

# An equity–interest rate hybrid model with stochastic volatility and the interest rate smile

**Lech A. Grzelak**

CWI – National Research Institute for Mathematics and Computer Science,  
PO Box 94079, 1090 GB Amsterdam, The Netherlands;

email: l.a.grzelak@cwi.nl

and

Rabobank International, Croeselaan 18, 3521 CB Utrecht, The Netherlands

**Cornelis W. Oosterlee**

CWI – National Research Institute for Mathematics and Computer Science,  
PO Box 94079, 1090 GB Amsterdam, The Netherlands;

email: c.w.oosterlee@cwi.nl

and

Delft Institute of Applied Mathematics, Delft University of Technology,  
Mekelweg 4, 2628CD Delft, The Netherlands

*We define an equity–interest rate hybrid model in which the equity part is driven by Heston stochastic volatility and the interest rate is generated by the displaced diffusion stochastic volatility LIBOR market model. We assume a nonzero correlation between the main processes. A number of approximations lead to an approximating model which falls within the class of affine processes described by Duffie, for which we then provide the corresponding forward characteristic function. By using the appropriate change of measure and freezing the LIBOR rates, the dimension of the corresponding pricing partial differential equation can be greatly reduced. We discuss the accuracy of the approximations and the efficient calibration in detail. Finally, using experiments, we show the effect of the correlations and interest rate smile/skew on typical equity–interest rate hybrid product prices. This approximate hybrid model can be evaluated for a whole strip of strikes for equity plain vanilla options in milliseconds.*

## 1 INTRODUCTION

Over the past decade the Heston equity model (Heston (1993)) with deterministic interest rates has established itself as one of the benchmark models for pricing equity derivatives. The assumption of deterministic interest rates in the Heston model is rather harmless when equity products with a short time to maturity need to be priced. For long-term equity contracts or equity–interest rate hybrid products, however, a

## 2 L. A. Grzelak and C. W. Oosterlee

deterministic interest rate might be unacceptable. The extension of the Heston model with stochastic interest rates is established for basic short-rate processes, like Hull–White or multifactor models, in, for example, Grzelak *et al* (2011) and Grzelak and Oosterlee (2011). These interest rate models cannot generate implied volatility smiles or skews as commonly observed in the interest rate market. They can therefore mainly be used for long-term equity options, or for fairly simple equity–interest rate hybrid products. For hybrid products that are exposed to the interest rate smile, more involved models are required. In this paper we develop such a hybrid model.

For several years the lognormal LIBOR market model (LMM) (see Brace *et al* (1997); Jamshidian (1997); and Miltersen *et al* (1997)) has been an established benchmark for interest rate derivatives. Without enhancements, this stochastic differential equation (SDE) system is unable to model strike-dependent volatilities of fixed income derivatives such as caps and swaptions. An important step in the modeling process came with the local volatility (Andersen and Andreasen (2000)) and stochastic volatility extensions (Andersen and Brotherton-Ratcliffe (2005); Andersen and Andreasen (2002); and Rebonato (2002)), with which a model can be fitted reasonably well to market data while guaranteeing the model’s stability.

A number of stochastic volatility extensions of the LMM have been presented in the literature (see, for example, Brigo and Mercurio (2007)). The model on which our work is based is the displaced diffusion–stochastic volatility (DD–SV) model developed by Andersen and Andreasen (2002). It was Piterbarg (2005) who connected the time-dependent model volatilities and skews for LIBOR and swap rates to the market implied quantities. The concept in Piterbarg (2005) of effective skew and effective volatility enables the calibration of the volatility smiles for a grid of swaptions.

In this paper we develop an equity–interest rate hybrid model with equity modeled by the Heston model and the interest rate driven by the LMM (specifically, by the DD–SV model (Andersen and Andreasen (2002))). The model is relevant for pricing equity–interest rate hybrid products that are exposed to the interest rate smile. In practice, the equity calibration is performed with an *a priori* calibrated interest rate model. For the remaining calibration task a very efficient and fast model evaluation is mandatory. We develop and present such a model here.

By changing measures from the risk-neutral to the forward measure, associated with the zero-coupon bond as the numeraire, the dimension of the approximating characteristic function can be reduced significantly. This – along with freezing the LIBOR rates and appropriate linearizations of the nonaffine terms arising in the corresponding instantaneous covariance matrix – is key to the efficient model evaluation and pricing of equity options of European type. For a whole strip of strikes the approximate hybrid model developed can be evaluated for equity plain vanilla options in just milliseconds.

The approximating model, denoted by H1–LMM, is very practical for calibration, because of the high speed of the calculations involved, the available characteristic function and existing Fourier inverse algorithms (see Fang and Oosterlee (2008)). Moreover, for typical instruments on which the calibration is carried out, it appears to be very precise.

However, the approximate model used for calibration cannot be a new stand-alone model for hybrid products, as the approximations introduced modify the Girsanov kernel when changing between the measures. This means that, in fact, the approximation we have derived is not guaranteed to be arbitrage-free.

The paper is organized as follows. First, in Section 2, we discuss the generalization of the Heston model and provide details about the DD–SV interest rate model. In Section 3 the dynamics for the equity forward model are derived. An approximation leading to the corresponding characteristic function is developed in Section 4. Numerical experiments, in which the accuracy of the approximations is checked, are presented in Section 5.

## 2 THE EQUITY AND INTEREST RATE MODELS

### 2.1 The Heston model and extensions

With state vector  $X(t) = [S(t), \xi(t)]^T$ , under the risk-neutral pricing measure, the Heston stochastic volatility model (Heston (1993)) is specified by the following system of SDEs:

$$\left. \begin{aligned} \frac{dS(t)}{S(t)} &= r(t) dt + \sqrt{\xi(t)} dW_x(t), & S(0) > 0 \\ d\xi(t) &= \kappa(\bar{\xi} - \xi(t)) dt + \gamma\sqrt{\xi(t)} dW_\xi(t), & \xi(0) > 0 \end{aligned} \right\} \quad (2.1)$$

with  $r(t)$  a deterministic time-dependent interest rate, a correlation  $dW_x(t) dW_\xi(t) = \rho_{x,\xi} dt$ , and  $|\rho_{x,\xi}| < 1$ . The variance process,  $\xi(t)$ , of the stock,  $S(t)$ , is a mean-reverting square-root process, in which  $\kappa > 0$  determines the speed of adjustment of the volatility toward its theoretical mean,  $\bar{\xi} > 0$ , and  $\gamma > 0$  is the second-order volatility, ie, the variance of the volatility.

As already indicated in Heston (1993), under the log transform for the stock,  $x(t) = \log S(t)$ , the model belongs to the class of affine processes (see Duffie *et al* (2000)). For  $\tau = T - t$ , the characteristic function (CF) is therefore given by:

$$\phi_H(u, X(t), \tau) = \exp(A(u, \tau) + B_x(u, \tau)x(t) + B_\xi(u, \tau)\xi(t)) \quad (2.2)$$

where the complex-valued functions  $A(u, \tau)$ ,  $B_x(u, \tau)$  and  $B_\xi(u, \tau)$  are known in closed form (see Heston (1993)).

4 L. A. Grzelak and C. W. Oosterlee

The CF is explicit, but its inverse also has to be found for pricing purposes. Because of the form of the CF, we cannot get it analytically, and a numerical method for integration has to be used (see, for example, Carr and Madan (1999), Fang and Oosterlee (2008), Lee (2004) and Lewis (2001) for Fourier methods).

Since a deterministic interest rate is not sufficient for our pricing purposes, we relax this assumption and assume the rates to be stochastic. A first extension of the framework can be made by defining a correlated short-rate process,  $r(t)$ , of the following form:

$$dr(t) = \mu_r(t, r(t)) dt + \sigma_r(t, r(t)) dW_r(t), \quad r(0) > 0$$

with  $dW_r(t) dW_x(t) = \rho_{x,r} dt$ . Depending on the functions  $\mu_r(t, r(t))$  and  $\sigma_r(t, r(t))$ , many different interest rate models are available. Popular single-factor versions include the Hull and White (1996), Cox–Ingersoll–Ross (Cox *et al* (1985)) and Black and Karasinski (1991) models. Multifactor models arise by extending the single-factor processes with additional sources of randomness (see Brigo and Mercurio (2007) for a survey).

Clearly, even for nonzero correlation between the equity process and the interest rates, the extension of the plain Heston model with an additional (correlated) stochastic interest rate process is rather straightforward. However, the standard techniques for determining the corresponding CF are not applicable (the model is not affine, see Duffie *et al* (2000)), so model calibration can become a cumbersome task.

Previously, in Grzelak and Oosterlee (2011), we proposed linear approximations for the nonaffine terms in the instantaneous covariance matrix related to a short-rate-based hybrid model in order to determine a CF. With such a short-rate model, however, the interest rate can only be calibrated well to at-the-money products like caps and swaptions. Those models can therefore only be used for relatively basic hybrid products, which are insensitive to the interest rate smile.

When developing a more advanced hybrid model, moving away from the short-rate processes to the market models, the main difficulty is linking the discrete tenor LIBOR rates,  $L(t, T_i, T_j)$  for  $T_i < T_j$ , to the continuous equity process,  $S(t)$ . This issue is addressed here.

In the next section we present the main concepts of the market models.

## 2.2 The market model with stochastic volatility

Here, we build the basis for the interest rate process in the Heston hybrid model.

For a given set of maturities  $\mathcal{T} = \{T_0, T_1, T_2, \dots, T_N\}$  with a tenor structure  $\tau_k = T_k - T_{k-1}$  for  $k = 1, \dots, N$ , we define  $P(t, T_i)$  to be the price of a zero-coupon treasury bond maturing at time  $T_i$  ( $\geq t$ ), with face-value €1 and the forward

LIBOR rate  $L_k(t) := L(t, T_{k-1}, T_k)$ :

$$L(t, T_{k-1}, T_k) \equiv \frac{1}{\tau_k} \left( \frac{P(t, T_{k-1})}{P(t, T_k)} - 1 \right) \quad \text{for } t < T_{k-1} \quad (2.3)$$

For modeling the LMM, we take the DD–SV model. The LIBOR rate  $L_k(t)$  is defined under its natural measure by the following system of SDEs:

$$\left. \begin{aligned} dL_k(t) &= \sigma_k(t)(\beta_k(t)L_k(t) + (1 - \beta_k(t))L_k(0))\sqrt{V(t)} dW_k^k(t), & L_k(0) > 0 \\ dV(t) &= \lambda(V(0) - V(t)) dt + \eta\sqrt{V(t)} dW_V^k(t), & V(0) > 0 \end{aligned} \right\} \quad (2.4)$$

with:

$$\left. \begin{aligned} dW_i^k(t) dW_j^k(t) &= \rho_{i,j} dt & \text{for } i \neq j \\ dW_V^k(t) dW_i^k(t) &= 0 \end{aligned} \right\} \quad (2.5)$$

where  $\sigma_k(t)$  determines the level of the volatility smile. Parameter  $\beta_k(t)$  controls the slope of the volatility smile, and  $\lambda$  determines the speed of mean reversion for the variance and influences the speed at which the volatility smile flattens as the swaption expiry increases (Piterbarg (2005)). Parameter  $\eta$  determines the curvature of the smile. Subscript “ $i$ ” and superscript “ $j$ ” in  $dW_i^j(t)$  indicate the associated process and the corresponding measure, respectively. Throughout this paper we assume that the DD–SV model in (2.4) is already in the effective parameter framework developed in Piterbarg (2005). This means that approximate time-homogeneous parameters are used instead of time-dependent parameters. For this reason we set  $\beta_k(t) \equiv \beta_k$  and  $\sigma_k(t) \equiv \sigma_k$ .

An important feature, which will be shown in the next section, is that it is convenient in our framework to work under the  $T_N$ -terminal measure associated with the last zero-coupon bond,  $P(t, T_N)$ .

By taking:

$$\phi_k(t) = \beta_k L_k(t) + (1 - \beta_k)L_k(0) \quad (2.6)$$

under the  $T_N$ -terminal measure and for  $k < N$ , the LIBOR dynamics are given by:

$$\left. \begin{aligned} dL_k(t) &= -\phi_k(t)\sigma_k V(t) \sum_{j=k+1}^N \frac{\tau_j \phi_j(t) \sigma_j}{1 + \tau_j L_j(t)} \rho_{k,j} dt + \sigma_k \phi_k(t) \sqrt{V(t)} dW_k^N(t) \\ dV(t) &= \lambda(V(0) - V(t)) dt + \eta\sqrt{V(t)} dW_V^N(t) \end{aligned} \right\} \quad (2.7)$$

with:

$$\left. \begin{aligned} dW_i^N(t) dW_j^N(t) &= \rho_{i,j} dt & \text{for } i \neq j \\ dW_k^N(t) dW_V^N(t) &= 0 \end{aligned} \right\} \quad (2.8)$$

In the DD–SV model in (2.4) the change of measure does not affect the drift in the process for the stochastic variance,  $V(t)$ . This is due to the assumption of independence between the variance process,  $V(t)$ , and the LIBORs,  $L_k(t)$ . Although a generalization to a nonzero correlation is possible (see Wu and Zhang (2008)), it is not strictly necessary. The model, by the displacement construction and the stochastic variance, already provides a satisfactory fit to market data.

Note that, for  $k = N$ , the dynamics for  $L(t, T_{k-1}, T_k)$  do not contain a drift term (as LIBOR  $L(t, T_{N-1}, T_N)$  is a martingale under the  $T_N$  measure).

When changing measures for the stock process from the risk-neutral to the  $T_N$ -forward measure, one needs to find the form for the zero-coupon bond,  $P(t, T_N)$ . By the recursive Equation (2.3) it is easy to find the following expression for the last bond (needed in Equation (3.3)):

$$P(t, T_N) = P(t, T_{m(t)}) \left( \prod_{j=m(t)+1}^N (1 + \tau_j L(t, T_{j-1}, T_j)) \right)^{-1} \quad (2.9)$$

with  $m(t) = \min(k : t \leq T_k)$  (empty products in (2.9) are defined to be equal to 1). The bond  $P(t, T_N)$  in (2.9) is fully determined by the LIBOR rates  $L_k(t)$ ,  $k = 1, \dots, N$ , and the bond  $P(t, T_{m(t)})$ . Although the LIBORs  $L_k(t)$  are defined in (2.7), the bond  $P(t, T_{m(t)})$  is not yet well-defined in the current framework.

In the following subsection we discuss possible interpolation methods for the short-dated bond  $P(t, T_{m(t)})$ .

### 2.3 Interpolations of short-dated bonds

Let us consider the discrete tenor structure  $\mathcal{T}$  and the LIBOR rates  $L_k(t)$  as defined in (2.3). As already indicated in Brace *et al* (1997) and Musiela and Rutkowski (1997), the main problem with market models is that they do not provide continuous time dynamics for any bond in the tenor structure. Therefore, it is rather difficult, without additional assumptions, to define a short-rate process,  $r(t)$ , which can be used in combination with the Heston model for equity.

In this section we discuss how to extend the market model so that the no-arbitrage conditions are met and the bonds  $P(t, T_i)$  for  $t \notin \mathcal{T}$  are well-defined.

We start with the interpolation technique introduced in Schlögl (2002). In this approach a linear interpolation that produces a piecewise deterministic short rate for  $t \in (T_{m(t)-1}, T_{m(t)})$  is used. The method is equivalent to the assumption of a zero volatility for all zero-coupon bonds,  $P(t, T_i)$ , maturing at a (future) date in the tenor structure  $\mathcal{T}$ , ie,  $t \leq T_{m(t)}$ , the zero-coupon bond  $P(t, T_{m(t)})$  is well-defined and arbitrage-free (see Schlögl (2002) and Beveridge and Joshi (2010)) if:

$$P(t, T_{m(t)}) \stackrel{\text{def}}{=} (1 + (T_{m(t)} - t)L(T_{m(t)-1}, T_{m(t)}))^{-1} \quad \text{for } T_{m(t)-1} < t < T_{m(t)} \quad (2.10)$$

Representation (2.10) satisfies the main features of the zero-coupon bond, ie, for  $t \rightarrow T_{m(t)}$ , the bond  $P(t, T_{m(t)}) \rightarrow 1$ . Since Equation (2.10) implies a zero volatility interpolation for the intermediate intervals, a deterministic interest rate is assumed for intermediate time points,  $T_{m(t)-1} < t < T_{m(t)}$ .

However, the assumption of a locally deterministic interest rate in short-dated bonds may be unsatisfactory, for example, for pricing path-sensitive products in which the payment does not occur at the prespecified dates,  $T_i \in \mathcal{T}$ . In such a case, one can use an interpolation that incorporates some internal volatility. An alternative arbitrage-free interpolation for zero-coupon bonds is, for example, given by:

$$P(t, T_{m(t)}) \stackrel{\text{def}}{=} (1 + (T_{m(t)} - t)\psi(t))^{-1} \quad \text{for } t \leq T_{m(t)} \quad (2.11)$$

with:

$$\psi(t) = \alpha(t)L_{m(t)}(T_{m(t)-1}) + (1 - \alpha(t))L_{m(t)+1}(t)$$

and where  $\alpha(t)$  is a (chosen) deterministic function that controls the level of the volatility in the short-dated bonds.

More details on interpolation approaches can be found in Schlögl (2002), Piterbarg (2004), Davis and Mataix-Pastor (2009) and Beveridge and Joshi (2010).

**REMARK 2.1** When calibrating the equity–interest rate hybrid model, the interest rate part is usually calibrated to market data, independent of the equity part. Afterward, the calibrated interest rate model is combined with the equity component. With suitable correlations imposed, the remaining parameters are then determined. Obviously, in the last step the hybrid parameters are determined by calibration to equity option values. By assuming that the equity maturities,  $T_i$ , are defined to be the same dates as the zero-coupon bonds in the LMM, there is no need for advanced zero-coupon bond interpolations. The interpolation routines are, however, often used when pricing the hybrids themselves. The hybrid product pricing is typically performed with a short-step Monte Carlo simulation, for which the assumption of a constant short-term interest rate may not be satisfactory (especially if the hybrid payments occur at dates that are not specified in the tenor structure  $\mathcal{T}$ ).

### 3 THE HYBRID HESTON–LIBOR MARKET MODEL

In this section we present the full-scale hybrid model.

As indicated in, for example, Morini and Mercurio (2007), when pricing interest rate derivatives the usual reference measure is the spot measure  $\mathbb{Q}$  associated with a directly rebalanced bank account numeraire  $B(t)$ . However, when dealing with an equity–interest rate hybrid model, after calibrating the interest rate part, one needs to price the European equity options in order to determine the unknown equity parameters.

8 L. A. Grzelak and C. W. Oosterlee

The price of a European call option is given by:

$$\Pi(t) = B(t)\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{B(T_N)}(S(T_N) - K)^+ \mid \mathcal{F}_t\right) \quad \text{with } t < T_N \quad (3.1)$$

where  $K$  is the strike,  $S(T_N)$  is the stock price at time  $T_N$ ,  $\mathcal{F}_t$  is the filtration and  $B(T_N)$  is the numeraire. Since the money-saving account,  $B(T_N)$ , is a stochastic quantity, the joint distribution of  $1/B(T_N)$  and  $S(T_N)$  is required to determine the value in (3.1). This may be a difficult task. Obviously this issue is avoided when switching between the appropriate measures. From the risk-free measure  $\mathbb{Q}$  to the forward measure associated with the zero-coupon bond maturing at the payment day,  $T_N$ ,  $P(t, T_N)$  (see Jamshidian (1991)). With the Radon–Nikodým derivative we obtain:

$$\begin{aligned} \Pi(t) &= P(t, T_N)\mathbb{E}^{T_N}\left(\frac{(S(T_N) - K)^+}{P(T_N, T_N)} \mid \mathcal{F}_t\right) \\ &= P(t, T_N)\mathbb{E}^{T_N}((F^{T_N}(T_N) - K)^+ \mid \mathcal{F}_t) \quad \text{with } t < T_N \end{aligned} \quad (3.2)$$

with  $F^{T_N}(t)$  the forward of the stock  $S(t)$ , defined as:

$$F^{T_N}(t) = \frac{S(t)}{P(t, T_N)} \quad (3.3)$$

### 3.1 Derivation of the hybrid model

Under the  $T_N$ -forward measure we assume that the equity process is driven by the Heston stochastic volatility model, given by the following dynamics:

$$\left. \begin{aligned} \frac{dS(t)}{S(t)} &= (\dots) dt + \sqrt{\xi(t)} dW_x^N(t), & S(0) > 0 \\ d\xi(t) &= \kappa(\bar{\xi} - \xi(t)) dt + \gamma\sqrt{\xi(t)} dW_\xi^N(t), & \xi(0) > 0 \end{aligned} \right\} \quad (3.4)$$

Note that the drift in (3.4) is not yet specified.

For the interest rate model we choose the DD–SV LMM under the  $T_N$ -measure generated by the numeraire  $P(t, T_N)$ , given by:

$$\left. \begin{aligned} dL_k(t) &= -\phi_k(t)\sigma_k V(t) \sum_{j=k+1}^N \frac{\tau_j \phi_j(t)\sigma_j}{1 + \tau_j L_j(t)} \rho_{k,j} dt + \sigma_k \phi_k(t) \sqrt{V(t)} dW_k^N(t) \\ dV(t) &= \lambda(V(0) - V(t)) dt + \eta \sqrt{V(t)} dW_V^N(t) \end{aligned} \right\} \quad (3.5)$$

with a nonzero correlation between the stock process,  $S(t)$ , and its variance process,  $\xi(t)$ , between the LIBORs  $L_i(t) \equiv L(t, T_{i-1}, T_i)$  and  $L_j(t) \equiv L(t, T_{j-1}, T_j)$  for



$i, j = 1 \dots N, i \neq j$ , and between the stock  $S(t)$  and LIBOR rates, ie:

$$\left. \begin{aligned} dW_x^N(t) dW_\xi^N(t) &= \rho_{x,\xi} dt \\ dW_x^N(t) dW_j^N(t) &= \rho_{x,j} dt \\ dW_i^N(t) dW_j^N(t) &= \rho_{i,j} dt \end{aligned} \right\} \quad (3.6)$$

We assume a zero correlation between the LIBORs  $L_i(t)$  and their variance process  $V(t)$ , between the LIBORs and the variance process for equity  $\xi(t)$ , between the variance processes  $\xi(t)$  and  $V(t)$ , and between the stock  $S(t)$  and the variance of the LIBORs  $V(t)$ .

For the calculation of the value of the European option given in (3.2), we first need to determine the dynamics for the forward,  $F^{T_N}(t)$ . From Ito’s lemma we obtain:

$$\begin{aligned} dF^{T_N}(t) &= \frac{1}{P(t, T_N)} dS(t) - \frac{S(t)}{P^2(t, T_N)} dP(t, T_N) \\ &\quad + \frac{S(t)}{P^3(t, T_N)} (dP(t, T_N))^2 - \frac{1}{P^2(t, T_N)} (dS(t))(dP(t, T_N)) \end{aligned}$$

Since the forward is a martingale under the  $T_N$ -measure generated by the zero-coupon bond,  $P(t, T_N)$ , the forward dynamics do not contain a drift term. This implies that we should not encounter any  $dt$  terms<sup>1</sup> in the dynamics of  $dF^{T_N}(t)$ , ie:

$$dF^{T_N}(t) = (\dots) dt + \frac{1}{P(t, T_N)} dS(t) - \frac{S(t)}{P^2(t, T_N)} dP(t, T_N) \quad (3.7)$$

Equation (3.7) shows that, in order to find the dynamics for process  $dF^{T_N}(t)$ , the dynamics for  $P(t, T_N)$  also need to be determined. With the approximation introduced in Section 2.3, the bond  $P(t, T_N)$  is given by:

$$\frac{1}{P(t, T_N)} = (1 + (T_{m(t)} - t)L_{m(t)}(T_{m(t)-1})) \prod_{j=m(t)+1}^N (1 + \tau_j L(t, T_{j-1}, T_j))$$

Before we derive the Ito dynamics for the zero-coupon bond,  $P(t, T_N)$ , for ease of notation we define the following “support variables”:

$$\begin{aligned} f(t) &= 1 + (T_{m(t)} - t)L(T_{m(t)-1}, T_{m(t)-1}, T_{m(t)}) \\ g_j(t, L_j(t)) &= 1 + \tau_j L(t, T_{j-1}, T_j) \end{aligned}$$

By taking the log transform of the bond,  $\log P(t, T_N)$ , we find:

$$\log P(t, T_N) = -\log(f(t)) - \sum_{j=m(t)+1}^N \log g_j(t, L_j(t)) \quad (3.8)$$

<sup>1</sup> Note that the  $dt$  term in (3.7) should compensate for the drift appearing from  $dS(t)$ .

10 L. A. Grzelak and C. W. Oosterlee

so that the dynamics for the log bond read:

$$d \log P(t, T_N) = -d \log(f(t)) - \sum_{j=m(t)+1}^N d \log g_j(t, L_j(t)) \quad (3.9)$$

On the other hand, by applying Ito's lemma to  $\log P(t, T_N)$ , we obtain:

$$d \log P(t, T_N) = \frac{1}{P(t, T_N)} dP(t, T_N) - \frac{1}{2} \left( \frac{1}{P(t, T_N)} \right)^2 (dP(t, T_N))^2 \quad (3.10)$$

By neglecting the  $dt$  terms (as we do not encounter any  $dt$  terms in the dynamics of  $dF^{T_N}(t)$ ) and by matching Equations (3.9) and (3.10), we obtain:

$$\frac{dP(t, T_N)}{P(t, T_N)} = - \sum_{j=m(t)+1}^N d \log g_j(t, L_j(t)) \quad (3.11)$$

with the dynamics for  $d \log g_j(t, L_j(t))$ :

$$d \log g_j(t, L_j(t)) = \frac{\tau_j}{1 + \tau_j L_j(t)} dL_j(t) \quad (3.12)$$

After substitution of (3.11), (3.12) and (3.5) and neglecting the  $dt$  terms, the dynamics for the bond  $P(t, T_N)$  are given by:

$$\frac{dP(t, T_N)}{P(t, T_N)} = - \sum_{j=m(t)+1}^N \frac{\tau_j \sigma_j \phi_j(t) \sqrt{V(t)}}{1 + \tau_j L_j(t)} dW_j^N(t) \quad (3.13)$$

Now we return to the derivations for the forward,  $F^{T_N}(t)$ , in Equation (3.7). By Equation (3.4) these can be expressed as:

$$\frac{dF^{T_N}(t)}{F^{T_N}(t)} = \sqrt{\xi(t)} dW_x^N(t) - \frac{1}{P(t, T_N)} dP(t, T_N) \quad (3.14)$$

Finally, by combining Equations (3.14) and (3.13), the dynamics for the forward  $F^{T_N}(t)$  are determined:

$$\frac{dF^{T_N}(t)}{F^{T_N}(t)} = \sqrt{\xi(t)} dW_x^N(t) + \sum_{j=m(t)+1}^N \frac{\tau_j \sigma_j \phi_j(t) \sqrt{V(t)}}{1 + \tau_j L_j(t)} dW_j^N(t) \quad (3.15)$$

Since the forward  $F^{T_N}(t)$  is a martingale under the  $T_N$ -measure (ie, fully determined in terms of the volatility structure), the interpolation with zero volatility does not affect the dynamics for the forward  $F^{T_N}(t)$ . As indicated in Rebonato (2004), under the forward measure, the forward price (3.15) includes components arising from the volatility of the zero-coupon bonds that connect the spot and the forward prices.

#### 4 APPROXIMATION FOR THE HYBRID MODEL

With the stock process  $S(t)$  under the  $T_N$ -terminal measure to be driven by the Heston model with a stochastic, correlated variance process  $\xi(t)$ , we obtain the dynamics in (3.15) for the forward prices  $F^{T_N}(t)$ , with  $dW_x^N(t) dW_\xi^N(t) = \rho_{x,\xi} dt$  and the parameters as defined in (2.1). The LIBOR rates  $L_j(t)$  are defined in (3.5).

We call this model the Heston–LIBOR market model (H–LMM) here. This is the full-scale model, which requires approximations for efficient pricing of European equity options.

The model in (3.15) is not of the affine form, as it involves terms like  $\phi_j(t)/(1 + \tau_j L_j(t))$ . Therefore, we cannot use the standard techniques from Duffie *et al* (2000) to determine the CF. The availability of a CF is especially important for the model calibration, where fast pricing for equity plain vanilla products is essential. For this reason we freeze the LIBOR rates (Glasserman and Zhao (1999); Hull and White (2000); and Jäckel and Rebonato (2000)), ie:

$$L_j(t) \approx L_j(0) \quad (4.1)$$

As a consequence,  $\phi_j(t) \approx L_j(0)$  (with  $\phi_j(t)$  in (2.6)) and the dynamics for the forward  $F^{T_N}(t)$  read:

$$\frac{dF^{T_N}(t)}{F^{T_N}(t)} \approx \sqrt{\xi(t)} dW_x^N(t) + \sum_{j=m(t)+1}^N \frac{\tau_j \sigma_j L_j(0) \sqrt{V(t)}}{1 + \tau_j L_j(0)} dW_j^N(t) \quad (4.2)$$

with the correlations and the remaining processes given in (3.6). We now determine the log transform of the forward  $x(t) := \log F^{T_N}(t)$ . With  $\mathcal{A} = \{m(t) + 1, \dots, N\}$  and application of Ito’s lemma, the dynamics for  $x(t)$  are given by:

$$\begin{aligned} dx(t) \approx & -\frac{1}{2} \left( \sum_{j \in \mathcal{A}} \psi_j \sqrt{V(t)} dW_j^N(t) + \sqrt{\xi(t)} dW_x^N(t) \right)^2 \\ & + \sqrt{\xi(t)} dW_x^N(t) + \sum_{j \in \mathcal{A}} \psi_j \sqrt{V(t)} dW_j^N(t) \quad (4.3) \end{aligned}$$

with:

$$\psi_j = \frac{\tau_j \sigma_j L_j(0)}{1 + \tau_j L_j(0)}$$

The square of the sum in the drift can be reformulated using the following formula:

$$\left( \sum_{j=1}^N z_j \right)^2 = \sum_{j=1}^N z_j^2 + \sum_{\substack{i,j=1,\dots,N \\ i \neq j}} z_i z_j \quad \text{for } N > 0$$

12 L. A. Grzelak and C. W. Oosterlee

By taking  $z_j = \psi_j \sqrt{V(t)} dW_j^N$  the dynamics can now be expressed as:

$$\begin{aligned} dx(t) &\approx -\frac{1}{2} \left\{ \xi(t) + V(t) \left( \sum_{j \in \mathcal{A}} \psi_j^2 + \sum_{\substack{i, j \in \mathcal{A} \\ i \neq j}} \psi_i \psi_j \rho_{i,j} \right) + 2\sqrt{V(t)} \sqrt{\xi(t)} \sum_{j \in \mathcal{A}} \psi_j \rho_{x,j} \right\} dt \\ &\quad + \sqrt{\xi(t)} dW_x^N(t) + \sqrt{V(t)} \sum_{j \in \mathcal{A}} \psi_j dW_j^N(t) \end{aligned}$$

By setting:

$$\left. \begin{aligned} A_1(t) &:= \sum_{j \in \mathcal{A}} \psi_j^2 + \sum_{\substack{i, j \in \mathcal{A} \\ i \neq j}} \psi_i \psi_j \rho_{i,j} \\ A_2(t) &:= \sum_{j \in \mathcal{A}} \psi_j \rho_{x,j} \end{aligned} \right\} \quad (4.4)$$

we obtain:

$$\begin{aligned} dx(t) &\approx -\frac{1}{2} (\xi(t) + V(t)A_1(t) + 2\sqrt{V(t)}\sqrt{\xi(t)}A_2(t)) dt \\ &\quad + \sqrt{\xi(t)} dW_x^N(t) + \sqrt{V(t)} \sum_{j \in \mathcal{A}} \psi_j dW_j^N(t) \end{aligned} \quad (4.5)$$

On the other hand, the frozen LIBOR dynamics are given by:

$$dL_k(t) \approx -\sigma_k L_k(0) V(t) \sum_{j=k+1}^N \psi_j \rho_{k,j} dt + \sigma_k L_k(0) \sqrt{V(t)} dW_k^N(t)$$

which, by taking:

$$B_1(k) = \sum_{j=k+1}^N \psi_j \rho_{k,j}$$

equal to:

$$dL_k(t) \approx -\sigma_k L_k(0) V(t) B_1(k) dt + \sigma_k L_k(0) \sqrt{V(t)} dW_k^N(t) \quad (4.6)$$

with the variance process  $V(t)$  given in (3.5).

Here, we derive the instantaneous covariance for the stochastic model given by (4.5) and (4.6) with the variance processes in (3.4) and (3.5). Since the dynamics for the forward  $F^{TN}(t)$  involve the LIBOR rates, the dimension of the covariance matrix will be dependent on time  $t$ . For a given state vector:

$$X(t) = [x(t), \xi(t), L_1^N(t), L_2^N(t), \dots, L_N^N(t), V(t)]^T$$

the covariance matrix will be of the following form:

$$\Sigma(X(t))\Sigma(X(t))^T = \begin{bmatrix} \Sigma_{x,x} & \Sigma_{x,\xi} & \Sigma_{x,L_1} & \Sigma_{x,L_2} & \cdots & \Sigma_{x,L_N} & 0 \\ \Sigma_{\xi,x} & \Sigma_{\xi,\xi} & 0 & 0 & \cdots & 0 & 0 \\ \Sigma_{L_1,x} & 0 & \Sigma_{L_1,L_1} & \Sigma_{L_1,L_2} & \cdots & \Sigma_{L_1,L_N} & 0 \\ \Sigma_{L_2,x} & 0 & \Sigma_{L_2,L_1} & \Sigma_{L_2,L_2} & \cdots & \Sigma_{L_2,L_N} & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \Sigma_{L_N,x} & 0 & \Sigma_{L_N,L_1} & \Sigma_{L_N,L_2} & \cdots & \Sigma_{L_N,L_N} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \Sigma_{V,V} \end{bmatrix} dt \quad (4.7)$$

with:

$$\Sigma_{x,x} = \xi(t) + V(t)A_1(t) + 2\sqrt{V(t)}\sqrt{\xi(t)}A_2(t) \quad (4.8)$$

$$\Sigma_{L_i,L_j} = \rho_{i,j}\sigma_i\sigma_jL_i(0)L_j(0)V(t) \quad (4.9)$$

$$\Sigma_{x,L_i} = \rho_{x,i}\sigma_iL_i(0)\sqrt{\xi(t)}\sqrt{V(t)} + \sigma_iL_i(0)V(t)\sum_{j \in \mathcal{A}} \psi_j\rho_{i,j} \quad (4.10)$$

and:

$$\left. \begin{aligned} \Sigma_{\xi,\xi} &= \gamma^2\xi(t), & \Sigma_{L_i,L_i} &= \sigma_i^2L_i^2(0)V(t) \\ \Sigma_{V,V} &= \eta^2V(t), & \Sigma_{x,\xi} &= \rho_{x,\xi}\gamma\xi(t) \end{aligned} \right\} \quad (4.11)$$

Zeros are present in the covariance matrix due to the assumption of zero correlation for  $\rho_{x,V}$ ,  $\rho_{\xi,L_i}$ ,  $\rho_{L_i,V}$  and  $\rho_{\xi,V}$ . The covariance matrix and the drift in Equation (4.5) include the nonaffine terms  $\sqrt{\xi(t)}\sqrt{V(t)}$ . Therefore, the resulting model is not affine and we cannot easily derive the corresponding CF. Appropriate approximations will be introduced in the next subsection.

#### 4.1 The hybrid model linearization

In order to derive the system in an affine form, approximations for the nonaffine terms in the instantaneous covariance matrix (4.7) are necessary (as done in Grzelak and Oosterlee (2011) for a hybrid with stochastic volatility for equity and a short-rate model for the interest rate). In the present work, we linearize these terms by projection on the first moments, as follows:

$$\begin{aligned} \sqrt{\xi(t)}\sqrt{V(t)} &\approx \mathbb{E}(\sqrt{\xi(t)}\sqrt{V(t)}) \\ &\stackrel{\perp}{=} \mathbb{E}(\sqrt{\xi(t)})\mathbb{E}(\sqrt{V(t)}) \\ &=: \vartheta(t) \end{aligned} \quad (4.12)$$

14 L. A. Grzelak and C. W. Oosterlee

with  $\perp$  indicating independence between the processes  $\xi(t)$  and  $V(t)$ . By Dufresne (2001) and simplifications as in Kummer (1936), the closed-form expression for the expectation of the square root of the square-root process  $\mathbb{E}(\sqrt{\xi(t)})$  can be found:<sup>2</sup>

$$\mathbb{E}(\sqrt{\xi(t)}) = \sqrt{2c(t)}e^{-\omega(t)/2} \sum_{k=0}^{\infty} \frac{1}{k!} (\frac{1}{2}\omega(t))^k \frac{\Gamma(\frac{1}{2}(1+d)+k)}{\Gamma(\frac{1}{2}d+k)} \quad (4.13)$$

with:

$$c(t) = \frac{1}{4\kappa} \gamma^2 (1 - e^{-\kappa t}), \quad d = \frac{4\kappa \bar{\xi}}{\gamma^2}, \quad \omega(t) = \frac{4\kappa \xi(0) e^{-\kappa t}}{\gamma^2 (1 - e^{-\kappa t})} \quad (4.14)$$

and gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt$$

Parameters  $\kappa$ ,  $\bar{\xi}$ ,  $\xi(0)$  and  $\gamma$  are given in (3.4).

Although the expectation is in closed form, its evaluation is rather expensive. One may prefer to use a suitable proxy, given by:

$$\mathbb{E}(\sqrt{\xi(t)}) \approx a_1 + b_1 e^{-c_1 t}, \quad \mathbb{E}(\sqrt{V(t)}) \approx a_2 + b_2 e^{-c_2 t} \quad (4.15)$$

with constant coefficients  $a_i$ ,  $b_i$  and  $c_i$  for  $i = 1, 2$  that can easily be determined (see Appendix A).

## 4.2 The forward characteristic function

With the approximations introduced, the nonaffine terms in the drift and in the instantaneous covariance matrix have been linearized. This approximate model is therefore in the class of affine processes. With the approximations, under the log transform, the forward,  $x(t)$ , is governed by the following SDE:

$$dx(t) = -\frac{1}{2}(\xi(t) + V(t)A_1(t) + 2\vartheta(t)A_2(t)) dt + \sqrt{\xi(t)} dW_x^N(t) + \sqrt{V(t)} \sum_{j \in \mathcal{A}} \psi_j dW_j^N(t)$$

(with  $A_1$  and  $A_2$  as in (4.4)) which is of the affine form. We call this approximation of the full-scale hybrid model the approximate Heston–LIBOR market model, or H1–LMM.

Now we derive the corresponding forward characteristic function. Since the dimension of the hybrid changes over time, the number of coefficients in the corresponding characteristic function will also change. For a given time to expiry,  $\tau = T_N - t$

<sup>2</sup> The expectation for  $\mathbb{E}(\sqrt{V(t)})$  is found analogously.

and  $\mathcal{B} = \{m(T_N - \tau) + 1, \dots, T_N\}$ , the forward characteristic function for the approximate hybrid model is of the following form:

$$\begin{aligned} \phi^{T_N}(u, \mathbf{X}(t), \tau) = \exp \left( A(u, \tau) + B_x(u, \tau)x(t) + B_\xi(u, \tau)\xi(t) \right. \\ \left. + \sum_{j \in \mathcal{B}} B_j(u, \tau)L_j(t) + B_V(u, \tau)V(t) \right) \quad (4.16) \end{aligned}$$

subject to the terminal condition  $\phi^{T_N}(u, \mathbf{X}(T_N), 0) = \exp(iu x(T_N))$ , which, according to Equation (3.3), equals  $\phi^{T_N}(u, \mathbf{X}(T_N), 0) = \exp(iu \log S(T_N))$ . The coefficients  $A(u, \tau)$ ,  $B_x(u, \tau)$ ,  $B_\xi(u, \tau)$ ,  $B_j(u, \tau)$  and  $B_V(u, \tau)$  satisfy the system of ordinary differential equations (ODEs) in the following lemma.

LEMMA 4.1 *The functions  $B_x(u, \tau) =: B_x$ ,  $B_\xi(u, \tau) =: B_\xi$ ,  $B_j(u, \tau) =: B_j$ ,  $B_V(u, \tau) =: B_V$  and  $A(u, \tau) =: A$  for the forward characteristic function given in (4.16) satisfy the following ODEs:*

$$\begin{aligned} \frac{d}{d\tau} B_x(u, \tau) &= 0 \\ \frac{d}{d\tau} B_j(u, \tau) &= 0 \\ \frac{d}{d\tau} B_\xi(u, \tau) &= \frac{1}{2} B_x(B_x - 1) + (\rho_{x,\xi} \gamma B_x - \kappa) B_\xi + \frac{1}{2} \gamma^2 B_\xi^2 \\ \frac{d}{d\tau} B_V(u, \tau) &= \frac{1}{2} A_1(t) B_x(B_x - 1) - \sum_{j \in \mathcal{A}} \sigma_j L_j(0) B_x B_j \sum_{k \in \mathcal{A}} \psi_k \rho_{k,j} \\ &\quad - \lambda B_V + \frac{1}{2} \sum_{j \in \mathcal{A}} \sigma_j^2 L_j^2(0) B_j^2 \\ &\quad + \sum_{\substack{i,j \in \mathcal{A} \\ i \neq j}} \rho_{i,j} \sigma_i \sigma_j L_i(0) L_j(0) B_i B_j + \frac{1}{2} \eta^2 B_V^2 \\ \frac{d}{d\tau} A(u, \tau) &= \vartheta(t) A_2(t) B_x(B_x - 1) + \kappa \bar{\xi} B_\xi + \lambda V(0) B_V \\ &\quad + \sum_{j \in \mathcal{A}} \rho_{x,j} \sigma_j L_j(0) \vartheta(t) B_x B_j \end{aligned}$$

where  $\mathcal{A} = \{m(t) + 1, \dots, N\}$ ,  $t = T_N - \tau$ , with boundary conditions  $B_x(u, 0) = iu$ ,  $B_j(u, 0) = 0$ ,  $B_\xi(u, 0) = 0$ ,  $B_V(u, 0) = 0$  and  $A(u, 0) = 0$ .

PROOF The proof can be found in Appendix B. □

COROLLARY 4.2 *Under the  $T_N$ -forward measure, the characteristic function for  $x(t)$  in (4.16) does not contain terms like  $B_j(u, \tau)$  and  $L_j(t)$  for  $j = 1, \dots, N$ ,*

16 L. A. Grzelak and C. W. Oosterlee

which implies a dimension reduction for the corresponding pricing partial differential equation, if we move forward in time. This is a consequence of the fact that the LIBOR dynamics have the property of “dying” after the reset date, ie, the dimension of the underlying model is reduced as particular LIBORs have been determined.

In fact, the dimension reduction is a natural consequence of freezing the LIBOR rates at their initial values. The approximating equity model for  $x(t) = \log F^{TN}(t)$  is given by the following system of SDEs:

$$dx(t) \approx -\frac{1}{2}\hat{f}(t, V(t), \xi(t)) dt + \sqrt{\xi(t)} dW_x^N(t) + \sqrt{V(t)} \sum_{j \in \mathcal{A}} \psi_j dW_j^N(t) \quad (4.17)$$

with:

$$\left. \begin{aligned} d\xi(t) &= \kappa(\bar{\xi} - \xi(t)) dt + \gamma \sqrt{\xi(t)} dW_\xi^N(t) \\ dV(t) &= \lambda(V(0) - V(t)) dt + \eta \sqrt{V(t)} dW_V^N(t) \end{aligned} \right\} \quad (4.18)$$

where  $\hat{f}(t, V(t), \xi(t)) := \xi(t) + V(t)A_1(t) + 2\vartheta(t)A_2(t)$  and the functions  $A_1(t)$  and  $A_2(t)$  are defined in (4.4). We see that, by a number of approximations, the functional dependence of  $x(t)$  on the LIBOR dynamics is reduced to a dependence on the initial LIBOR rates  $L_k(0)$ , for  $k = 1, \dots, N$ , and volatility processes  $\xi(t)$  and  $V(t)$ .

As mentioned, the approximate model used for calibration cannot become a new stand-alone model for hybrid products, as the approximations introduced modify the Girsanov kernel when changing between the measures. This means that, in fact, the approximation we have derived is not guaranteed to be arbitrage-free.

Lemma 4.1 indicates that  $B_x(u, \tau) = iu$  and  $B_j(u, \tau) = 0$ , giving rise to a simplification of the forward CF:

$$\phi^{TN}(u, X(t), \tau) = \exp(A(u, \tau) + iux(t) + B_\xi(u, \tau)\xi(t) + B_V(u, \tau)V(t)) \quad (4.19)$$

with  $B_\xi(u, \tau)$ ,  $B_V(u, \tau)$  and  $A(u, \tau)$  given by:

$$\left. \begin{aligned} \frac{d}{d\tau} B_\xi(u, \tau) &= -\frac{1}{2}(u^2 + iu) + (\rho_{x,\xi}\gamma iu - \kappa)B_\xi + \frac{1}{2}\gamma^2 B_\xi^2 \\ \frac{d}{d\tau} B_V(u, \tau) &= -\frac{1}{2}A_1(t)(u^2 + iu) - \lambda B_V + \frac{1}{2}\eta^2 B_V^2 \\ \frac{d}{d\tau} A(u, \tau) &= -\vartheta(t)A_2(t)(u^2 + iu) + \kappa\bar{\xi}B_\xi + \lambda V(0)B_V \end{aligned} \right\} \quad (4.20)$$

subject to the initial conditions:

$$B_\xi(u, 0) = 0, \quad B_V(u, 0) = 0, \quad A(u, 0) = 0$$



With the help of the Feynman–Kac theorem, one can show that the forward characteristic function  $\phi^{TN} := \phi^{TN}(u, \mathbf{X}(t), \tau)$  given in (4.19) with functions  $B_\xi(u, \tau)$ ,  $B_V(u, \tau)$  and  $A(u, \tau)$  in (4.20) satisfies the following Kolmogorov backward equation:

$$0 = \frac{\partial \phi^{TN}}{\partial t} + \frac{1}{2}(\xi + A_1(t)V + 2A_2(t)\vartheta(t)) \left( \frac{\partial^2 \phi^{TN}}{\partial x^2} - \frac{\partial \phi^{TN}}{\partial x} \right) + \kappa(\bar{\xi} - \xi) \frac{\partial \phi^{TN}}{\partial \xi} + \lambda(V(0) - V) \frac{\partial \phi^{TN}}{\partial V} + \frac{1}{2}\eta^2 V \frac{\partial^2 \phi^{TN}}{\partial V^2} + \frac{1}{2}\gamma^2 \xi \frac{\partial^2 \phi^{TN}}{\partial \xi^2} + \rho_{x,\xi} \gamma \xi \frac{\partial^2 \phi^{TN}}{\partial x \partial \xi} \quad (4.21)$$

subject to  $\phi^{TN}(u, \mathbf{X}(T), 0) = \exp(iu x(T_N))$ , with  $\vartheta(t)$  in (4.12), and  $A_1(t)$ ,  $A_2(t)$  in (4.4).

Since  $\vartheta(t)$  is a deterministic function of time, the partial differential equation coefficients in (4.21) are all affine.

The complex-valued functions  $B_\xi(u, \tau)$ ,  $B_V(u, \tau)$  and  $A(u, \tau)$  in Lemma 4.1 are of Heston type (see Heston (1993)). An analytic closed-form solution is available for constant parameters. However, since the functions  $A_1(t)$  and  $A_2(t)$  are not constant but piecewise constant, an alternative approach needs to be used. As indicated in Andersen and Andreasen (2000), an analytic but recursive solution is also available for piecewise constant parameters. We provide the solutions in Proposition 4.3.

**PROPOSITION 4.3** (Piecewise complex-valued functions  $A(u, \tau)$ ,  $B_\xi(u, \tau)$ ,  $B_V(u, \tau)$ ) *For a given grid,  $0 = \tau_0 < \tau_1 < \dots < \tau_N = \tau$ , and time interval,  $s_j = \tau_j - \tau_{j-1}$ ,  $j = 1, \dots, N$ , the piecewise constant complex-valued coefficients  $B_\xi(u, \tau)$  and  $B_V(u, \tau)$  are given by the following recursive expressions:*

$$B_\xi(u, \tau_j) = B_\xi(u, \tau_{j-1}) + \frac{(\kappa - \rho_{x,\xi} \gamma i u - d_j^1 - \gamma^2 B_\xi(u, \tau_{j-1}))(1 - e^{-d_j^1 s_j})}{\gamma^2 (1 - g_j^1 e^{-d_j^1 s_j})}$$

$$B_V(u, \tau_j) = B_V(u, \tau_{j-1}) + \frac{(\lambda - d_j^2 - \eta^2 B_V(u, \tau_{j-1}))(1 - e^{-d_j^2 s_j})}{\eta^2 (1 - g_j^2 e^{-d_j^2 s_j})}$$

and:

$$A(u, \tau_j) = A(u, \tau_{j-1}) + \frac{\kappa \bar{\xi}}{\gamma^2} \left( (\kappa - \rho_{x,\xi} \gamma i u - d_j^1) s_j - 2 \log \left( \frac{1 - g_j^1 e^{-d_j^1 s_j}}{1 - g_j^1} \right) \right) + \frac{\lambda V(0)}{\eta^2} \left( (\lambda - d_j^2) s_j - 2 \log \left( \frac{1 - g_j^2 e^{-d_j^2 s_j}}{1 - g_j^2} \right) \right) - A_2(t)(u^2 + iu) \int_{\tau_{j-1}}^{\tau_j} \vartheta(t) dt$$

18 L. A. Grzelak and C. W. Oosterlee

with:

$$\begin{aligned}
 d_j^1 &= \sqrt{(\rho_{x,\xi}\gamma iu - \kappa)^2 + \gamma^2(iu + u^2)} \\
 d_j^2 &= \sqrt{\lambda^2 + \eta^2 A_1(t)(u^2 + iu)} \\
 g_j^1 &= \frac{(\kappa - \rho_{x,\xi}\gamma iu) - d_j^1 - \gamma^2 B_\xi(u, \tau_{j-1})}{(\kappa - \rho_{x,\xi}\gamma iu) + d_j^1 - \gamma^2 B_\xi(u, \tau_{j-1})} \\
 g_j^2 &= \frac{\lambda - d_j^2 - \eta^2 B_V(u, \tau_{j-1})}{\lambda + d_j^2 - \eta^2 B_V(u, \tau_{j-1})}
 \end{aligned}$$

and the initial conditions  $B_\xi(u, \tau_0) = 0$ ,  $B_V(u, \tau_0) = 0$  and  $A(u, \tau_0) = 0$ . Moreover, for  $t = T_N - \tau_j$ , the functions  $A_1(t)$  and  $A_2(t)$  are defined in (4.4) and  $\vartheta(t)$  is defined in (4.12), with the parameters  $\kappa$ ,  $\gamma$ ,  $\lambda$ ,  $\eta$  and  $\rho_{x,\xi}$  given in (3.4)–(3.6).

PROOF The proof can be found in Appendix C. □

With a characteristic function available for the log transformed forward  $x(t)$ , we can compute European option prices for equity maturing at the terminal time,  $T_N$ . In the case of an option maturing at a time different from the terminal time  $T_N$  (say, at  $T_i$  with  $i < N$ ), we need to price the equity forward  $F^{T_i}(t)$ , and therefore an appropriate change of measure for the H–LMM (3.15) should be applied. Since the forward  $F^{T_i}$  is a martingale under the  $T_i$ -forward measure, it does not contain a drift term. On the other hand, the variance process  $\xi(t)$  for the Heston model is neither correlated with the LIBORs nor with the LIBORs' variance process  $V(t)$ . The change of measure therefore does not affect the variance process  $\xi(t)$ . In Appendix D we present a proof for this statement.

## 5 NUMERICAL RESULTS

In this section several numerical experiments are presented. First, the accuracy of the approximate model H1–LMM is compared with the full-scale H–LMM for European call option prices. Furthermore, the sensitivity to the interest rate skew for both models is checked. Finally, we use a typical equity–interest rate hybrid payoff function and compare the performance of the new H–LMM with the Heston–Hull–White (HHW) hybrid model.

### 5.1 Approximation accuracy of the H1–LMM

We check here the accuracy of the developed approximation H1–LMM. We compare the Monte Carlo European call prices from the full-scale H–LMM with the corresponding prices obtained by the Fourier inverse algorithm (Fang and Oosterlee

(2008)) for the H1–LMM. In the Monte Carlo simulation we work under one measure, the  $T_N$ -terminal measure. So, the prices for different option maturities are calculated by the following expression:

$$\Pi_{\text{MC}}(t) = P(t, T_N) \mathbb{E}^{T_N} \left( \frac{(S_{T_i} - K)^+}{P(T_i, T_N)} \mid \mathcal{F}_t \right) \quad \text{for } i \leq N$$

which, by Equation (3.3), equals:

$$\Pi_{\text{MC}}(t) = P(t, T_N) \mathbb{E}^{T_N} \left( \left( F^{T_N}(T_i) - \frac{K}{P(T_i, T_N)} \right)^+ \mid \mathcal{F}_t \right)$$

where  $K$  is the strike price, and the bond  $P(T_i, T_N)$  is given by (2.9).

The prices calculated by the Fourier inverse algorithm are obtained with the following expression:

$$\Pi_{\text{F}}(t) = P(t, T_i) \mathbb{E}^{T_i} ((F^{T_i}(T_i) - K)^+ \mid \mathcal{F}_t)$$

with the CF from Proposition 4.3. As mentioned, the change of measure does not affect the volatility of the Heston process. Pricing under different measures is therefore consistent.

When calibrating the plain Heston model in practice, the parameters obtained rarely satisfy the Feller condition  $\gamma^2 < 2\kappa\bar{\xi}$  (if the Feller condition is satisfied, this ensures that the variance process is positive). In order to mimic a realistic setting, we also choose parameters that do not satisfy this inequality, ie:

$$\kappa = 1.2, \quad \bar{\xi} = 0.1, \quad \gamma = 0.5, \quad S(0) = 1, \quad \xi(0) = 0.1$$

For the interest rate model, we take:

$$\beta_k = 0.5, \quad \sigma_k = 0.25, \quad \lambda = 1, \quad V(0) = 1, \quad \eta = 0.1$$

In the correlation matrix a number of model correlations need to be specified. For the correlations between the LIBOR rates, we set large positive values, as frequently observed in the fixed income markets (see, for example, Brigo and Mercurio (2007)),  $\rho_{i,j} = 0.98$  for  $i, j = 1, \dots, N, i \neq j$ . For the correlation between  $S(t)$  and  $\xi(t)$  we set a negative correlation,  $\rho_{x,\xi} = -0.3$ , which corresponds to the skew in the implied volatility for equity. Finally, the correlation between the stock and the LIBORs we set  $\rho_{x,i} = 0.5$  for  $i = 1, \dots, N$ . In practice, this correlation would be estimated from

**TABLE 1** The European equity call option prices of the H1–LMM compared with those of the H–LMM.

Strike $K$	$T_2$		$T_5$		$T_{10}$	
	CF	MC	CF	MC	CF	MC
$K = 40\%$	0.6418	0.6424 (0.0035)	0.7017	0.7014 (0.0034)	0.7821	0.7833 (0.0081)
$K = 80\%$	0.3299	0.3316 (0.0030)	0.4638	0.4648 (0.0034)	0.6203	0.6226 (0.0082)
$K = 100\%$	0.2149	0.2167 (0.0027)	0.3730	0.3742 (0.0034)	0.5562	0.5588 (0.0083)
$K = 120\%$	0.1332	0.1345 (0.0024)	0.2993	0.3004 (0.0034)	0.5008	0.5036 (0.0083)
$K = 160\%$	0.0483	0.0486 (0.0016)	0.1933	0.1941 (0.0034)	0.4109	0.4140 (0.0082)
$K = 200\%$	0.0184	0.0184 (0.0010)	0.1268	0.1273 (0.0031)	0.3419	0.3452 (0.0080)
$K = 240\%$	0.0078	0.0076 (0.0006)	0.0850	0.0852 (0.0026)	0.2878	0.2913 (0.0079)

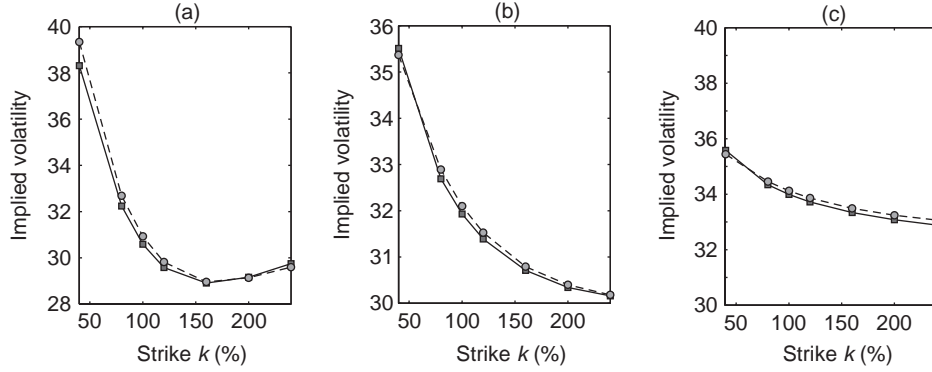
The H–LMM Monte Carlo experiment was performed with 200 000 paths and twenty intermediate points between dates  $T_{i-1}$  and  $T_i$ , for  $i = 1, \dots, N$ . The tenor structure was chosen to be  $\mathcal{T} = \{T_1, \dots, T_{10}\}$  with the terminal measure  $T_N = T_{10}$ . Numbers in parentheses are sample standard deviations.

historical data. The following correlation matrix results:

$$\begin{bmatrix}
 1 & \rho_{x,\xi} & \rho_{x,1} & \cdots & \rho_{x,N} & \rho_{x,V} \\
 \rho_{\xi,x} & 1 & \rho_{\xi,1} & \cdots & \rho_{\xi,N} & \rho_{\xi,V} \\
 \rho_{1,x} & \rho_{1,\xi} & 1 & \cdots & \rho_{1,N} & \rho_{1,V} \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 \rho_{N,x} & \rho_{N,\xi} & \rho_{N,1} & \cdots & 1 & \rho_{N,V} \\
 \rho_{V,x} & \rho_{V,\xi} & \rho_{V,1} & \cdots & \rho_{V,N} & 1
 \end{bmatrix}
 \begin{bmatrix}
 1 & -0.3 & 0.5 & \cdots & 0.5 & 0 \\
 -0.3 & 1 & 0 & \cdots & 0 & 0 \\
 0.5 & 0 & 1 & \cdots & 0.98 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0.5 & 0 & 0.98 & \cdots & 1 & 0 \\
 0 & 0 & 0 & \cdots & 0 & 1
 \end{bmatrix}
 \tag{5.1}$$

The accuracy and the associated standard deviations, in terms of the European call option prices for equity (with the Monte Carlo simulation versus the Fourier inversion of the CF), are presented in Table 1. In Figure 1 on the facing page the corresponding implied volatility plots are presented. The accuracy of the approximations introduced (H1–LMM) is highly satisfactory for this experiment.

**FIGURE 1** Comparison of implied Black–Scholes volatilities for the European equity option, obtained by Fourier inversion of the H1–LMM and by Monte Carlo simulation of the H–LMM.



Black–Scholes implied volatility for (a)  $T = 2$ , (b)  $T = 5$  and (c)  $T = 10$ . Solid line: Fourier inverse. Dashed line: Monte Carlo.

### 5.2 Interest rate skew

The approximation H1–LMM was based on freezing the appropriate LIBOR rates and on linearizations in the instantaneous covariance matrix. By freezing the LIBORs, ie,  $\ell_k(t) \equiv L_k(0)$ , we have that  $\phi_k(t) = \beta_k L_k(t) + (1 - \beta_k)L_k(0) = L_k(0)$ .

In the DD–SV model, parameter  $\beta_k$  controls the slope of the interest rate volatility smile, so by freezing the LIBORs to  $L_k(0)$  the information about the interest rate skew is not included in the approximation H1–LMM.

Here we perform an experiment with the full-scale model (H–LMM). Using a Monte Carlo simulation, we check the influence of parameter  $\beta_k$  on the equity implied volatilities (Black and Scholes (1973)). The equity implied volatilities for the European call option for H–LMM are presented in Table 2 on the next page. The experiment displays a small impact of the different  $\beta_k$ s on the equity implied volatilities, which implies that our approximation, H1–LMM, makes sense for various parameters  $\beta_k$  in the interest rate modeling in the present setting.

To explain the small effect of the variation in  $\beta_k$  on the equity implied volatility we need to return to the equity forward equation in (3.15), ie:

$$\frac{dF^{TN}(t)}{F^{TN}(t)} = \sqrt{\xi(t)} dW_x^N(t) + \sum_{j=m(t)+1}^N \frac{\tau_j \sigma_j \phi_j(t) \sqrt{V(t)}}{1 + \tau_j L_j(t)} dW_j^N(t)$$

The equity forward is based on two types of correlated volatilities: the equity with  $dW_x^N(t)$  and the interest rate with  $dW_j^N(t)$  for  $j = 1, \dots, N$ . Since, in the experi-

**TABLE 2** Equity implied volatilities.

Strike	$\beta_k = 0$	$\beta_k = 0.5$	$\beta_k = 1$
$K = 40\%$	35.71% (0.0290)	35.50% (0.0221)	34.60% (0.0460)
$K = 80\%$	34.63% (0.0109)	34.49% (0.0086)	34.26% (0.0175)
$K = 100\%$	34.23% (0.0087)	34.15% (0.0066)	33.99% (0.0139)
$K = 120\%$	33.90% (0.0073)	33.89% (0.0055)	33.78% (0.0119)
$K = 160\%$	33.40% (0.0058)	33.53% (0.0045)	33.47% (0.0097)
$K = 200\%$	33.05% (0.0052)	33.28% (0.0041)	33.26% (0.0088)
$K = 240\%$	32.81% (0.0048)	33.09% (0.0039)	33.12% (0.0085)

The effect of the interest rate skew, controlled by  $\beta_k$ , on the equity implied volatilities. The Monte Carlo simulation was performed with the setup from Table 1 on page 20. The maturity is  $T_N = 10$ . Values in parentheses indicate implied volatility standard deviations (the experiment was repeated ten times).

ment, we have chosen a realistic set of parameters (as in Section 5.1) with a rather large parameter  $\gamma = 0.5$ , the first term in the forward SDE above,  $\sqrt{\xi(t)} dW_x^N(t)$ , is dominating. The other volatilities contribute in particular when large maturities are considered. The theoretical proof for this statement is rather involved, but we can illustrate it by setting  $t = 0$ . For the equity part we then have:

$$\sqrt{\xi(0)} \approx 0.3162$$

and for the interest rate we have:

$$\sqrt{V(0)} \sum_{j=1}^N \frac{\tau_j \sigma_j L_j(0)}{1 + \tau_j L_j(0)} \approx 0.0122N$$

where  $N$  corresponds to the number of LIBORs considered.

The experiments performed show that the equity option prices are not strongly influenced by the value of  $\beta_k$ , which indicates that freezing the LIBORs in the H1-LMM may not influence the calibration procedure significantly.

### 5.3 Pricing a hybrid product

Although the interest rate skew parameter  $\beta_k$  does not strongly influence the equity prices, it may still have an impact on the hybrid contract price. In this subsection we use the H-LMM and price a typical exotic payoff.

As indicated in Hunter and Picot (2006), an investor interested in structured products may look for higher expected return (higher coupons) than available from basic market instruments. By trading hybrid products they can also trade the correlation, for example, by including multiple assets in a structured derivatives product, and therefore the basket volatility can be reduced. This typically makes the corresponding option cheaper.

The main advantage of the H–LMM lies in its capability to price hybrid products that are sensitive to an equity smile, an interest rate smile and the correlation between the assets. A hybrid payoff that involves the equity and interest rate assets is the so-called “minimum of several assets payoff” (see Hunter and Picot (2006)). The contract is made for an investor willing to take some risk in one asset class in order to become a participant in a different asset class. If the investor wants to be involved in an  $n$ -year constant maturity swap (CMS), by taking some risk in equity, this can be expressed by the following payoff:

$$\text{payoff} = \max \left( 0, \min \left( C_n(T), k\% \times \frac{S(T)}{S(t)} \right) \right)$$

where  $S(t)$  is the stock price at time  $t$  and  $C_n(t)$  is an  $n$ -year CMS. By setting the tenor structure  $\mathcal{T} = \{1, \dots, 10\}$ , with payment for  $P(T_N, T_M)$  at time  $T_N = 5$  and with maturity  $T_M = 10$ , we obtain the following pricing equation:

$$\Pi_H(t) = P(t, T_5) \mathbb{E}^{T_5} \left( \max \left( 0, \min \left( \frac{1 - P(T_5, T_{10})}{\sum_{k=6}^{10} P(T_5, T_k)}, k\% \times \frac{S(T_5)}{S(t)} \right) \right) \middle| \mathcal{F}_t \right) \quad (5.2)$$

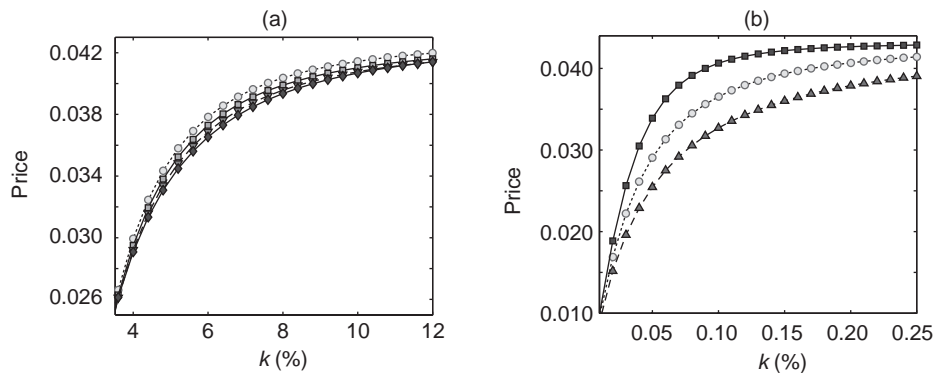
In our simulation, the bonds  $P(T_i, T_j)$  are obtained from the DD–SV LMM and determined by (2.9) for  $t = T_i$  and  $T_N = T_j$ . As a first test we check the sensitivity to the interest rate skew (by changing  $\beta$  and keeping the correlation  $\rho_{x,i} = 0$  for all  $i$ ) and to the correlation between the stock,  $S_t$ , and the LIBOR rates,  $L_i(t)$ , by varying the correlation,  $\rho_{x,i} = \{0, -0.7, 0.7\}$  for all  $i$ . Figure 2 on the next page shows the corresponding results. We see a significant impact on the hybrid prices, which suggests that plain equity models, or equity short-rate hybrid models, may lead to different prices for such hybrid products.

Insight into the added value of the H–LMM can be gained by comparing the H–LMM results with, for example, the HHW hybrid model. In the HHW model the equity part is driven by the Heston process, as in Equation (2.1), but the interest rate is driven by a Hull–White short-rate process given by the following SDE:

$$dr(t) = \lambda(\theta(t) - r(t)) dt + \eta dW_r(t) \quad \text{with } r(0) > 0 \quad (5.3)$$

with positive parameters  $\lambda$ ,  $\theta(t)$ ,  $\eta$  and  $dW_x(t) dW_r(t) = \rho_{x,r} dt$ .

**FIGURE 2** The value for a “minimum of several assets” hybrid product.



The prices are obtained by Monte Carlo simulation with 20 000 paths and 20 intermediate points. (a) Influence of  $\beta$ . Dotted circle line,  $\beta = 0$ ; solid square line,  $\beta = 0.3$ ; dashed triangle line,  $\beta = 0.6$ ; solid diamond line,  $\beta = 1$ . (b) Influence of  $\rho_{x,L}$ . Dotted circle line,  $\rho_{x,L} = 0$ ; dashed triangle line,  $\rho_{x,L} = -0.7$ ; solid square line,  $\rho_{x,L} = 0.7$ .

Before performing the pricing of the hybrid product the model parameters need to be determined. The models were calibrated to data sets provided in Appendix E. For the H-LMM, the parameters from Section 5.1 were used. In the calibration of the HHW model, we first calibrated the Hull-White process, for which we obtained:

$$\lambda = 0.0614, \quad \eta = 0.0133, \quad r_0 = 0.05$$

Then, with an imposed correlation between the stock and the short rate,  $\rho_{x,r} = 0.5$ , the remaining parameters were found to be:

$$\begin{aligned} \kappa &= 0.650, & \gamma &= 0.469, & \bar{\xi} &= 0.090 \\ \rho_{x,\xi} &= -0.222, & \xi(0) &= 0.114 \end{aligned}$$

In part (a) of Figure 3 on the facing page we present the pricing results with the two hybrid models. For  $k > 5\%$  (with  $k$  in Equation (5.2)) a significant difference between the obtained prices is observed, although the two models were calibrated to the same data set.

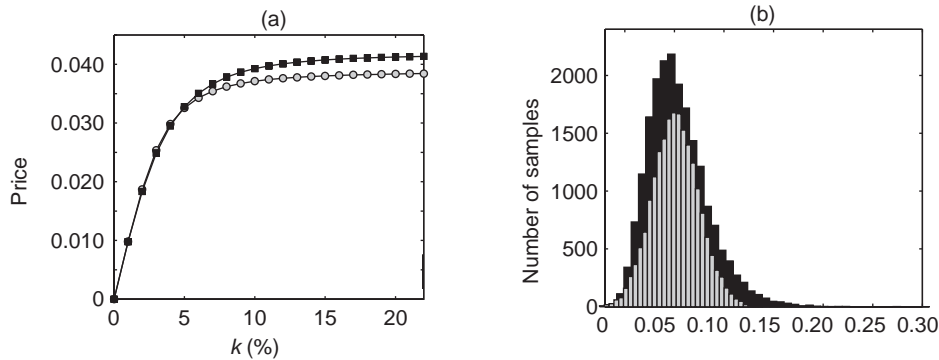
Payoff equation (5.2) shows that, as the percentage  $k$  increases, the dominating part of the product will be the CMS rate. We conclude that the Hull-White underlying model for the short rate indeed does not take into account the interest rate smile/skew and therefore gives different prices for a smile/skew sensitive product.

In part (b) of Figure 3 on the facing page we present the histograms of the CMS rate for both models. The histograms show a significantly fatter tail in the case of the DD-SV model than the one for the Hull-White short-rate model.

I don't quite understand the caption here. You say in the figure heading that the graphs are for the H-LMM and HHW models, but in the notes Heston-DD-SV is mentioned. Please clarify.



**FIGURE 3** Hybrid prices and CMS for the H–LMM and HHW models.



(a) Hybrid prices obtained by two different hybrid models: H–LMM and HHW. The models were calibrated to the same data set. Gray circle line: HHW model. Black square line: Heston–DD–SV. (b) Constant maturity swap rate for the H–LMM and HHW models. Black bars: CMS–SV–LMM. Gray bars: CMS–HW.

## 6 CONCLUSION

As well as requiring models that are well-defined and that capture the important features in the market, the financial industry also needs models that are efficiently calibrated to market data.

We have proposed an equity–interest rate hybrid model with stochastic volatility for stock and for the interest rates. To bring the model into the class of affine processes, we projected the nonaffine terms on time-dependent functions. This approximation to the full-scale model is affine, and we have determined a closed-form forward characteristic function. By this the approximate hybrid model, H1–LMM, can be used for calibration purposes.

The main advantage of the model developed here lies in its ability to price hybrid products exposed to the interest rate smile accurately and efficiently.

We have focused on the calibration aspects. In future research we will aim for theoretical analysis of the impact of the various approximations made.

## APPENDIX A: APPROXIMATIONS FOR $\mathbb{E}(\sqrt{v(t)})$

Expectation  $\mathbb{E}(\sqrt{\xi(t)})$  (as well as  $\mathbb{E}(\sqrt{V(t)})$ ) can be approximated by a function of the following form:

$$\mathbb{E}(\sqrt{\xi(t)}) \approx a + be^{-ct} =: \tilde{\Lambda}(t) \tag{A.1}$$

with  $a$ ,  $b$  and  $c$  as constants. Appropriate values for  $a$ ,  $b$  and  $c$  in (A.1) can be obtained via an optimization problem of the form  $\min_{a,b,c} \|\Lambda(t) - \tilde{\Lambda}(t)\|_n$ , where  $\|\cdot\|_n$  is any

26 L. A. Grzelak and C. W. Oosterlee

$n$ th norm, where:

$$\Lambda(t) = \sqrt{c(t)(\omega(t) - 1) + c(t)d + \frac{c(t)d}{2(d + \omega(t))}}$$

for  $c(t)$ ,  $d$  and  $\omega(t)$  defined in (4.14). Here, instead of a numerical approximation for these coefficients, we use a simple analytic expression in Result A 1.

**RESULT A 1** By matching the functions  $\Lambda(t)$  and  $\tilde{\Lambda}(t)$  for  $t \rightarrow +\infty$ ,  $t \rightarrow 0$  and  $t = 1$ , we find that:

$$\left. \begin{aligned} \lim_{t \rightarrow +\infty} \Lambda(t) &= \sqrt{\bar{v} - \frac{\gamma^2}{8\kappa}} = a_1 = \lim_{t \rightarrow +\infty} \tilde{\Lambda}(t) \\ \lim_{t \rightarrow 0} \Lambda(t) &= \sqrt{v(0)} = a_1 + b_1 = \lim_{t \rightarrow 0} \tilde{\Lambda}(t) \\ \lim_{t \rightarrow 1} \Lambda(t) &= \Lambda(1) = a_1 + b_1 e^{-c_1} = \lim_{t \rightarrow 1} \tilde{\Lambda}(t) \end{aligned} \right\} \quad (\text{A.2})$$

The values  $a_1$ ,  $b_1$  and  $c_1$  can now be estimated by:

$$a_1 = \sqrt{\bar{v} - \frac{\gamma^2}{8\kappa}}, \quad b_1 = \sqrt{v(0)} - a_1, \quad c_1 = -\log(b_1^{-1}(\Lambda(1) - a_1)) \quad (\text{A.3})$$

## APPENDIX B: PROOF OF LEMMA 4.1

**PROOF** For affine processes  $X(t)$ , the forward CF  $\phi^{T_N}(\mathbf{u}, X(t), \tau)$  is given by (Duffie *et al* (2000)):

$$\begin{aligned} \phi^{T_N}(\mathbf{u}, X(t), \tau) &= \mathbb{E}^{T_N}(e^{i\mathbf{u}^T X(T)} \mid \mathcal{F}_t) \\ &= \exp(A(\mathbf{u}, \tau) + \mathbf{B}^T(\mathbf{u}, \tau)X(t)) \end{aligned}$$

with time lag  $\tau = T_N - t$ . Here, the expectation is taken under the  $T_N$ -forward measure,  $\mathbb{Q}^{T_N}$ . The complex-valued functions  $A(\mathbf{u}, \tau)$  and  $\mathbf{B}^T(\mathbf{u}, \tau)$  have to satisfy the following complex-valued ODEs:

$$\left. \begin{aligned} \frac{d}{d\tau} \mathbf{B}(\mathbf{u}, \tau) &= a_1^T \mathbf{B} + \frac{1}{2} \mathbf{B}^T c_1 \mathbf{B} \\ \frac{d}{d\tau} A(\mathbf{u}, \tau) &= \mathbf{B}^T a_0 + \frac{1}{2} \mathbf{B}^T c_0 \mathbf{B} \end{aligned} \right\} \quad (\text{B.1})$$

with  $a_i, c_i, i = 0, 1$ , defined in:

$$\mu(X(t)) = a_0 + a_1 X(t)$$

for any  $(a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n \times n}$ , and:

$$\Sigma(X(t)) \Sigma(X(t))^T = (c_0)_{ij} + (c_1)_{ij}^T X(t)$$

for arbitrary  $(c_0, c_1) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n \times n}$ . Index  $n$  indicates the dimension,  $\mu(\mathbf{X}(t))$  is the drift of processes  $\mathbf{X}(t)$  and  $\Sigma(\mathbf{X}(t))\Sigma(\mathbf{X}(t))^T$  corresponds to the instantaneous covariance matrix.

Under the log transform we find that the state vector  $\mathbf{X}(t)$  has  $N + 3$  elements ( $n = N + 3$ ):

$$\mathbf{X}(t) = [x(t), \xi(t), L_1(t), \dots, L_N(t), V(t)]^T$$

With the Heston equity model (4.5) and the stochastic volatility LMM in (4.6) we set vector  $\mathbf{u} = [u, 0, \dots, 0]^T$ . In order to find the functions  $A(u, \tau)$  and  $\mathbf{B}^T(u, \tau)$  in (B.1) we need to determine the matrices  $a_1^T$ ,  $c_0$ ,  $c_1$  and the vector  $a_0$ . By the approximations in (4.1) and (4.12), the drifts in the LIBORs  $L_i(t)$  and in the forward dynamics do not contain any nonaffine terms. For  $\mathcal{A} = \{m(t) + 1, \dots, N\}$ ,  $t = T_N - \tau$ , the nonzero elements in matrix  $a_1^T$  are given by:

$$\begin{aligned} a_1^T(2, 1) &= -\frac{1}{2}, & a_1^T(2, 2) &= -\kappa \\ a_1^T(N + 3, 1) &= -\frac{1}{2}A_1(t), & a_1^T(N + 3, N + 3) &= -\lambda \end{aligned}$$

with:

$$a_1^T(N + 3, j + 2) = -\sigma_j L_j(0) B_1(j) \quad \text{for } j \in \mathcal{A}$$

To determine the matrices  $c_1$  and  $c_0$  we use the instantaneous covariance matrix from (4.7). For matrix  $c_1$  the nonzero elements are given by:

$$\begin{aligned} c_1(1, 1, 2) &= 1, & c_1(1, 1, N + 3) &= A_1(t) \\ c_1(2, 1, 2) &= \rho_{x, \xi} \gamma, & c_1(1, 2, 2) &= \rho_{x, \xi} \gamma \\ c_1(2, 2, 2) &= \gamma^2, & c_1(N + 3, N + 3, N + 3) &= \eta^2 \end{aligned}$$

and:

$$\begin{aligned} c_1(j + 2, j + 2, N + 3) &= \sigma_j^2 L_j^2(0) & \text{for } j \in \mathcal{A} \\ c_1(i + 2, j + 2, N + 3) &= \rho_{i, j} \sigma_i \sigma_j L_i(0) L_j(0) & \text{for } i, j \in \mathcal{A}, i \neq j \\ c_1(1, j + 2, N + 3) &= \sigma_j L_j(0) \sum_{k \in \mathcal{A}} \psi_k \rho_{j, k} \\ c_1(j + 2, 1, N + 3) &= c_1(1, j + 2, N + 3) \end{aligned}$$

In essence, the first and the second index of  $c_1$  indicate which covariance term we deal with, whereas the third term indicates which variable is defined. The unspecified matrix values are equal to zero.

For matrix  $c_0$  and vector  $a_0$  we get:

$$\begin{aligned} c_0(1, 1) &= 2\vartheta(t)A_2(t) \\ c_0(1, j + 2) &= c_0(j + 2, 1) = \rho_{x, j} \sigma_j \vartheta(t) L_j(0) \quad \text{for } j \in \mathcal{A} \end{aligned}$$

28 L. A. Grzelak and C. W. Oosterlee

and:

$$a_0(1) = -\vartheta(t)A_2(t), \quad a_0(2) = \kappa\bar{\xi}, \quad a_0(N+3) = \lambda V(0)$$

By substitutions and appropriate matrix multiplications in (B.1) the proof is complete.  $\square$

### APPENDIX C: PROOF OF PROPOSITION 4.3

**PROOF** We note that the functions  $A_1(t)$  and  $A_2(t)$  are constant between the times  $\tau_i$ . For simplicity, we set  $\tau_0 = 0$  and  $\tau = T - t$ . Since  $B_j(u, \tau) = 0$ , the equations which need to be solved are given by:

$$\frac{d}{d\tau} B_\xi(u, \tau) = b_{1,0} + b_{1,1} B_\xi + b_{1,2} B_\xi^2 \quad (C.1)$$

$$\frac{d}{d\tau} B_V(u, \tau) = b_{2,0} + b_{2,1} B_V + b_{2,2} B_V^2 \quad (C.2)$$

$$\frac{d}{d\tau} A(u, \tau) = a_0 B_\xi + a_1 B_V + f(t) \quad (C.3)$$

with certain initial conditions for  $B_\xi(u, \tau_0)$ ,  $B_V(u, \tau_0)$  and  $A(u, \tau_0)$  and coefficients:

$$\left. \begin{aligned} b_{1,0} &= -\frac{1}{2}(u^2 + iu), & b_{1,1} &= \rho_{x,\xi}\gamma iu - \kappa, & b_{1,2} &= \frac{1}{2}\gamma^2 \\ b_{2,0} &= -\frac{1}{2}A_1(t)(u^2 + iu), & b_{2,1} &= -\lambda, & b_{2,2} &= \frac{1}{2}\eta^2 \end{aligned} \right\} \quad (C.4)$$

and the coefficients for  $A(u, \tau)$ :

$$a_0 = \kappa\bar{\xi}, \quad a_1 = \lambda V(0), \quad f(t) = -\vartheta(t)A_2(t)(u^2 + iu) \quad (C.5)$$

Since  $B_\xi(u, \tau)$  and  $B_V(u, \tau)$  do not depend on  $A(u, \tau)$ , a closed-form solution is available (see, for example, Heston (1993) and Wu and Zhang (2008)). For  $\tau > 0$  we find:

$$B_\xi(u, \tau) = B_\xi(u, \tau_0) + \frac{(-b_{1,1} - d_1 - 2b_{1,2}B_\xi(u, \tau_0))}{2b_{1,2}(1 - g_1 e^{-d_1(\tau-\tau_0)})} (1 - e^{-d_1(\tau-\tau_0)}) \quad (C.6)$$

$$B_V(u, \tau) = B_V(u, \tau_0) + \frac{(-b_{2,1} - d_2 - 2b_{2,2}B_V(u, \tau_0))}{2b_{2,2}(1 - g_2 e^{-d_2(\tau-\tau_0)})} (1 - e^{-d_2(\tau-\tau_0)}) \quad (C.7)$$

with:

$$\left. \begin{aligned} d_1 &= \sqrt{b_{1,1}^2 - 4b_{1,0}b_{1,2}}, & d_2 &= \sqrt{b_{2,1}^2 - 4b_{2,0}b_{2,2}} \\ g_1 &= \frac{-b_{1,1} - d_1 - 2B_\xi(u, \tau_0)b_{1,2}}{-b_{1,1} + d_1 - 2B_\xi(u, \tau_0)b_{1,2}}, & g_2 &= \frac{-b_{2,1} - d_2 - 2B_V(u, \tau_0)b_{2,2}}{-b_{2,1} + d_2 - 2B_V(u, \tau_0)b_{2,2}} \end{aligned} \right\} \quad (C.8)$$

For  $A(u, \tau)$  we have:

$$A(u, \tau) = A(u, \tau_0) + a_0 \int_0^\tau B_\xi(u, s) ds + a_1 \int_0^\tau B_V(u, s) ds + \int_0^\tau f(\tau - s) ds$$

The first two integrals can be solved analytically:

$$\begin{aligned} \int_0^\tau B_\xi(u, s) ds &= \frac{1}{2b_{1,2}} \left( (-b_{1,1} + d_1)(\tau - \tau_0) - 2 \log \left( \frac{1 - g_1 e^{-d_1(\tau - \tau_0)}}{1 - g_1} \right) \right) \\ \int_0^\tau B_V(u, s) ds &= \frac{1}{2b_{2,2}} \left( (-b_{2,1} + d_2)(\tau - \tau_0) - 2 \log \left( \frac{1 - g_2 e^{-d_2(\tau - \tau_0)}}{1 - g_2} \right) \right) \end{aligned} \quad (\text{C.9})$$

For the last integral we have:

$$\int_0^\tau f(\tau - s) ds = -(u^2 + iu) \int_0^\tau \vartheta(\tau - s) A_2(\tau - s) ds \quad (\text{C.10})$$

Since  $A_2(\tau - s)$  is constant between 0 and  $\tau$ , function  $A_2(\tau - s)$  can be taken outside the integral. The proof is finished by the appropriate substitutions.  $\square$

## APPENDIX D: EQUITY VARIANCE DYNAMICS UNDER MEASURE CHANGE

**PROPOSITION D 1** *The dynamics of the variance process  $\xi(t)$  given in (3.4) are not affected by changing the forward measure generated by numeraire  $P(t, T_i)$ , for  $i = 1, \dots, N$ .*

**PROOF** Under the  $T_N$ -forward measure, the model with the forward stock  $F^{T_N}(t)$  in (3.15), the variance process  $\xi(t)$  in (3.4), and the LIBOR rates as given in (3.5) can, in terms of the independent Brownian motions, be expressed as:

$$\begin{bmatrix} dL_1(t) \\ dL_2(t) \\ \vdots \\ dL_N(t) \\ dV(t) \\ \frac{dF^{T_N}(t)}{F^{T_N}(t)} \\ d\xi(t) \end{bmatrix} = \begin{bmatrix} \mu_1(t) \\ \mu_2(t) \\ \vdots \\ \mu_N(t) = 0 \\ \lambda(V(0) - V(t)) \\ 0 \\ \kappa(\bar{\xi} - \xi(t)) \end{bmatrix} dt + \mathbf{A}\mathbf{H} \begin{bmatrix} d\tilde{W}_1^N(t) \\ d\tilde{W}_2^N(t) \\ \vdots \\ d\tilde{W}_N^N(t) \\ d\tilde{W}_V^N(t) \\ d\tilde{W}_x^N(t) \\ d\tilde{W}_\xi^N(t) \end{bmatrix} \quad (\text{D.1})$$

30 L. A. Grzelak and C. W. Oosterlee

with:

$$A = \begin{bmatrix} \sigma_1 \phi_1(t) \sqrt{V(t)} & \cdots & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \sigma_N \phi_N(t) \sqrt{V(t)} & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & \eta \sqrt{V(t)} & 0 & 0 & 0 \\ \Upsilon_1(t) \sqrt{V(t)} & \cdots & \Upsilon_N(t) \sqrt{V(t)} & 0 & \sqrt{\xi(t)} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & \gamma \sqrt{\xi(t)} & 0 \end{bmatrix} \quad (D.2)$$

where:

$$\Upsilon_j(t) = \frac{\tau_j \sigma_j \phi_j(t)}{1 + \tau_j L_j(t)}$$

and  $H$  is the Cholesky lower triangular of the correlation matrix  $C$ , which is given by:

$$C = \begin{bmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,N} & 0 & \rho_{x,1} & 0 \\ \rho_{2,1} & 1 & \cdots & \rho_{2,N} & 0 & \rho_{x,2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho_{N,1} & \rho_{N,2} & \cdots & 1 & 0 & \rho_{x,N} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \rho_{x,1} & \rho_{x,2} & \cdots & \rho_{x,N} & 0 & 1 & \rho_{x,\xi} \\ 0 & 0 & \cdots & 0 & 0 & \rho_{x,\xi} & 1 \end{bmatrix} \quad (D.3)$$

With  $\zeta_k(t)$  the  $k$ th row vector from matrix  $M = AH$ , the dynamics for the LIBOR  $L_N(t)$  can be expressed as:

$$dL_N(t) = \zeta_N(t) d\tilde{W}^N(t)$$

The Radon–Nikodým derivative,  $\Lambda_N^{N-1}(t)$ , is given by:

$$\Lambda_N^{N-1}(t) = \frac{d\mathbb{Q}^{N-1}}{d\mathbb{Q}^N} = \frac{P(T_0, T_N)}{P(T_0, T_{N-1})} (1 + \tau_N L_N(t)) \quad (D.4)$$

and therefore, the dynamics for  $\Lambda_{N-1}^N(t)$  read:

$$d\Lambda_N^{N-1}(t) = \Lambda_N^{N-1}(t) \frac{\tau_N \zeta_N(t)}{1 + \tau_N L_N(t)} d\tilde{W}^N(t) \quad (D.5)$$

By the Girsanov theorem this implies that the change of measure is given by:

$$d\tilde{W}^N(t) = \frac{\tau_N \zeta_N(t)^T}{1 + \tau_N L_N(t)} dt + d\tilde{W}^{N-1}(t) \quad (D.6)$$

where the first  $N + 1$  elements correspond to  $N$  LIBORs with their volatility process  $V(t)$ . The remaining two elements are related to the equity part driven by the Heston model.

We wish to find the dynamics for process  $\xi(t)$  under the measure  $\mathbb{Q}^{N-1}$ . In terms of the independent Brownian motions the variance process  $\xi(t)$  is given by:

$$d\xi(t) = \kappa(\bar{\xi} - \xi(t)) dt + \zeta_{N+3}(t) d\tilde{W}^N(t)$$

with:

$$\zeta_{N+3}(t) = \underbrace{[0, 0, 0, \dots, 0]}_{N+1}, \gamma \sqrt{\xi(t)} \rho_{x,\xi}, \gamma \sqrt{\xi(t)} \sqrt{1 - \rho_{x,\xi}^2} \quad (D.7)$$

By Equation (D.6) the dynamics for  $\xi(t)$  under  $\mathbb{Q}^{N-1}$  are given by:

$$d\xi(t) = \kappa(\bar{\xi} - \xi(t)) dt + \zeta_{N+3}(t) \left( \frac{\tau_N \zeta_N(t)^T}{1 + \tau_N L_N(t)} dt + d\tilde{W}^{N-1}(t) \right) \quad (D.8)$$

Since:

$$\zeta_N(t) = \underbrace{[\dots, \dots, \dots]}_{N+1}, 0, 0 \quad (D.9)$$

so the scalar product  $\zeta_{N+3}(t) \zeta_N(t)^T = 0$ . This results in the following dynamics for the process  $\xi(t)$  under the  $\mathbb{Q}^{N-1}$  measure:

$$d\xi(t) = \kappa(\bar{\xi} - \xi(t)) dt + \zeta_{N+3}(t) d\tilde{W}^{N-1}(t) \quad (D.10)$$

Since, for all  $j = 1, \dots, N$ , the scalar product  $\zeta_{N+3}(t) \zeta_j(t)^T = 0$ , changing the corresponding forward measures does not affect the drift of the variance process  $\xi(t)$ . This observation concludes the proof.  $\square$

## APPENDIX E: REFERENCE MARKET DATA

In Table E.1 on the next page we present the reference market data to which the models have been calibrated.

The zero-coupon bonds are given by:  $P(0, 1) = 0.9512$ ,  $P(0, 2) = 0.9048$ ,  $P(0, 3) = 0.8607$ ,  $P(0, 4) = 0.8187$ ,  $P(0, 5) = 0.7788$ ,  $P(0, 6) = 0.7408$ ,  $P(0, 7) = 0.7047$ ,  $P(0, 8) = 0.6703$ ,  $P(0, 9) = 0.6376$  and  $P(0, 10) = 0.6065$ .

## REFERENCES

- Andersen, L. B. G., and Andreasen, J. (2000). Volatility skews and extensions of the LIBOR market model. *Applied Mathematical Finance* **7**(1), 1–32.  
 Andersen, L. B. G., and Andreasen, J. (2002). Volatile volatilities. *Risk* **15**(12), 163–168.

**TABLE E.1** European equity call option price.

Strike $K$	$T = 0.5$	$T = 1$	$T = 2$	$T = 3$	$T = 4$	$T = 5$	$T = 10$
40%	0.610	0.620	0.642	0.663	0.683	0.702	0.779
80%	0.235	0.271	0.329	0.378	0.421	0.461	0.612
100%	0.098	0.143	0.212	0.271	0.322	0.368	0.546
120%	0.030	0.067	0.131	0.190	0.244	0.293	0.489
160%	0.003	0.015	0.051	0.095	0.141	0.188	0.397
200%	0.000	0.004	0.023	0.051	0.086	0.125	0.328
240%	0.000	0.001	0.012	0.030	0.055	0.086	0.275
260%	0.000	0.001	0.009	0.024	0.045	0.073	0.253
300%	0.000	0.000	0.005	0.016	0.031	0.053	0.216

The standardized European equity call option values for different maturities ( $T$ [y]) and strikes ( $K$ [%]).

- Andersen, L. B. G., and Brotherton-Ratcliffe, R. (2005). Extended LIBOR market models with stochastic volatility. *The Journal of Computational Finance* **9**(1), 1–27.
- Beveridge, C., and Joshi, M. (2010). Interpolation schemes in the displaced-diffusion LIBOR market model and the efficient pricing and Greeks for callable range accruals. Working Paper, SSRN.
- Black, F., and Karasinski, P. (1991). Bond and option pricing when short rates are lognormal. *Financial Analysts Journal* **4**(47), 52–59.
- Black, F., and Scholes, M. (1973). The pricing of options and corporate liabilities. *Journal of Political Economy* **81**(3), 637–654.
- Brace, A., Gatarek, D., and Musiela, M. (1997). The market model of interest rate dynamics. *Mathematical Finance* **7**(2), 127–155.
- Brigo, D., and Mercurio, F. (2007). *Interest Rate Models: Theory and Practice*, 2nd edn. Springer.
- Carr, P. P., and Madan, D. B. (1999). Option valuation using the fast fourier transform. *The Journal of Computational Finance* **2**(4), 61–73.
- Cox, J. C., Ingersoll, J. E., and Ross, S. A. (1985). A theory of the term structure of interest rates. *Econometrica* **53**(2), 385–407.
- Davis, M. H. A., and Mataix-Pastor, V. (2009). Arbitrage-free interpolation of the swap curve. *International Journal of Theoretical and Applied Finance* **12**(7), 969–1005.
- Duffie, D., Pan, J., and Singleton, K. (2000). Transform analysis and asset pricing for affine jump-diffusions. *Econometrica* **68**(6), 1343–1376.
- Dufresne, D. (2001). The integrated square-root process. Working Paper, University of Montreal.
- Fang, F., and Oosterlee, C. W. (2008). A novel option pricing method based on Fourier-cosine series expansions. *SIAM Journal on Scientific Computing* **31**(2), 826–848.
- Glasserman, P., and Zhao, X. (1999). Fast Greeks by simulation in forward LIBOR models. *The Journal of Computational Finance* **3**(1), 5–39.
- Grzelak, L. A., and Oosterlee, C. W. (2011). On the Heston model with stochastic interest rates. *SIAM Journal on Financial Mathematics* **2**(1), 255–286.



- Grzelak, L. A., Oosterlee, C. W., and Van Weeren, S. (2011). The affine Heston model with correlated Gaussian interest rates for pricing hybrid derivatives. *Quantitative Finance* **11**(11), 1647–1663.
- Heston, S. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6**(2), 327–343.
- Hull, J., and White, A. (1996). Using Hull–White interest rate trees. *Journal of Derivatives* **3**(3), 26–36.
- Hull, J., and White, A. (2000). Forward rate volatilities, swap rate volatilities and the implementation of the LIBOR market model. *Journal of Fixed Income* **10**(2), 46–62.
- Hunter, C., and Picot, G. (2006). *Hybrid Derivatives: The Euromoney Derivatives and Risk Management Handbook*. BNP Paribas, London.
- Jäckel, P., and Rebonato, R. (2000). Linking caplet and swaption volatilities in a BGM framework: approximate solutions. *The Journal of Computational Finance* **6**(4), 41–60.
- Jamshidian, F. (1991). Bond and option evaluation in the Gaussian interest rate model. *Research in Finance* **9**, 131–170.
- Jamshidian, F. (1997). LIBOR and swap market models and measures. *Financial Stochastics* **1**(4), 293–330.
- Kummer, E. E. (1936). Über die hypergeometrische Reihe  $F(a; b; x)$ . *Journal für die Reine und Angewandte Mathematik* **15**, 39–83.
- Lee, R. (2004). Option pricing by transform methods: extensions, unification, and error control. *The Journal of Computational Finance* **7**(3), 51–86.
- Lewis, A. (2001). *Option Valuation Under Stochastic Volatility*. Finance Press, Newport Beach, CA.
- Miltersen, K. R., Sandmann, K., and Sondermann, D. (1997). Closed form solutions for term structure derivatives with lognormal interest rates. *Journal of Finance* **52**(1), 409–430.
- Morini, M., and Mercurio, F. (2007). No-arbitrage dynamics for a tractable SABR term structure LIBOR model. Working Paper, SSRN eLibrary.
- Musiela, M., and Rutkowski, M. (1997). *Martingale Methods in Financial Modelling*. Springer.
- Piterbarg, V. V. (2004). Computing deltas of callable LIBOR exotics in forward LIBOR models. *The Journal of Computational Finance* **7**(2), 107–144.
- Piterbarg, V. V. (2005). Stochastic volatility model with time dependent skew. *Applied Mathematical Finance* **12**(2), 147–185.
- Rebonato, R. (2002). *Modern Pricing of Interest-Rate Derivatives*, 1st edn. Princeton University Press.
- Rebonato, R. (2004). *Volatility and Correlation: The Perfect Hedger and the Fox*, 2nd edn. Wiley.
- Schlögl, E. (2002). Arbitrage-free interpolation in models of market observable interest rates. In *Advances in Finance and Stochastics: Essays in Honour of Dieter Sondermann*, pp. 197–218. Springer.
- Wu, L., and Zhang, F. (2008). Fast swaption pricing under the market model with a square-root volatility process. *Quantitative Finance* **8**(2), 163–180.