A SIMPLE AND EFFICIENT SEGREGATED SMOOTHER FOR THE DISCRETE STOKES EQUATIONS *

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Abstract. We consider the multigrid solution of the generalized Stokes equations from incompressible fluid dynamics. We introduce a segregated (i.e., equation-wise) Gauss-Seidel smoother based on a Uzawa-type iteration. We analyze it in the framework of local Fourier analysis. We obtain an analytic bound on the smoothing factor showing uniform performance for a family of Stokes problems, ranging from stationary to time-dependent with small time step. These results are confirmed by the numerical computation of the two-grid convergence factor for different types of grids and discretizations. Numerical results also show that the actual convergence of the W-cycle is roughly the same as that obtained with the Vanka smoother, despite this latter is significantly more costly per iteration step.

Key words. Multigrid methods, generalized Stokes equations, Uzawa smoother, smoothing theory, local Fourier analysis, staggered and nonstaggered grids

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1. Introduction.

The need to numerically solve the generalized Stokes equations appears for example in incompressible fluid dynamics and in some structural mechanics problems such as plasticity, beam and shell studies. Furthermore, in the large-scale simulation of fluid flow by using the nonlinear time-dependent Navier-Stokes equations, a generalized Stokes problem has to be solved at each nonlinear iteration, becoming a time-consuming task.

The Stokes equations form a saddlepoint problem, and depending on the choice of discretization method, one may end up, after discretization, with a matrix in 2×2 block form, in which the lower diagonal block is either a zero block or a block containing very small matrix elements. Saddlepoint problems are well-known and well-studied in numerical analysis. A clear overview of this topic has been presented in [2].

The tradition of solving Stokes equations with multigrid is long, and many interesting approaches have been presented during the last 35 years. Basically two established well-known robust and efficient approaches remained. Their difference lies in the type of smoothing operator adopted. A state-of-the-art smoother for the Stokes equations is a Gauss-Seidel-type coupled Vanka smoother [20], in which the primary unknowns, pressure and the velocities in a grid cell are updated simultaneously. Then, each smoothing step requires to solve as many small dense systems as there are cells in the grid.

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Decoupled, i.e., equation-wise, smoothing is often preferred for reasons of cost efficiency. For Stokes problems, one first transforms the discrete system such that equation-wise smoothing on the transformed discrete system, followed by a back-transformation to the original unknowns, brings excellent smoothing factors. Equation-wise, decoupled smoothing on a transformed system is called distributive smoothing [4, 24].

Both state-of-the-art smoothers are somewhat involved, as compared to the basic smoothing techniques applied to the scalar elliptic PDEs. Straightforward generalization of the basic point-wise smoothing to systems of equations, like the Stokes equations, implies a segregated, decoupled smoothing scheme in which the individual equations of the PDE system are considered to be scalar equations. In this paper we present a smoother which belongs to this category. Issues associated with standard equation-wise smoothing procedures like damped Jacobi and Gauss-Seidel are avoided by considering an Uzawa-type iteration. A similar, Uzawa smoother was presented in a PhD thesis by P. Nigon, as well as in a conference proceedings paper in the nineteen-eighties [14]. Here we present a new analysis of this approach leading to both a deeper understanding of the multigrid performance and to the formulation of a more efficient variant.

More precisely, the segregated smoother that we propose uses two different smoothing procedures for the two different types of unknowns. For the velocity components, any basic smoothing techniques applicable to scalar elliptic PDEs is possible, but our analysis favors symmetric schemes. This leads us to focus on symmetric Gauss-Seidel smoothing; that is, one forward pointwise Gauss-Seidel sweep for all velocity unknowns followed by one backward sweep. In [14], two forward Gauss-Seidel sweeps were used instead, and numerical results reveal that, everything else being equal, our choice is indeed more effective.

On the other hand the Uzawa-like procedure amounts to a simple Richardson iteration for the smoothing of the pressure unknowns. As usual, such iteration involves a relaxation parameter, and in [14] results were given for the optimal parameter, without giving hints on how to select it.

Here we provide an upper bound on the smoothing factor [19] associated with the proposed Uzawa smoother that clearly indicates how the different parameters may affect the convergence, and we deduce a rule of thumb to select the aforementioned relaxation parameter as a function of the main problems characteristics.

We then consider three typical discretizations of a family of two-dimensional Stokes problems, ranging from stationary to time-dependent with small time step. In each case, we perform numerical experiments whose results confirm the relevance of our analysis and the efficiency of the proposed approach. On the one hand, our upper bound on the smoothing factor gives an excellent estimation of the exact smoothing factor, and selecting the parameter according the proposed rule of is indeed optimal or near optimal. On the other hand, the smoothing factor indeed reflects well the two-grid convergence factor computed with local Fourier analysis (LFA), as well as the actual convergence factor associated with practical multigrid cycles. Finally, the convergence is roughly the same as that obtained with the Vanka smoother; since this latter is significantly more costly per step, it means that our approach is more effective, at least for the selected examples.

The outline of the paper is as follows. In §2, we present the generalized (i.e., parametrized) Stokes equations and give the general properties of the linear systems resulting from their discretizations. The Uzawa smoother is introduced in §3, where

we also develop our analysis of the associated smoothing factor. This general analysis is then particularized and numerically illustrated in the subsequent sections, where we consider successively a staggered MAC discretization (§4), a collocated grid discretization with an artificial pressure term (§5), and a stabilized linear finite element method on a equilateral triangular mesh (§6). Concluding remarks are given in §7.

2. The generalized Stokes equations and their discretization. Given a bounded polygonal domain $\Omega \subseteq \mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\Omega$, the generalized Stokes problem in two dimensions requires finding the velocity vector $\mathbf{u} : \Omega \to \mathbb{R}^2$, and the kinematic pressure field $p : \Omega \to \mathbb{R}$, satisfying

$$\begin{aligned} \xi\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega, \end{aligned} \tag{2.1}$$

where $\mathbf{f}$ represents a prescribed force, and the parameters $\nu > 0$ (viscosity) and $\xi \geq 0$ are given. This latter is often a proportional quantity to the inverse of the time-step in an implicit time integration method applied to a nonstationary Stokes problem; $\xi = 0$ corresponds to the classical stationary Stokes problem. Whatever the chosen scheme, the discretization of (2.1) leads to a linear system of the form

$$K = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}. \tag{2.2}$$

In this matrix, A is the discrete representation of the operator $\xi - \nu\Delta$; more precisely, A is block diagonal with one diagonal block per spatial dimension, each of them being the discrete operator acting on one of the velocity components. It further follows that A is symmetric positive definite (SPD). The matrix block B^T is the discrete gradient and B the discrete divergence; C is a stabilization term that is needed by some discretization schemes to avoid spurious solutions. Such spurious solutions arise when the discrete gradient admits more than the constant vector in its null space or near null space; i.e., when the discrete gradient is zero or near zero for some spurious pressure modes. The existence of such modes depends on which combination of discretization scheme is used for velocities and pressure. We refer to, e.g., [22] and [8] for more details on, respectively, finite difference and finite element discretization. Note a required property of the stabilization operator: if B is not full rank, C has to be positive definite on the null space of B^T, which further entails that the system matrix is nonsingular [2].

An important exception to this latter rule is when the boundary conditions are such that the physical pressure is only determined up to a constant. To make the problem well-posed, one needs then to impose some additional condition, such as

$$\int_\Omega p \, \mathrm{d}\mathbf{x} = 0. \tag{2.3}$$

At the discrete level, this is traduced by the fact that $B^T\mathbf{1} = \mathbf{0}$, where $\mathbf{1}$ is the vector of all ones. Hence, matrix K may be singular with a null space spanned by $(\mathbf{0} \;\; \mathbf{1}^T)^T$. Whether K is singular or not depends on C, but, to preserve accuracy, good stabilization operators need be small for smooth modes, and often satisfy $C\,\mathbf{1} = 0$ as well (see §5 below for an example).

The singularity of K raises however no particular difficulties when solving the linear system with an iterative method, see, e.g., [8, Section 8.3] for a detailed discussion in the context of the Navier-Stokes equations. Basically, the system is compatible

because the right hand side of (2.1) together with the lack of boundary conditions for the pressure entail that the right hand side of the discrete system is zero for the second block of equations. Further, starting from, say, the zero vector as initial approximation, all updates are kept orthogonal to the kernel vector $(\mathbf{0} \quad \mathbf{1}^T)^T$; that is, all updates are such that the discrete pressure unknowns have mean value zero, implying that the constraint (2.3) is satisfied at the discrete level by all successive approximations. Hence the error vector has no component against the singular mode, which thus plays no role in the convergence. Regarding the iteration matrix, it means that the eigenvalue 1 associated with the singular mode does not have to be taken into account. The convergence is governed by the "effective" spectral radius, which corresponds to the maximum in modulus of *other* eigenvalues.

Considering more particularly multigrid methods, prolongations operators are often such that a discrete pressure with mean value zero on the coarse grid remains with mean value zero once prolongated on the fine grid; i.e., no particular treatment is needed if the coarse solver picks up the "right" solution. On the other hand, as will be seen, the smoother that we propose in the next section is such that the relaxation preserves the mean value of the discrete pressure as soon as $B^T\mathbf{1} = \mathbf{0}$ and $C\,\mathbf{1} = \mathbf{0}$; i.e., here too, no particular treatment is needed to enforce (2.3) at the discrete level when the system is singular.

3. The Uzawa smoother. A well-known statement in the multigrid community is that a good indication for the appropriate choice of relaxation method for a system of equations is found in the systems' *determinant* [6]. If the operators on the main diagonal form the determinant of the system, smoothing is easy. In that case, the differential operator that corresponds to the primary unknown in each equation is the leading operator, indicating that equation-wise relaxation methods are likely efficient.

For the Stokes operator, however, the dominating terms in the determinant come from offdiagonal blocks. As a general rule, this indicates that coupled relaxation is likely needed, which motivated the focus on Vanka and distributive smoothers mentioned in the Introduction. Both indeed ensure a form of coupling between the relaxation processes applied to the unknowns belonging to a same grid cell. Interestingly, the smoother proposed here does not fall into these two categories of relaxation methods, and is truly decoupled. So, the determinant may be a good indication, but alternative smoothing methods, based on a different analysis, may be efficient as well.

Our approach is in fact not rooted in multigrid research, but in the Uzawa method (e.g., [2, Section 8.1]), which is an iterative scheme to solve linear systems with a matrix of the form (2.2). It amounts to performing stationary iterations with the preconditioner

$$\begin{pmatrix} A & \\ B & -\omega^{-1} I \end{pmatrix},$$

where ω is some positive parameter. The presence of the matrix block A in this preconditioner implies an exact solve for velocity at each iteration. This makes the approach costly, and the Uzawa method is in fact popular thanks to the "inexact" variants that replace this block with some preconditioner for A.

In the multigrid context, it seems then natural to consider as smoother this operator when A is replaced by a typical smoother M_A; that is, to consider the smoother

$$M = \begin{pmatrix} M_A & 0 \\ B & -\frac{1}{\omega} I \end{pmatrix}, \tag{3.1}$$

Notice that, because of the structure of A (block diagonal with diagonal blocks corresponding to discrete $\xi - \nu\Delta$), all commonly used smoothers for scalar elliptic PDEs can be used to define M_A.

The smoother (3.1) is a decoupled smoother, and a single smoothing step can be described as follows: given $(\mathbf{u}^T, \mathbf{p}^T)^T$ an approximation of the solution to the system, compute the relaxed approximation $(\widehat{\mathbf{u}}^T, \widehat{\mathbf{p}}^T)^T$ according to:

- apply smoother M_A to relax the system $A\mathbf{u} = \mathbf{f} - B^T\mathbf{p}$; i.e., $\hat{\mathbf{u}} = \mathbf{u} + M_A^{-1}\left(\mathbf{f} - A\,\mathbf{u} - B^T\mathbf{p}\right)$;
- update the pressure: $\hat{\mathbf{p}} = \mathbf{p} + \omega\ (B\,\hat{\mathbf{u}} - C\,\mathbf{p})$.

Observe that, when K is singular because $B^T\mathbf{1} = \mathbf{0}$ and $C\mathbf{1} = \mathbf{0}$, one has $\mathbf{1}^T B = \mathbf{1}^T C = 0$ and therefore $\mathbf{1}^T\hat{\mathbf{p}} = \mathbf{1}^T\mathbf{p}$, i.e., the relaxation associated with the Uzawa smoother preserves the mean value of the discrete pressure as discussed at the end of §2.

When M_A consists in two forward Gauss-Seidel sweeps for velocities, this scheme is the one suggested in [14]. In this work, we shall mainly consider another variant, where M_A is based on symmetric Gauss-Seidel iterations for A; i.e.,

$$M_A \;=\; (D_A + L_A)\ D_A^{-1}\,(D_A + U_A)\ , \tag{3.2}$$

where D_A, L_A and U_A are, respectively, the diagonal, the strict lower, and the strict upper parts of A. Indeed, besides being a good smoother, such M_A satisfies two important properties needed by our theoretical analysis. Firstly, it is SPD when A is SPD, and, next, the associated largest eigenvalue satisfies (see, e.g., [1, Theorem 7.17])

$$\lambda_{\max}(M_A^{-1}A) \;\leq\; 1. \tag{3.3}$$

Numerical experiments will also reveal that, for essentially the same cost, the convergence associated with such M_A is often faster than that obtained with two SOR sweeps as in [14].

Now, our analysis of the Uzawa smoother requires to first make the assumptions and simplifications associated with the LFA framework. LFA (also known as local mode analysis) is the most powerful tool for the quantitative analysis and design of efficient multigrid methods for general problems. In this analysis an infinite regular grid is considered and boundary conditions are not taken into account. LFA was introduced by Brandt in [3] and afterwards extended in [5]. A good introduction can be found in the paper by Stüben and Trottenberg [18] and in the books by Wesseling [21], Trottenberg et al. [19], and Wienands and Joppich [23].

In this framework, the discrete operators can be expressed in a basis (the Fourier basis) in which they have a simpler form. For instance, discrete representations of $\xi - \nu\Delta$ are typically diagonal, each component corresponding to a particular "frequency". LFA is further based on the classification of these frequencies in "Low frequency" components and "High frequency" components. The idea behind this is the following: multigrid methods work if the smoother and the coarse grid operator interact properly, and it is generally expected that the smoother damps the high frequency components of the error whereas the coarse grid correction damps the low frequency ones. Hence, regarding the smoother, most important is its action on the high frequency components.

It is then convenient to permute these components so that the high frequency ones are labeled first. Then A, expressed in the Fourier basis, admits the block diagonal

form

$$A_{\mathcal{F}} = \begin{pmatrix} A_{high} & \\ & A_{low} \end{pmatrix} .$$

The exact structure of B in this basis depends on the discretization scheme, but, as a general rule, the different types of frequency do not mismatch. That is, the discrete gradient of a low (resp. high) frequency pressure vector involve only low (resp. high) frequency velocity components. Moreover, typical stabilization matrices C are also block diagonal with respect to this partitioning of frequencies. Hence, the system matrix in the Fourier basis has the form

$$K_{\mathcal{F}} = \begin{pmatrix} A_{high} & & B_{high}^* & \\ & A_{low} & & B_{low}^* \\ B_{high} & & -C_{high} & \\ & B_{low} & & -C_{low} \end{pmatrix} . \tag{3.4}$$

For B_{high} and B_{low} , the conjugate transpose appears in the top right block because the basis transformation is orthogonal but in general complex. Hence the transformed matrix is no more real, and blocks that were transpose of each other become conjugate transpose of each other. On the other hand, blocks that were symmetric become Hermitian, and remain definite if they were previously definite. In particular, if C was positive definite on the null space of B^* , C_{high} is positive definite on the null space of B_{high}^* , and C_{low} is positive definite on the null space of B_{low}^* ,

The smoother has of course to be expressed in the same basis. In fact, standard smoothers for scalar elliptic PDEs are also diagonal when expressed in the Fourier basis; hence, applying to M_A the same transformations and permutations as to A yields

$$M_{\mathcal{F}}^{(A)} = \begin{pmatrix} M_{high}^{(A)} & \\ & M_{low}^{(A)} \end{pmatrix} .$$

On the other hand, because the basis transformation is orthogonal, the identity remains the identity in the new basis. Hence the smoother (3.1) becomes

$$M_{\mathcal{F}} = \begin{pmatrix} M_{high}^{(A)} & & & \\ & M_{low}^{(A)} & & \\ B_{high} & & -\omega^{-1} I & \\ & B_{low} & & -\omega^{-1} I \end{pmatrix} . \tag{3.5}$$

Now, despite the simpler form of the operators in the new basis, performing a complete analysis of the multigrid iteration matrix remains often out of reach for systems of PDEs. If one is interested in smoothers' performances, insight can be gained by considering a simplified (and, in some sense, idealized) scheme, in which the coarse grid corrections erase exactly all low frequency components of the error, while leaving the high frequency ones unchanged [19]. The corresponding convergence factor is referred to as the smoothing factor, and the following Theorem allows to bound this latter for the Uzawa smoother (3.1).

THEOREM 3.1. *Let $K_{\mathcal{F}}$ be defined by* (3.4)*, where A_{high} , A_{low} , C_{high} and C_{low} are, respectively, $n_{high} \times n_{high}$, $n_{low} \times n_{low}$, $m_{high} \times m_{high}$ and $m_{low} \times m_{low}$ Hermitian matrices with $m_{high} \leq n_{high}$. Assume that A_{high} is positive definite, that C_{high} is*

nonnegative definite, and that either B_{high} has full rank or C_{high} is positive definite on the null space of B_{high}^* .

Let $M_{\mathcal{F}}$ be defined by (3.5), where $M_{high}^{(A)}$, $M_{low}^{(A)}$ are, respectively, $n_{high} \times n_{high}$ and $n_{low} \times n_{low}$ Hermitian positive definite matrices.

Define the smoothing factor μ by

$$\mu \;=\; \rho\Big(\left(I - M_{\mathcal{F}}^{-1} K_{\mathcal{F}}\right) T_c^{(\mathcal{F})} \Big) \;,$$

where

$$T_c^{(\mathcal{F})} \;=\; \begin{pmatrix} I_{n_{high}} & & & \\ & 0_{n_{low}\times n_{low}} & & \\ & & I_{m_{high}} & \\ & & & 0_{m_{low}\times m_{low}} \end{pmatrix} \;.$$

If $\lambda_{\max}\left({M_{high}^{(A)}}^{-1} A_{high} \right) \leq 1$, letting

$$\mu_A \;=\; \rho\left(I - {M_{high}^{(A)}}^{-1} A_{high} \right)$$

and

$$\mu_S \;=\; \rho\left(I - \omega \left(C_{high} + B_{high} A_{high}^{-1} B_{high}^* \right) \right) \;,$$

there holds

$$\mu \;\leq\; \overline{\mu} \;=\; \max\left((\mu_A)^{1/2} \,,\, \mu_S \right) \;. \tag{3.6}$$

Proof. Using

$$M_{\mathcal{F}}^{-1} \;=\; \begin{pmatrix} {M_{high}^{(A)}}^{-1} & & & \\ & M_{low}^{(A)-1} & & \\ \omega\, B_{high}\, {M_{high}^{(A)}}^{-1} & & -\omega I_{m_{high}} & \\ & \omega\, B_{low}\, M_{low}^{(A)-1} & & -\omega I_{m_{low}} \end{pmatrix} \;,$$

one can check that

$$\mu \;=\; \rho(I - M_{high}^{-1} K_{high}) \;,$$

where

$$K_{high} \;=\; \begin{pmatrix} A_{high} & B_{high}^* \\ B_{high} & -C_{high} \end{pmatrix} \;, \qquad M_{high} \;=\; \begin{pmatrix} M_{high}^{(A)} & \\ B_{high} & -\omega^{-1} I_{m_{high}} \end{pmatrix} \;.$$

The required result then follows from Corollary 4.5 in [15]. Strictly speaking, this latter result is proved only for real matrices. But, examining thoroughly its proof as well as that of the theorems it is based on, it turns out that the extension to the complex case runs smoothly, reading "symmetric" as "Hermitian" and "transpose" as "conjugate transpose". □

Note that (3.3) implies a fortiori $\lambda_{\max}\left(M_{high}^{(A)\ -1} A_{high}\right) \leq 1$. Hence the only additional assumption in Theorem 3.1 always holds with the symmetric Gauss-Seidel smoother. This theorem provides then a bound on the smoothing factor involving only the smoothing factor μ_A associated with the smoother for velocities, and μ_S, which can be seen as the smoothing factor associated with Richardson iterations for the Schur complement

$$S \;=\; C + B\,A^{-1}B^T \;. \tag{3.7}$$

Bounding μ_A raises no particular difficulties. Often one can reuse available results for scalar elliptic PDEs [19, 23]. The analysis of μ_S is more tricky. Moreover, even if one is not interested in bounds, one needs to know something about the eigenvalues of S to be able to select ω in a sensible way. The following theorem is helpful in this respect.

THEOREM 3.2. *Let the assumptions of Theorem 3.1 hold. Let $\nu > 0$ and $\xi \geq 0$ be real numbers such that*

$$A_{high} \;=\; \nu\, A_{high}^{(0)} + \xi\, G_{high} \qquad \text{and} \qquad C_{high} \;=\; \nu^{-1}\, C_{high}^{(0)} \tag{3.8}$$

for some Hermitian positive definite matrices $A_{high}^{(0)}$, G_{high}, and Hermitian nonnegative definite matrix $C_{high}^{(0)}$. Let $h > 0$, $\beta > 0$, $\eta > 0$ and $\gamma \geq 0$ be real numbers such that

$$\lambda_{\max}\left(C_{high} + B_{high}\, A_{high}^{(0)\ -1} B_{high}^*\right) \leq\; \beta \;, \tag{3.9}$$

$$\lambda_{\max}\left(G_{high}^{-1} A_{high}^{(0)}\right) \leq\; \frac{1}{\eta\, h^2} \;, \tag{3.10}$$

$$\lambda_{\max}\left(C_{high}^{(0)}\right) \leq\; \gamma \;. \tag{3.11}$$

Then, setting, for some positive real number τ,

$$\omega \;=\; \frac{\tau\,\nu\;\left(1+\eta\,\frac{\xi\,h^2}{\nu}\right)}{\beta+\gamma\;\eta\,\frac{\xi\,h^2}{\nu}} \;, \tag{3.12}$$

one has $\mu_S < 1$ if $\tau < 2$. Moreover, letting

$$\kappa_\beta = \frac{\beta}{\lambda_{\min}\left(C_{high} + B_{high}\, A_{high}^{(0)\ -1} B_{high}^*\right)} \;, \tag{3.13}$$

$$\kappa_\eta = \frac{1}{\eta\, h^2\, \lambda_{\min}\left(G_{high}^{-1} A_{high}^{(0)}\right)} \;, \tag{3.14}$$

$$\kappa_\gamma = \begin{cases} \frac{\gamma}{\lambda_{\min}\left(C_{high}^{(0)}\right)} & \text{if } C_{high} \text{ is positive definite}, \\ \infty & \text{otherwise}, \end{cases} \tag{3.15}$$

there holds

$$\mu_S \;\leq\; \max\left(\tau-1\,,\,1-\tau\;\frac{1+\eta\,\frac{\xi\,h^2}{\nu}}{1+\kappa_\eta\,\eta\,\frac{\xi\,h^2}{\nu}}\;\frac{\frac{\beta}{\kappa_\beta}+\frac{\gamma}{\kappa_\gamma}\;\kappa_\eta\,\eta\,\frac{\xi\,h^2}{\nu}}{\beta+\gamma\;\eta\,\frac{\xi\,h^2}{\nu}}\right) \;. \tag{3.16}$$

In particular, if $\xi = 0$ *:*

$$\mu_S \;\leq\; \max\left(\tau - 1\,,\, 1 - \frac{\tau}{\kappa_\beta}\right)\;. \tag{3.17}$$

Proof. Reading inequalities in the nonnegative definite sense ($Q \geq R$ if and only if $Q - R$ is nonnegative definite), one first checks that

$$\begin{aligned} C_{high} + B_{high}\,A_{high}^{-1}\,B_{high}^{*} \;&=\; \nu^{-1}\,C_{high}^{(0)} + B_{high}\left(\nu\,A_{high}^{(0)} + \xi\,G_{high}\right)^{-1} B_{high}^{*} \\ &\leq\; \nu^{-1}\left(C_{high}^{(0)} + B_{high}\left(1 + \tfrac{\eta\,\xi\,h^2}{\nu}\right)^{-1}\,{A_{high}^{(0)}}^{-1} B_{high}^{*}\right) \\ &=\; \tfrac{\nu^{-1}}{1+\eta\,\frac{\xi\,h^2}{\nu}}\,\left(C_{high}^{(0)} + B_{high}\,{A_{high}^{(0)}}^{-1} B_{high}^{*} + \eta\,\tfrac{\xi\,h^2}{\nu} C_{high}^{(0)}\right) \\ &\leq\; \tfrac{\nu^{-1}}{1+\eta\,\frac{\xi\,h^2}{\nu}}\left(\beta + \gamma\;\eta\,\tfrac{\xi\,h^2}{\nu}\right)\;I\;. \end{aligned} \tag{3.18}$$

Hence, with ω given by (3.12), all eigenvalues of $\omega\left(C_{high} + B_{high}\,A_{high}^{-1}\,B_{high}^{*}\right)$ are in the interval $(0\,,\,\tau)$, entailing $\mu_S < 1$ if $0 < \tau < 2$. Moreover,

$$\mu_S \;\leq\; \max\left(\tau - 1\,,\, 1 - \frac{\tau\,\nu\left(1 + \eta\,\frac{\xi\,h^2}{\nu}\right)\;\lambda_{\min}\left(C_{high} + B_{high}\,A_{high}^{-1}\,B_{high}^{*}\right)}{\beta + \gamma\;\eta\,\frac{\xi\,h^2}{\nu}}\right)\;,$$

whereas, reusing (3.18)

$$\begin{aligned} C_{high} + B_{high}\,A_{high}^{-1}\,B_{high}^{*} \;&\geq\; \nu^{-1}\left(C_{high}^{(0)} + B_{high}\left(1 + \kappa_\eta\,\eta\,\tfrac{\xi\,h^2}{\nu}\right)^{-1}\,{A_{high}^{(0)}}^{-1} B_{high}^{*}\right) \\ &=\; \tfrac{\nu^{-1}}{1+\kappa_\eta\,\eta\,\frac{\xi\,h^2}{\nu}}\,\left(C_{high}^{(0)} + B_{high}\,{A_{high}^{(0)}}^{-1} B_{high}^{*} + \kappa_\eta\,\eta\,\tfrac{\xi\,h^2}{\nu} C_{high}^{(0)}\right) \\ &\geq\; \tfrac{\nu^{-1}}{1+\kappa_\eta\,\eta\,\frac{\xi\,h^2}{\nu}}\left(\tfrac{\beta}{\kappa_\beta} + \tfrac{\gamma}{\kappa_\gamma}\;\kappa_\eta\,\eta\,\tfrac{\xi\,h^2}{\nu}\right)\;I\;. \end{aligned}$$

Then (3.16) and (3.17) follow by considering, respectively, the second and the first term in the maximum. □

We now discuss the application of this result to the discretization of (2.1). Thus we let $A^{(0)}$ be a discrete representation of $(-\Delta)$, G that of the identity, and $C^{(0)}$ the proper stabilization term when $\nu = 1$.[1] Then, (3.8) holds with $A_{high}^{(0)}$, G_{high}, $C_{high}^{(0)}$ equal to the high frequency part of these matrices when expressed in the Fourier basis. Note that $G = I$ if a finite difference scheme is used, whereas G is the mass matrix for velocity components if a finite element method is used.

First observe that (3.9), (3.10), (3.11) a fortiori hold if

$$\lambda_{\max}\left(C^{(0)} + B{A^{(0)}}^{-1}B^{T}\right) \leq\; \beta\;, \tag{3.19}$$

$$\lambda_{\max}\left(G^{-1}A^{(0)}\right) \leq\; \frac{1}{\eta\,h^2}\;, \tag{3.20}$$

$$\lambda_{\max}\left(C^{(0)}\right) \leq\; \gamma\;. \tag{3.21}$$

[1] Because the quality of the stabilization depends on the eigenvalue distribution of the Schur complement [8], it is clear that the stabilization term has to be kept proportional to ν^{-1}.

For the infinite grid, (3.19) holds with $\beta = 1$ if a finite difference scheme is used, and $\beta = \lambda_{\max}(Q)$ for a finite element method, where Q is the pressure mass matrix. Further, finding a relevant η is most often easy; e.g., $\eta = 1/8$ for a five point finite difference scheme. Finally, γ depends on the stabilization scheme, with, of course, $\gamma = 0$ when the discretization used is stable. In the finite element context, "ideal" stabilization implies $\gamma = \lambda_{\max}(Q)$, see [8, Section 5.5.2]. On the other hand, the simple form of stabilization used for finite difference methods (see an example below) makes it easy to find a relevant γ. Note also that η is really needed only if ξ is positive and, moreover, large enough so that $\xi\, h^2/\nu$ is not fairly small. When the corresponding term in (2.1) comes from finite differences in time with an implicit scheme, this means using a time step of $\mathcal{O}(h^2)$, which is relatively rare in practice.

The analysis of κ_β, κ_η and κ_γ in (3.13), (3.14), (3.15) can be more tricky. In general, meaningful bounds cannot be obtained considering the matrices coming directly from the discretization, like in (3.19), (3.20), (3.21). At least for κ_η and κ_γ, it is important to restrict the operators to the high frequency components to obtain estimates that remain bounded for decreasing h. However, doing so, it is clear that these constants will be only moderately larger than 1 and independent of h and other problem parameters; see below for some examples.

On the other hand, it is not necessary to know κ_β, κ_η and κ_γ to properly implement the method. More precisely, they are needed if one wants to define ω using the rule (3.12) with τ such that the corresponding bound on μ_S is minimal. However, recall that the final bound $\overline{\mu}$ on μ cannot be better than the square root of μ_A – and it would be disappointing if it would be much larger. Hence we have a good rule of thumb if one uses (3.12) with τ somehow larger than one, so that this square root and $\tau - 1$ are roughly comparable. For instance, this approach with $\tau = 1.4$ has been found effective in all examples considered below.

A last but not least remark in this section. We found that ω can be defined using (3.12) with some default value for τ and parameters β, η, γ defined from (3.19), (3.20), (3.21); i.e., without involving anything specifically connected to the simplifications associated with LFA. It suggests that, if our analysis is restricted to that context, the approach itself is not, and the Uzawa smoother with the same rule to define ω can be proposed as a general approach to solve systems of the form (2.2) with a multigrid method. Note here that the value of β in (3.19) can be larger than that indicated above, depending on the boundary conditions. For finite element methods, see [8], $\beta = \lambda_{\max}(Q)$ remains valid if Dirichlet boundary conditions are used everywhere for velocity, whereas, otherwise, β can increase up to $2\,\lambda_{\max}(Q)$ for two-dimensional problems, and $3\,\lambda_{\max}(Q)$ for three-dimensional ones. The constant β may similarly increase for finite difference discretizations, but remains $\mathcal{O}(1)$. Since, for typical stabilization schemes, γ is also $\mathcal{O}(h^2)$ for finite element discretizations and $\mathcal{O}(1)$ for finite difference ones, it means that, when $\xi = 0$, one will have $\omega = \mathcal{O}(h^{-2})$ in the former case (recall that $\lambda_{\max}(Q) = \mathcal{O}(h^2)$), and $\omega = \mathcal{O}(1)$ in the latter.

These results are applied in the following sections, where the Uzawa smoother is analyzed, implemented and tested for three different types of discretizations.

4. Finite difference discretization on staggered grid. As first example of discretization, we consider the Marker-and-Cell (MAC) finite difference scheme [12]. We assume that Ω is a unit square region $(0,1) \times (0,1)$ divided into a uniform grid of cells of size h. The discrete velocities and pressures are distributed in a staggered arrangement. That is, the discrete pressure unknowns p are defined at the cell centers (the $\times$–points), and the discrete values of u and v, the components of the velocity

vector, are located at the grid cell faces in the ∘– and •–points, respectively, see Figure 4.1

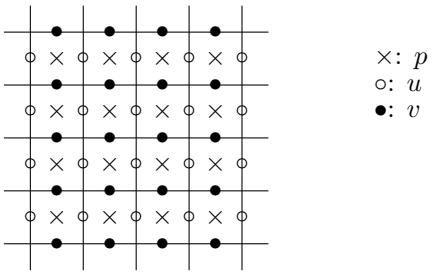

FIG. 4.1. *Staggered grid location of unknowns for Stokes equations.*

It is well-known that the MAC scheme is naturally stable with $C = 0$; see for example [17]. Hence the matrix of the discrete system writes

$$K_h = \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix}. \tag{4.1}$$

Notice that B_h is full rank, except that $B_h^T \mathbf{1} = \mathbf{0}$, but, as discussed at the end of §2, this singularity entails no practical difficulty.

In order to solve the linear system, a geometric multigrid method based on the Uzawa smoother described above can be designed. Regarding the coarse-grid correction part of the algorithm, in multigrid methods for Cartesian grid discretizations, one chooses standard geometric grid coarsening, i.e., the sequence of coarse grids is obtained by doubling the mesh size in each space direction. In this framework, an appropriate coarse-grid correction consists of geometric transfer operators $R_{h,2h}$, $P_{2h,h}$, and a direct coarse-grid discretization. The inter-grid transfer operators that act on the different unknowns are dictated by the staggered grid. For the Stokes equations, they are defined as follows: At grid-points corresponding to velocity unknowns u and v, one can consider 6-point restrictions and at grid-points associated with pressure unknowns p_h a 4-point cell-centered restriction can be applied. In stencil notation these restriction operators are given by

$$R^u_{h,2h} = \frac{1}{8}\begin{bmatrix} 1 & 2 & 1 \\ & * & \\ 1 & 2 & 1 \end{bmatrix}, \quad R^v_{h,2h} = \frac{1}{8}\begin{bmatrix} 1 & & 1 \\ 2 & * & 2 \\ 1 & & 1 \end{bmatrix}, \quad R^p_{h,2h} = \frac{1}{4}\begin{bmatrix} 1 & & 1 \\ & * & \\ 1 & & 1 \end{bmatrix},$$

respectively. For the prolongation of the corrections, we have applied the corresponding adjoint operators multiplied by a factor of 4.

To apply the results of the preceding section, we need to set up the framework of LFA analysis. In this context, we define an extension of the staggered grid to an infinite grid $G_h = G_h^1 \cup G_h^2 \cup G_h^3$, where

$$G_h^j := \{\mathbf{x}^j_{k_1,k_2} = (k_1,k_2)\,h + \boldsymbol{\delta}^j \mid k_1,k_2 \in \mathbb{Z}\}, \text{ with } \boldsymbol{\delta}^j = \begin{cases} (0,h/2), \text{ if } j=1, \\ (h/2,0), \text{ if } j=2, \\ (h/2,h/2), \text{ if } j=3. \end{cases}$$

The velocity unknowns u, and v, are located at points $\mathbf{x}^1_{k_1,k_2} \in G_h^1$ and $\mathbf{x}^2_{k_1,k_2} \in G_h^2$, respectively, whereas pressure unknowns are situated at nodes $\mathbf{x}^3_{k_1,k_2} \in G_h^3$. From the

definition of the occurring operators on G_h, the discrete solution, its current approximation and the corresponding error or residual can be represented by formal linear combinations of complex exponential functions, the Fourier modes. These functions form a unitary basis of the space of bounded infinite grid functions, and in G_h they are given by

$$\boldsymbol{\varphi}_h(\boldsymbol{\theta}, \mathbf{x}_{k_1,k_2}) := \begin{pmatrix} e^{\imath\boldsymbol{\theta}\cdot\mathbf{x}^1_{k_1,k_2}/h} \\ e^{\imath\boldsymbol{\theta}\cdot\mathbf{x}^2_{k_1,k_2}/h} \\ e^{\imath\boldsymbol{\theta}\cdot\mathbf{x}^3_{k_1,k_2}/h} \end{pmatrix},$$

where $\boldsymbol{\theta} \in \boldsymbol{\Theta} := (-\pi, \pi]^2$, $\mathbf{x}_{k_1,k_2} = (\mathbf{x}^1_{k_1,k_2}, \mathbf{x}^2_{k_1,k_2}, \mathbf{x}^3_{k_1,k_2})$, $\mathbf{x}^j_{k_1,k_2} \in G^j_h$. In this way, the Fourier space is defined by

$$\mathcal{F}(G_h) := span\{\boldsymbol{\varphi}_h(\boldsymbol{\theta}, \cdot) \,|\, \boldsymbol{\theta} \in \boldsymbol{\Theta}\},$$

and the behavior of the multigrid components can be analyzed by evaluating their effect of the Fourier modes. The subset of low frequencies is defined as $\boldsymbol{\Theta}^{2h}_{low} = (-\pi/2, \pi/2]^2$, and the subset of high frequencies is $\boldsymbol{\Theta}\backslash\boldsymbol{\Theta}^{2h}_{low}$. In the transition from G_h to G_{2h}, each low-frequency $\boldsymbol{\theta} = \boldsymbol{\theta}^{00} \in \boldsymbol{\Theta}^{2h}_{low}$ is coupled with three high-frequencies $\boldsymbol{\theta}^{11}$, $\boldsymbol{\theta}^{10}$, $\boldsymbol{\theta}^{01}$, given by

$$\boldsymbol{\theta}^{ij} = \boldsymbol{\theta}^{00} - (i\, sign(\theta_1), j\, sign(\theta_2))\pi,\ i, j = 0, 1.$$

Because of this, the Fourier space can be subdivided into the corresponding four-dimensional subspaces, known as $2h-$harmonics:

$$\mathcal{F}^{2h}(\boldsymbol{\theta}) := \{\boldsymbol{\varphi}_h(\boldsymbol{\theta}^{00}, \cdot), \boldsymbol{\varphi}_h(\boldsymbol{\theta}^{11}, \cdot), \boldsymbol{\varphi}_h(\boldsymbol{\theta}^{10}, \cdot), \boldsymbol{\varphi}_h(\boldsymbol{\theta}^{01}, \cdot)\}, \quad \text{with} \quad \boldsymbol{\theta} = \boldsymbol{\theta}^{00} \in \boldsymbol{\Theta}^{2h}_{low}.$$

Now, the two-grid iteration matrix is

$$M_{h,2h} = S^{\nu_2}_h (I_h - P_{2h,h}(K_{2h})^{-1} R_{h,2h} L_h) S^{\nu_1}_h,$$

where $S_h = I - M^{-1}_h K_h$ is the iteration matrix associated with the smoother, and ν_1 , ν_2 are, respectively, the number of pre- and post-smoothing steps. The LFA two-grid convergence factor

$$\rho = \rho\,(M_{h,2h}) \tag{4.2}$$

is easy to compute because $M_{h,2h}$ is block diagonal with respect to the partitioning in $2h-$harmonics; that is, only four coupled frequencies have to be considered at a time.

Often this convergence factor is well approximated by $\mu^{\nu_1+\nu_2}$, where μ is the smoothing factor already introduced in the preceding section and which can also be defined as

$$\mu = \sup_{\boldsymbol{\Theta}\backslash\boldsymbol{\Theta}^{2h}_{low}} \rho\,(S_h(\boldsymbol{\theta}))\,.$$

We now discuss the application of Theorems 3.1 and 3.2 to bound μ while selecting the parameter ω in the Uzawa smoother on a sensible basis. [2] It is well-known that the

[2]The theorems are proved for finite dimensional matrices only. However, if LFA matrices are infinite dimensional, they are block diagonal with respect to the partitioning in $2h-$harmonics, and one may check that applying the theorems separately to each diagonal block yields the same result as a formal extension of these theorems to the infinite dimensional case.

smoothing factor of the symmetric Gauss-Seidel method for the standard five-point discretization of Laplace operator is $\mu_A = 0.25$ [18]. Further, for the MAC-scheme β and κ_β in (3.9), (3.13) are equal to one. Hence, assuming that the parameter ω has been selected according to (3.12), one has, for $\xi = 0$,

$$\overline{\mu} \;=\; \max\left(0.5\,,\,\tau - 1\right)\,.$$

This implies that if $\tau \leq 1.5$, the upper bound on the smoothing factor is determined by the smoother for velocities, whereas, if $\tau > 1.5$, it corresponds to the smoothing factor associated with Richardson iterations for the Schur complement.

For $\xi > 0$, since

$$\lambda_{\min}\left(A^{(0)}_{high}\right) \geq \frac{2}{h^2}, \quad \lambda_{\max}\left(A^{(0)}_{high}\right) \leq \frac{8}{h^2},$$

it follows that $\eta = 1/8$ in formula (3.10), and $\kappa_\eta = 4$ in (3.14), so that (because $C = 0$, $\gamma = 0$ and therefore κ_γ is not important)

$$\mu_S \;\leq\; \max\left(\tau - 1\,,\,1 - \tau\,\frac{8\nu + \xi h^2}{8\nu + 4\xi h^2}\right)\,. \tag{4.3}$$

Notice that μ_A decreases with increasing ξ and becomes fairly small when ξ is fairly large; then, our bound $\overline{\mu}$ on μ amounts to the above value of μ_S.

To illustrate this numerically, we fix $h^{-1} = 256$ and $\nu = 1$ (observe that only the ratio ξ/ν matters), and consider two values of ξ, whereas the parameter τ is varied in the interval $[1, 1.6]$. In Table 4.1, the relaxation parameter ω given by (3.12), and the theoretical bound of the smoothing factor $\overline{\mu}$ provided by (3.6), together with the exact LFA smoothing factor μ are shown for $\nu_1 + \nu_2 = 1$.

ξ		$\tau = 1$	$\tau = 1.1$	$\tau = 1.2$	$\tau = 1.3$	$\tau = 1.4$	$\tau = 1.5$	$\tau = 1.6$
0	ω	1	1.1	1.2	1.3	1.4	1.5	1.6
	$\overline{\mu}$	0.5	0.5	0.5	0.5	0.5	0.5	0.6
	μ	0.5	0.5	0.5	0.5	0.5	0.5	0.6
10^5	ω	1.19	1.30	1.42	1.54	1.66	1.78	1.90
	$\overline{\mu}$	0.36	0.36	0.36	0.36	0.36	0.36	0.42
	μ	0.36	0.36	0.36	0.36	0.36	0.36	0.42

TABLE 4.1

LFA smoothing factor μ together with its theoretical bound $\overline{\mu}$ provided in (3.6), as a function of the relaxation parameter ω given in (3.12) for $\tau \in [1, 1.6]$.

One sees that our bound $\overline{\mu}$ accurately matches the actual smoothing factor μ. Regarding the choice of parameter τ, any value $\tau \in [1, 1.5]$ seems appropriate. From now on, we fix the value $\tau = 1.4$. This choice is based on the robustness of the resulting method for all cases analyzed in this work, including the different discretization schemes.

With parameter τ fixed, we analyze the behavior of the proposed smoother (3.1) based on symmetric Gauss-Seidel (SGS) relaxation for M_A, and compare its performance with variants where M_A is based instead on a standard (forward) Gauss-Seidel (GS) method; with two sweeps, this corresponds to the method suggested in [14], which has roughly the same cost, whereas one sweep represents a cheaper alternative. In Table 4.2, for different values of ξ and different numbers of smoothing steps, we

present the smoothing factor and the LFA two-grid convergence factor (4.2), using ω as in (3.12) with thus $\tau = 1.4$. Results are displayed for $h^{-1} = 256$ and $h^{-1} = 1024$. The method suggested in this work is significantly better than the method in [14], except in one case where both perform similarly. On the other hand doing just one forward sweep on velocities does not lead to a convergent method. Observe further that the convergence factors associated with the proposed approach are uniformly very small when using four smoothing steps,.

	$h^{-1} = 256$					$h^{-1} = 1024$				
		SGS for M_A		GS for M_A			SGS for M_A		GS for M_A	
				2 sw.	1 sw.				2 sw.	1 sw.
ξ	ω	μ	ρ	ρ	ρ	ω	μ	ρ	ρ	ρ
				$\nu_1 + \nu_2 = 1$						
0	1.40	0.50	0.44	0.87	1.42	1.40	0.50	0.44	0.87	1.42
100	1.40	0.50	0.44	0.87	1.42	1.40	0.50	0.44	0.87	1.42
10^5	1.66	0.36	0.80	0.80	1.24	1.41	0.48	0.43	0.84	1.42
				$\nu_1 + \nu_2 = 4$						
0	1.40	0.06	0.08	0.71	4.16	1.40	0.06	0.08	0.72	4.16
100	1.40	0.06	0.08	0.71	4.16	1.40	0.06	0.08	0.72	4.16
10^5	1.66	0.01	0.04	0.18	2.40	1.41	0.05	0.07	0.65	4.00

TABLE 4.2

Results for the MAC-scheme using the rule (3.12) *with* $\tau = 1.4$*; 1 (resp. 2) sw. stands for 1 (resp. 2) Gauss-Seidel sweep(s) for velocities in each smoothing step; 2 sw. is the strategy in [14].*

We now consider the performance of multigrid cycles on finite grids, and compare the Uzawa smoother as defined in this work with a state-of-the-art Vanka smoother [20]. Vanka smoothers are block Gauss-Seidel methods where one block consists of a small number of degrees of freedom. For the MAC scheme, the pressure and the velocities in the x- and y-directions in a grid cell are simultaneously updated, resulting in 5×5 blocks that are updated during the processing of a grid cell. A relaxation parameter $w = 0.7$ has been used in the implementation of the Vanka smoother for improving its results. In Table 4.3, for different numbers of smoothing steps, we show for both smoothers the number of iterations needed to reduce the initial residual norm by a factor of 10^{-10}, together with the asymptotic convergence factor (in brackets). In this table, the results for W-cycles are presented for $\xi = 0$ and $\xi = 10^5$. One sees that the performances are similar in all cases. Hence it does not pay off to incur the additional computational cost and implementation efforts related to a Vanka smoother.

5. Finite difference discretization on collocated grid. The second example of discretization is a standard finite difference discretization on collocated grids; i.e., all unknowns are located at the vertices of grid cells, which makes the discretization somewhat easier but induces the presence of spurious pressure modes with zero discrete divergence. To rule out these modes, the continuity equation is perturbed by adding an artificial elliptic pressure term. The matrix of the discrete system is then

$$K = \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}, \tag{5.1}$$

	$\xi = 0$				$\xi = 10^5$			
	Uzawa		Vanka		Uzawa		Vanka	
$(\nu_1 + \nu_2)$	#it.	(ρ)	#it.	(ρ)	#it.	(ρ)	#it.	(ρ)
1	27	(0.45)	26	(0.45)	35	(0.53)	22	(0.40)
2	17	(0.29)	15	(0.24)	13	(0.22)	11	(0.15)
3	11	(0.14)	11	(0.16)	9	(0.10)	8	(0.08)
4	8	(0.07)	10	(0.13)	8	(0.06)	7	(0.04)

TABLE 4.3

Comparison of Uzawa and Vanka smoother for the MAC scheme. Number of iterations and asymptotic convergence factor of W-cycle for different number of smoothing steps.

where $C = -\alpha h^2 \Delta_h$ is a scaled discrete Laplacian (with Neumann boundary conditions) acting on pressure unknowns. As this term is proportional to h^2, second order accuracy is maintained when all terms in the system are discretized with second order accuracy. Notice that K is singular because $B^T \mathbf{1} = \mathbf{0}$ and $C\mathbf{1} = \mathbf{0}$; this singularity is managed as discussed at the end of §2 and therefore entails no practical difficulty.

For the parameter α a well-balanced choice needs to be made. It should be chosen sufficiently small to maintain accuracy but at the same time sufficiently large to guarantee stable solutions; here we use $\alpha = 1/16$, which appears to be a reasonable choice in practice [13].

To solve the resulting system of algebraic equations, a geometric multigrid with standard coarse-grid correction components is adopted. Standard, h–$2h$, grid coarsening is employed and the inter-grid transfer operators are full-weighting restriction and bilinear interpolation. We analyze whether the proposed Uzawa smoother also gives satisfactory results for this type of collocated grid discretization. Here, basic local Fourier analysis [23] accompanies the numerical experiments, based on an infinite collocated grid G_h. Fourier modes are three-component vectors whose elements are identical to the scalar Fourier components $\varphi_h(\boldsymbol{\theta}, \mathbf{x}) = e^{\imath \boldsymbol{\theta} \mathbf{x}/h}$.

To define ω with the rule (3.12), we need information about the parameter β in (3.9), and, when $\xi > 0$, also on η and γ in (3.10), (3.11). After some algebraic calculations, Eq. (3.9) is satisfied with $\beta = 0.775$. On the other hand, for the five point finite difference scheme, we have $\eta = 1/8$ as in the preceding section, whereas, due to the simple form of the stabilization term, it is easy to derive that $\gamma = 1/2$. Since we deal with the same five-point discretization of the Laplacian, the smoothing factor of the symmetric Gauss-Seidel method remains $\mu_A = 0.25$ when $\xi = 0$ and smaller for increasing ξ. Because κ_β is larger than 1 in this example, it follows that, when using $\tau = 1.4$, the dominating term in our upper bound $\overline{\mu}$ is always μ_S ; i.e., (3.6) amounts to $\mu \leq \overline{\mu} = \mu_S$. In Table 5.1, for different values of ξ and different numbers of smoothing steps, we present the smoothing factor and the LFA two-grid convergence factor for the proposed smoother. As in the previous section, we compare with the same Uzawa smoother but with standard Gauss-Seidel (GS) instead of symmetric Gauss-Seidel for M_A. Similar conclusions apply. The method proposed here exhibits significantly better convergence results than the method in [14], except for very large values of ξ, for which both perform similarly for $h^{-1} = 256$.

As discussed at the end of §3, it may be easier or, in some sense, more general, to define ω based on β from (3.19) instead of (3.9). In the present case, this would lead to a slightly larger value, namely $\beta = 1$ instead of $\beta = 0.775$ considered above. Using as usual $\tau = 1.4$, this yields, e.g., $\omega = 1.40$ when $\xi = 0$, and a two-grid convergence

	$h^{-1}=256$					$h^{-1}=1024$				
		SGS for M_A		GS for M_A			SGS for M_A		GS for M_A	
				2 sw.	1 sw.				2 sw.	1 sw.
ξ	ω	μ	ρ	ρ	ρ	ω	μ	ρ	ρ	ρ
					$\nu_1+\nu_2=1$					
0	1.80	0.53	0.53	0.80	0.86	1.80	0.53	0.53	0.80	0.86
100	1.80	0.53	0.53	0.80	0.86	1.80	0.53	0.53	0.80	0.86
10^5	1.91	0.51	0.51	0.51	0.66	1.81	0.53	0.53	0.77	0.85
					$\nu_1+\nu_2=4$					
0	1.80	0.08	0.10	0.61	0.79	1.80	0.08	0.10	0.61	0.79
100	1.80	0.08	0.10	0.61	0.79	1.80	0.08	0.10	0.61	0.79
10^5	1.91	0.06	0.16	0.16	0.25	1.81	0.08	0.09	0.52	0.73

TABLE 5.1

Results for finite difference discretizations on collocated grids using the rule (3.12) *with* $\tau = 1.4$ *; 1 (resp. 2) sw. stands for 1 (resp. 2) Gauss-Seidel sweep(s) for velocities in each smoothing step; 2 sw. is the strategy in [14].*

factor equals 0.17 for $h^{-1} = 256$ and $\nu_1 + \nu_2 = 4$. Hence there is no significant difference with ω induced by $\beta = 0.775$. Similar conclusions are obtained for other values of ξ and h, suggesting that one may indeed define ω with β from (3.19) without significant loss of performance.

6. Linear finite element discretization on triangular grid. The third example of discretization deals with a linear finite element formulation of problem (2.1) on an equilateral structured triangular mesh. We therefore consider the weak formulation of (2.1).

Find $(\mathbf{u}, p) \in \mathcal{U} \times \mathcal{Q}$, such that

$$\xi(\mathbf{u}, \mathbf{v}) + \nu(\nabla\mathbf{u}, \nabla\mathbf{v}) - (p, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathcal{U}, \tag{6.1}$$

$$(\nabla\cdot\mathbf{u}, q) = 0, \quad \forall q \in \mathcal{Q}, \tag{6.2}$$

where $(\cdot,\cdot)$ denotes the L^2 scalar product, Ω is a bounded domain in $\mathbb{R}^2$, $\mathcal{U} = (H_0^1(\Omega))^2$, and $\mathcal{Q} = L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_\Omega q\mathrm{d}\mathbf{x} = 0\}$, i.e., $L^2(\Omega)$-functions which only differ by a constant are not distinguished.

Let $\mathcal{T}_h$ be an equilateral structured triangulation of Ω, and $\mathcal{U}_h \subset (H_0^1(\Omega))^2$, and $\mathcal{Q}_h \subset L_0^2(\Omega)$ be the corresponding spaces of piecewise linear functions on $\mathcal{T}_h$. Since the pair $(\mathcal{U}_h, \mathcal{Q}_h)$ provides an unstable finite element scheme, similarly to the case of finite differences on a collocated grid, we must add an additional term to the discrete equations for stabilization. To this end, we consider the bilinear form on $\mathcal{Q}_h \times \mathcal{Q}_h$, defined by

$$c(p_h, q_h) = \sum_{T\in\mathcal{T}_h} h^2 \int_T \nabla p_h\, \nabla q_h \mathrm{d}\mathbf{x},$$

where h denotes grid size (i.e. the length of the edge on T). The stabilized discrete formulation of the Stokes problem in its weak form reads as follows.

Find $(\mathbf{u}_h, p_h) \in \mathcal{U}_h \times \mathcal{Q}_h$, such that

$$\xi(\mathbf{u}_h, \mathbf{v}_h) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) - (p_h, \nabla \cdot \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathcal{U}_h, \tag{6.3}$$

$$(\nabla \cdot \mathbf{u}_h, q_h) + \bar{\alpha} c(p_h, q_h) = 0, \quad \forall q_h \in \mathcal{Q}_h, \tag{6.4}$$

where the term $\bar{\alpha} c(p_h, q_h)$ refers to the stabilization of the problem, and $\bar{\alpha} > 0$ is a given parameter. The choice $\bar{\alpha} = 1/12$ appears to be optimal for linear elements [7], so it is used here as well.

The discrete problem obtained gives rise to a saddle point problem like (5.1). A geometric multigrid method based on the proposed Uzawa smoother is employed to solve the discrete equations. The hierarchy of grids is obtained by a regular refinement, dividing the triangular domain into four congruent triangles, connecting the midpoints of their edges and so forth until the mesh has the desired scale. The inter-grid transfer operators are the canonical operators related to linear finite elements over triangles, i.e., the seven-point prolongation and its adjoint as the restriction [11].

Local Fourier analysis used to be applied to discretizations on rectangular grids, however, in [9] LFA was extended to discretizations on non-rectangular grids, in particular, to triangular grids. The key to this generalization was a two-dimensional Fourier transform using coordinates in non-orthogonal bases. A unitary basis fitting the structure of the grid is considered as the spatial basis, and its reciprocal basis is considered for the frequency space. This way, a discrete Fourier transform for discrete functions defined on non-rectangular grids can be defined and LFA on non-rectangular grids can be performed as on rectangular grids.

We first apply Theorem 3.2 to the FEM structured equilateral triangular grid discretization. The smoothing factor of the symmetric Gauss-Seidel method for this discretization of the Laplacian is $\mu_A = 0.173$. Further, we find $\eta = 1/24$, and $\gamma = \sqrt{3}h^2/4$. Finally, Eq. (3.9) is satisfied with $\beta = 0.68\, h^2$.

With the help of the triangular LFA, the convergence behavior of multigrid based on the Uzawa smoother is analyzed for different values of ξ and different numbers of smoothing steps. Its performance is compared to the Uzawa smoother resulting from standard Gauss-Seidel relaxation for M_A and the approach in [14]. These comparisons are presented in Table 6.1, where for different numbers of smoothing steps and different values of ξ the two-grid convergence factors computed by LFA are shown, together with the relaxation parameter given by the theory, as well as the corresponding smoothing factor. The results are displayed for $h = 1/256$ and $h = 1/1024$. For small ξ-values the proposed approach is superior to the other two multigrid strategies, when more than one smoothing step is considered. In the case of large ξ, the performances are comparable.

As in previous numerical sections, we are interested in the numerical convergence of the multigrid cycles. Moreover, we compare the behavior of the proposed strategy with multigrid based on a suitable Vanka smoother. It simultaneously updates, for each grid point, the corresponding pressure unknown and the twelve unknowns associated to the velocities located at the six points around it, resulting in 13×13 systems that are updated during the processing of a vertex of the triangulation. A relaxation parameter $w = 0.7$ has been chosen for improving the smoothing properties of the Vanka smoother. In Table 6.2, the number of iterations to reduce the initial residual by a factor of 10^{-10} are shown, together with the asymptotic convergence factors in brackets. The results are displayed for W-cycles with different numbers of smoothing steps, for $\xi = 0$. The convergence of the Uzawa-based multigrid method is comparable to the performance with Vanka smoothers, that are however more expensive.

	$h^{-1}=256$					$h^{-1}=1024$				
		SGS for M_A		GS for M_A			SGS for M_A		GS for M_A	
				2 sw.	1 sw.				2 sw.	1 sw.
ξ	ω	μ	ρ	ρ	ρ	ω	μ	ρ	ρ	ρ
					$\nu_1+\nu_2=1$					
0	$2.06/h^2$	0.33	0.66	0.72	0.58	$2.06/h^2$	0.33	0.66	0.72	0.58
100	$2.06/h^2$	0.33	0.66	0.72	0.58	$2.06/h^2$	0.33	0.66	0.72	0.58
10^5	$2.10/h^2$	0.24	0.44	0.48	0.44	$2.06/h^2$	0.32	0.65	0.70	0.58
					$\nu_1+\nu_2=4$					
0	$2.06/h^2$	0.01	0.10	0.27	0.43	$2.06/h^2$	0.01	0.10	0.27	0.43
100	$2.06/h^2$	0.01	0.10	0.27	0.43	$2.06/h^2$	0.01	0.10	0.27	0.43
10^5	$2.10/h^2$	0.003	0.19	0.19	0.19	$2.06/h^2$	0.01	0.08	0.23	0.40

TABLE 6.1

Results for the P1-P1-scheme for equilateral triangular grids using the rule (3.12) *with* $\tau = 1.4$ *; 1 (resp. 2) sw. stands for 1 (resp. 2) Gauss-Seidel sweep(s) for velocities in each smoothing step; 2 sw. is the strategy in [14].*

	$\xi=0$			
	Uzawa		Vanka	
$(\nu_1+\nu_2)$	#it.	(ρ)	#it.	(ρ)
1	41	(0.63)	21	(0.38)
2	14	(0.22)	10	(0.14)
3	10	(0.13)	8	(0.08)
4	9	(0.10)	7	(0.06)

TABLE 6.2

Comparison of Uzawa and Vanka smoother for the linear finite element method. Number of iterations and asymptotic convergence factor of W-cycle for different number of smoothing steps.

7. Conclusions. An Uzawa smoothing method for the multigrid solution of the generalized Stokes system has been introduced, discussed and analyzed. Detailed LFA smoothing analysis resulted in a formula to determine the appropriate relaxation parameter ω. The resulting multigrid method based on the Uzawa smoother with suitable parameter appears as highly efficient for staggered and collocated finite difference discretizations, as well as for a finite element Stokes discretization. The multigrid results are confirmed by two-grid LFA, and comparison with Vanka smoother based multigrid appears favorably for the Uzawa smoother in the Stokes case.

As any study based on LFA, our analysis is restricted to constant coefficient problems. Regarding situations where, like in [10], the viscosity ν (strongly) varies inside the domain, we further observe that our rule (3.12) to define the relaxation parameter cannot be used anymore without ambiguity, as it sets ω proportional to ν. A straightforward adaptation would then consists in letting ω be variable as well, to keep it proportional to ν. In practice, it amounts to apply a "local" value of ω according to the "local" value of ν when relaxing the pressure unknowns as indicated in the pseudo algorithm on page 5. Such generalization deserves however further

investigations that are outside the scope of the present paper.

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