsilon = O(\epsilon_F) + O(\sum_{j=1}^{M-1} e_{m-j} e^{-r\Delta t_j} \sum_{k=0}^{N-1}{}' Re(\phi_{Y_j}(\frac{k\pi}{b-a})e^{-ik\pi\frac{a}{b-a}})W_k^j).$$

We now can write for the overall error:

$$\epsilon = O(\epsilon_F) + O(\sum_{j=1}^{M-1} e_{m-j} A(S_0, \Delta t_j)),$$

where $A(S_0, \tau)$ stands for the Asian option value with initial underlying price S_0 and time to maturity τ. Then

$$|\epsilon| = O(|\epsilon_F|) + O(\sum_{j=1}^{M-1} |e_{m-j}| A(S_0, \Delta t_j)).$$

[2]Note that N varies linearly but ϵ_q decays exponentially, so that $N|\epsilon_q|$ also decays exponentially.

By Lemma 4.2 we find

$$|\epsilon| = O(|\epsilon_F|) + O((|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|)\sum_{j=1}^{M-1} A(S_0, \Delta t_j)). \quad (56)$$

Volatility inherent in an Asian option is smaller than that of an equivalent vanilla European option, due to the averaging feature. This makes Asian options cheaper than their plain vanilla equivalents. In other words, with the same maturity, the value of an Asian option, $A(S_0, \tau)$, is less or equal to that of the corresponding vanilla European option, denoted by $E(S_0, \tau)$, written on the same underlying asset. The European option value will be used as upper bound for the corresponding arithmetic Asian option value in (56) and we have:

$$|\epsilon| = O(|\epsilon_F|) + O((|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|)\sum_{j=1}^{M-1} E(S_0, \Delta t_j)). \quad (57)$$

We assume that

$$\max_{j=1,\cdots,M-1} E(S_0, j\Delta t_j) = E(S_0, \Delta t_{j^*}),$$

so that the error in the Asian option price satisfies

$$|\epsilon| = O(|\epsilon_F|) + O((|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|)(M-1)E(S_0, \Delta t_{j^*})). \quad (58)$$

What remains is an upper bound for the plain vanilla European option value, $E(S_0, (M-1)\Delta t_{j^*})$, which is given as follows.

Result 4.1. *The value of a plain vanilla European call option can be bounded by*

$$v_C(S_0, \tau) \leq S_0 e^{-q\tau},$$

with S_0, τ, q the initial underlying price, the time to maturity and the dividend rate, respectively.

The value of a vanilla European put option can be bounded by

$$v_P(S_0, \tau) \leq K e^{-r\tau},$$

with K, r the strike price and the interest rate, respectively.

Summarizing, the error in the arithmetic Asian option with M monitoring dates can be approximated by:

$$|\epsilon| \sim \begin{cases} O((|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|)(M-1)S_0 e^{-q\Delta t_{j^*}}), & \text{for a call,} \\ O((|\epsilon_F| + \frac{2}{b-a}N|\epsilon_q|)(M-1)K e^{-r\Delta t_{j^*}}), & \text{for a put.} \end{cases} \quad (59)$$

For $f(y|x) \in \mathbf{C}^\infty[a,b]$, ϵ_F and ϵ_q converge exponentially with respect to N and n_q, respectively. Therefore, as N and n_q increase, the error in the Asian option price decreases exponentially:

$$|\epsilon| \leq \bar{P}(N, n_q)(\exp(-(N-1)\nu_F) + \exp(-(n_q-1)\nu_q)),$$

where $\bar{P}(N, n_q)$ is a term which varies less than exponentially with respect to N and n_q, and $\nu_F > 0, \nu_q > 0$.

When the probability density function has a discontinuous derivative, the error in the Asian option price converges algebraically.

5 Numerical results

In this section numerical results for Asian options under the Black-Scholes (BS), CGMY [5] and Normal Inverse Gaussian (NIG) [1] models are presented. We use the same parameter sets as in [12], based on three test cases:

- *BS case:* $r = 0.0367, \sigma = 0.17801$.
- *CGMY case:* $r = 0.0367, C = 0.0244, G = 0.0765, M = 7.5515,$ $Y = 1.2945$.
- *NIG case:* $r = 0.0367, \alpha = 6.1882, \beta = -3.8941, \delta = 0.1622$.

These parameters have been obtained by calibration (see [12]). The characteristic functions for these processes are presented in appendix B. In all numerical examples we set time to maturity $T - t_0 = 1$, and $S_0 = 100$. Strike price, K, and the number of monitoring dates, M, vary among the different experiments.

MATLAB 7.7.0 is used and the CPU is an Intel(R) Core(TM)2 Duo CPU E6550 (@ 2.33GHz Cache size 4MB). CPU time is recorded in seconds.

The absolute error that we report below is defined as the absolute value of the difference between the approximate solution at t_0 and S_0, and a reference value which is computed by the ASCOS method with a large number of terms in the Fourier cosine expansions. The values have also been compared to reference values in the literature. With our own reference values however we can compare up to a higher accuracy.

5.1 Geometric Asian options

First of all, we confirm the exponential convergence of the ASCOS method for geometric Asian options under the Black–Scholes model, for which an analytic result is available, in Figure 1. For increasing N-values the error decreases exponentially.

The performance of the ASCOS pricing method for the NIG and CGMY test cases is presented in Table 1. Geometric Asian call option prices with 12, 50 and 250 monitoring dates are shown. Reference values are taken from ASCOS computations with $N = 4096$. In all examples our method also gives the same option prices, up to a basis point, as those presented in [12].

From Table 1 we see that the option prices have converged up to basis point precision with $N = 128$ and $N = 512$, respectively, for the NIG and CGMY test cases. Exponential convergence is observed for these Lévy processes and, as a result, geometric Asian options can be priced within milli-seconds by the

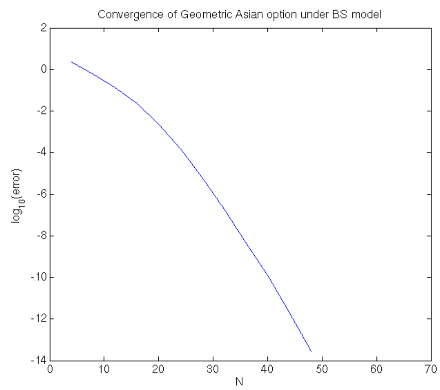

Figure 1: Convergence of geometric Asian options under the BS model with $M = 250, S_0 = 100, K = 90$.

NIG model				
M		$N = 64$	$N = 128$	$N = 192$
12	abs.error	1.42e-04	2.81e-05	1.33e-08
	CPU time	4.9e-04	7.7e-04	8.3e-04
50	abs.error	1.23e-04	3.07e-05	1.24e-08
	CPU time	9.3e-04	1.4e-03	2.1e-03
250	abs.error	1.13e-04	3.13e-05	2.11e-08
	CPU time	3.1e-03	5.8e-03	8.2e-03
CGMY model				
M		$N = 256$	$N = 512$	$N = 1024$
12	abs.error	2.1e-03	9.87e-06	6.27e-11
	CPU time	2.7e-03	4.1e-03	9.9e-03
50	abs.error	1.20e-02	1.24e-05	6.71e-11
	CPU time	1.2e-02	1.7e-02	4.3e-02
250	abs.error	1.16e-02	3.65e-05	3.84e-11
	CPU time	0.050	0.10	0.22

Table 1: Convergence of geometric Asian options for the NIG and CGMY test cases with $S_0 = 100, K = 110$.

ASCOS method. In a comparison with the results in [12], we found that our timing results are approximately 100 times faster for the NIG test case and 20 times for the CGMY case.

Table 2 presents the convergence behavior when we approximate continuously–monitored geometric Asian options ($M = \infty$) by discretely–monitored geometric Asian options combined with the 4–point Richardson extrapolation (30). Here d is as defined in (30), that is, discretely–monitored Asian options with $2^d, 2^{d+1}, 2^{d+2}, 2^{d+3}$ monitoring dates are used to approximate the continuously–monitored Asian options. The reference values have been obtained by employing the ASCOS method with $N = 4096, M = 512$.

	NIG		CGMY	
d	abs.error	CPU time	abs.error	CPU time
1	3.78e-04	0.0018	2.06e-04	0.0120
2	5.92e-05	0.0023	1.21e-04	0.0247
3	3.31e-05	0.0052	5.71e-05	0.0499

Table 2: Convergence of geometric Asian options for the NIG and CGMY cases with $S_0 = 100, K = 110$. For the NIG model we use $N = 128$, for the CGMY model $N = 512$.

The discretely–monitored Asian prices with 4, 8, 16 and 32 monitoring dates, i.e., $d = 2$ have converged to the continuously–monitored Asian price in Table 2. We need approximately 2 and 25 milliseconds to get the continuously–monitored Asian option prices for the NIG and CGMY test cases, respectively. As compared to [12], we achieve a speedup of 20 for the NIG test and the CPU time for the CGMY case is approximately one–third of that in [12].

5.2 Arithmetic Asian options

First, the error in an arithmetic Asian option under the Black–Scholes model with 50 monitoring dates is presented in Figure 1, where at the y–axis we have the logarithm (basis 10) of the absolute error in the Asian option price and at the x–axis the value of index d, where $N = 64d$ and $n_q = 100d$. The reference value is obtained by the ASCOS method with $N = 4096$ (the resulting values are as in [12]). Exponential convergence in the arithmetic Asian price with respect to N and n_q, increasing simultaneously, is observed in Figure 2.

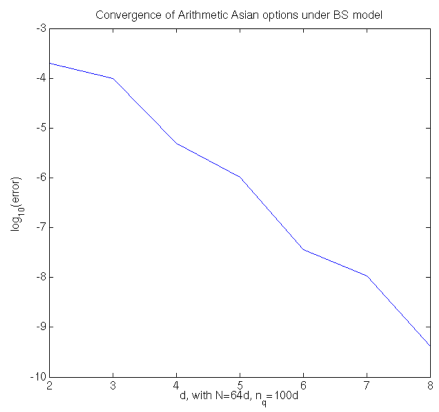

Figure 2: Convergence of arithmetic Asian options for the BS test case with $M = 50, S_0 = 100, K = 90$.

Table 3 then presents the convergence and the CPU time of an arithmetic

Asian option for the NIG test case with $M = 12$ and $M = 50$ (monthly and weekly-monitored, respectively). Reference values are again obtained by setting $N = 4096$. Exponential convergence can be seen in Table 3, as the error decreases exponentially, when n_q and N increase linearly. The Asian options for $M = 50$ converge up to basis point precision with $N = 128$ and $n_q = 200$, where the CPU time is approximately 2.5 seconds. Higher order accuracy can be achieved as N and n_q increase, but the CPU time grows with respect to $n_q N^2$.

The speed of convergence is *not* influenced significantly by an increase in the number of monitoring dates, M, neither is the CPU time.

M	time and error	$N = 128$ $n_q = 200$	$N = 256$ $n_q = 400$	$N = 384$ $n_q = 600$
12	abs.error	2.0e-3	1.71e-4	5.16e-6
	CPU time	2.41	15.13	46.09
50	abs.error	2.26e-4	6.94e-5	2.17e-6
	CPU time	2.43	15.16	46.22

Table 3: Convergence of arithmetic Asian options for the NIG test case with $S_0 = 100, K = 110$.

Furthermore, the convergence remains robust when the number of monitoring dates increases, which gave rise to convergence difficulties for other pricing methods. A larger number of Fourier cosine terms is required (thus resulting in a larger CPU time) as compared to monthly or weekly-monitored examples. This can be seen in Table 4, where arithmetic Asian options for the NIG and CGMY test cases, with 250 monitoring dates (daily-monitored), are presented. With $N = 256, n_q = 400$ and $N = 320, n_q = 500$, we find converged option prices (up to basis point precision) for the NIG and CGMY cases, respectively.

Due to the exponential convergence rate of the Clenshaw–Curtis quadrature and the Fourier cosine expansion, the number of terms needed remains limited, which influences the CPU time positively. In [12] an accuracy of $O(10^{-3})$ was reached in approximately 210 seconds for the same CGMY test case with $M = 250$. The ASCOS method reaches $O(10^{-4})$ accuracy in approximately 27 seconds.

A comparison of the CPU times in Tables 3 and 4 shows that the ASCOS CPU time does not increase from $M = 12$ to $M = 250$, because the quadrature rule, which dominates the CPU time, is used only once. This is especially beneficial for pricing continuously-monitored Asian options.

In Table 5 we finally compute continuously–monitored arithmetic Asian call options under the NIG model with $S_0 = 100$ and different strikes, by the repeated Richardson extrapolation based on discretely–monitored arithmetic Asian call options (30). The option prices converge somewhat slower with respect to parameter d, as compared to the geometric Asian case. However, the CPU time of the ASCOS method does not increase when d increases, so that

NIG	time and error	$N = 128$ $n_q = 200$	$N = 256$ $n_q = 400$	$N = 512$ $n_q = 800$
	abs.error	7.8e-3	9.33e-5	6.94e-7
	CPU time	2.42	15.23	104.28
CGMY	time and error	$N = 256$ $n_q = 400$	$N = 320$ $n_q = 500$	$N = 384$ $n_q = 600$
	abs.error	1.6e-3	4.69e-4	8.96e-5
	CPU time	14.92	26.61	44.41

Table 4: Convergence of arithmetic Asian options for the NIG and CGMY test cases with $S_0 = 100, K = 110, M = 250$.

we can use a larger value for d, for instance $d = 6$ ($M = 64, 128, 256, 512$) and obtain accurate results.

d	$K = 90$		$K = 100$	
	Option value	CPU time	Option value	CPU time
4	12.6748	60.05	5.1191	60.01
5	12.6744	60.13	5.1186	59.94
6	12.6743	60.35	5.1185	60.17

Table 5: Convergence of arithmetic Asian options under the NIG model with $S_0 = 100, N = 256, n_q = 400$.

6 Conclusions

In this article, we proposed an efficient pricing method for European-style Asian options, the ASCOS method, based on Fourier cosine expansions and Clenshaw–Curtis quadrature. The method performs well for different Lévy processes, different parameter values and different numbers of Asian option monitoring dates. The method is accompanied by a detailed error analysis, giving evidence for an exponential convergence rate for geometric and arithmetic Asian options. Due to the exponential convergence, our pricing method is highly efficient and significant speedup has been achieved compared to competitor pricing methods.

The ASCOS method performs in a robust manner when the number of monitoring dates increases, and, interestingly, the CPU time does not increase significantly. This makes the pricing method especially advantageous for weekly- and even daily-monitored arithmetic Asian options, as well as for continuously–monitored Asian options whose value is approximated by discretely–monitored Asian options in combination with Richardson extrapolation.

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A Beta function formation

After some manipulations with symbolic software, we find that integral (26) can be written in a form with incomplete Beta functions, as follows

$$\begin{aligned} & \int_a^b (e^x+1)^{i\frac{k\pi}{b-a}} \cos((x-a)\frac{l\pi}{b-a})dx \\ = \;& \frac{1}{2}e^{-\frac{l(ia+\pi)}{d}}(e^{\frac{2ial}{d}}(-\beta(-e^a,-\frac{il}{d},1+\frac{ik}{d})+\beta(-e^b,-\frac{il}{d},1+\frac{ik}{d})) \\ + \;& e^{\frac{2l\pi}{d}}(-\beta(-e^a,\frac{il}{d},1+\frac{ik}{d})+\beta(-e^b,\frac{il}{d},1+\frac{ik}{d}))), \end{aligned} \tag{60}$$

where $i=\sqrt{-1}$, $d=\frac{b-a}{\pi}$ and $\beta(x,y,z)$ is the incomplete Beta function

$$\beta(x,y,z) = \int_0^x t^{y-1}(1-t)^{z-1}dt.$$

The computation of the incomplete Beta functions in (60) is involved with these complex-valued arguments.

B Lévy processes and characteristic functions

One problem with the GBM model is that it is not able to reproduce the volatility skew or smile present in most financial markets. Over the past few years it has been shown that several exponential Lévy models are, at least to some extent, able to reproduce the skew or smile. One particular model we will consider is the *CGMY model* [5]. The underlying Lévy process is characterized by the triple $(\mu, \sigma, \nu_{\text{CGMY}})$, where the Lévy density is specified as:

$$\nu_{\text{CGMY}}(x) = \begin{cases} C\dfrac{\exp(-G|x|)}{|x|^{1+Y}} & \text{if} \quad x<0 \\ C\dfrac{\exp(-M|x|)}{|x|^{1+Y}} & \text{if} \quad x>0. \end{cases} \tag{61}$$

with parameters C, G, M and Y. Conveniently, the characteristic function of the log-asset price can be found in closed-form as:

$$\phi(u; x_0) = \exp\left(iu(x_0 + \mu t) - \frac{1}{2}u^2\sigma^2 t\right)\varphi_{\text{CGMY}}(u,t), \tag{62}$$

with $x_0 = \log(S_0)$ and

$$\varphi_{\text{CGMY}}(u,t) = \exp\left(tC\Gamma(-Y)\big((M - iu)^Y - M^Y + (G + iu)^Y - G^Y\big)\right),$$

where $\Gamma(x)$ is the gamma function.

When $C = 0$ the model reduces to the *GBM model*.

The *Normal Inverse Gaussian* (NIG) process [1] is a variance-mean mixture of a Gaussian distribution with an inverse Gaussian. The pure jump characteristic function of the NIG model reads

$$\varphi_{\text{NIG}}(u,t) = \exp\left(t\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right)\right),$$

with $\alpha, \delta > 0$ and $\beta \in (-\alpha, \alpha - 1)$. The α-parameter controls the steepness of the density; β is a skewness parameter: $\beta > 0$ implies a density skew to the right, $\beta < 0$ a density skew to the left, and $\beta = 0$ implies the density is symmetric around 0. δ is a scale parameter in the sense that the rescaled parameters $\alpha \to \alpha\delta$ and $\beta \to \beta\delta$ are invariant under location-scale changes of x.