WENO and Blended BDF Discretizations for Option Pricing Problems

Cornelis W. Oosterlee

Dept. of Applied Mathematical Analysis,
Delft University of Technology, the Netherlands.

Joint work with
Jörg Frisch (Fraunhofer SCAI, Sankt Augustin Germany),
Francisco J. Gaspar (University of Zaragoza, Spain)

Outline

- Brief introduction to different types of options.
- Quantities determining the sensitivity of a portfolio.
- Discretization in time of BDF type.
- Discretization in space of WENO type.
- Numerical result: simple Black-Scholes.
- Numerical result: Asian option.

Application: Option pricing

Types of options

- European Call option: at a certain time in the future (the expiry date), the holder of the option may purchase an asset (shares, stocks) for a prescribed amount (the exercise price).
  The other party of this contract (the writer) must sell the asset, if the holder decides to buy it.
- European Put option: gives the right to sell an asset on a certain date at a prescribed amount.
- American options (exercise permitted at any time during the life of an option)
- Path-dependent options (whose values does not only depend on the stock price, but also on other functions of the price development). Such options might depend on
  - some form of average (Asian options),
  - the maximum or the minimum of the asset price (lookback options),
  - if an asset reaches some prescribed value before expiry, the option can become valid or become worthless (barrier options)

Speculation and Hedging

- Options have two main uses: speculation and hedging.
- If an investor thinks that certain shares will go down, he can sell shares or buy put options.
- Writers of options typically use them as an insurance to reduce risk against unexpected movements in the market.
- Hedging:
  - Suppose a portfolio with shares and put options.
    If the share price s falls, the value of the portfolio depends on the ratio of s and option value u.

    A ratio exists, which results in no movement in the value of the portfolio. This ratio is instantaneously risk-free.

    A reduction of risk, for example by combining a number of shares and options in a portfolio is called hedging.

Main interesting questions are

- How much would one pay for this right ?
  (What's the value of an option ?)
- How can the writer minimize the risk associated ?

Notation

- The price of an asset (stock, share) is $s(t)$
- The value of an option is $u(s(t), t)$
- Its exercise price is E
- The expiry time is T, so $0 \leq t \leq T$
- The interest rate r, (here assumed to be constant), $r = 0.05$ means 5 % interest per year,
- The volatility σ, indicates the randomness of the asset price (standard deviation).

- Model for generating asset prices is, based on geometric Brownian motion:
$$\frac{ds}{s} = \sigma dw + \mu dt$$
with μdt a deterministic return similar to placing money in a bank, and σdw a term involving a stochastic change with dw sample from a normal distribution with mean 0.

Main Assumptions

There are different approaches for obtaining information on the prices of options

$\Rightarrow$ Set up a partial differential equation (with the help of Ito's lemma) and solve it by numerical methods, if necessary.

- The asset price follows the lognormal random walk.
- Interest rate r and volatility σ are constant.
- Transaction costs for hedging are not included in the model.
- No divident is paid during the life of the option.
- There are no arbitrage possibilities.

$\Rightarrow$ Black-Scholes partial differential equation:
(for a European option)

$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0$$

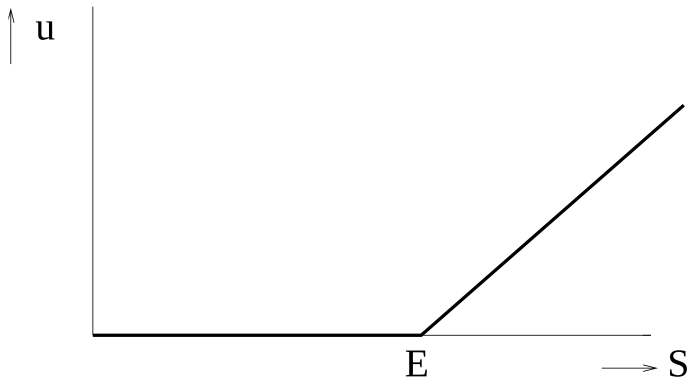
- The Black-Scholes equation is a parabolic PDE

Final and boundary conditions

Call

- The holder of a European call option will exercise his option, if $s > E$ at $t = T$, paying an amount E to obtain an asset worth s. The profit is $s - E$. If $s < E$ at $t = T$ one will not exercise the European option. The option is in this case valueless. So,

$$u(s, T) = \max(s - E, 0).$$



- Boundary conditions,
  - at $s = 0$: $u(0, t) = 0$.
  - at $s \to \infty$: $u(s, t) = s - Ee^{(-r(T-t))}$ (by arbitrage arguments).

$\Rightarrow$ With such conditions, the European option can often be solved analytically !

- A transformation $t := T - t$ transforms the equation backward in time to a forward equation with initial condition.

Continuous Asian Options

Options depending on averages

- The exercise price or the asset price is replaced by an average of the asset price:
  - Final conditions for an arithmetic-average floating strike call:
$$u(s,T) := g(s,T) = \max(s - \frac{1}{T}\int_0^T s(\tau)d\tau, 0)$$
  - Final conditions for an arithmetic-average fixed strike call:
$$u(s,T) := g(s,T) = \max(\frac{1}{T}\int_0^T s(\tau)d\tau - E, 0)$$
- With a new variable: $a(t) := (\int_0^t s(\tau)d\tau)/t$, one obtains a Black-Scholes type equation for Asian options:
$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} + \left(\frac{s-a}{t}\right)\frac{\partial u}{\partial a} - ru = 0$$
- There is no diffusion term in the a-direction.

$\Rightarrow$ In CFD, this situation often occurs and appropriate discretizations for such terms have been developed there.

$\Rightarrow$ Ultraparabolic equation.

Research

Discretizations and Solution Methods

Accurate “space” discretization for convection-dominated problems:

- central discretization $\rightarrow$ upwind discretization $\rightarrow$
  higher order upwind discretization $\rightarrow$
  Total Variation Diminishing schemes (TVD), with limiters $\rightarrow$
  (Weighted) Essentially Non-Oscillatory schemes (WENO).
- Appropriate time discretizations:
  Combinations of implicit and explicit schemes:
  Variations (Hundsdorfer, 2001) of the Backward Difference Formulae BDF, (Gear 1970)

Time Discretization

- Second order ($O(\delta t^2)$) accuracy: Backward difference formulae (BDF), combinations of implicit and explicit schemes (IMEX)
- BDF 2:

$$\frac{3/2u_h^{(m+1)} - 2u_h^{(m)} + 1/2u_h^{(m-1)}}{\delta t} + A_h u_h^{(m+1)} = 0$$

- BDF3:

$$\frac{\partial u(s,a,t)}{\partial t} \approx \frac{11u^{(m+1)} - 18u^{(m)} + 9u^{(m-1)} - 2u^{(m-2)}}{6\delta t}$$

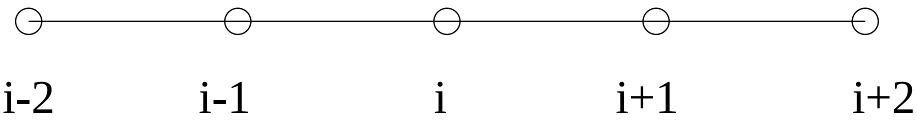
- IMEX (Partially implicit BFD2 blends, Hundsdorfer 2001:) $\mathcal{A} = \mathcal{C}$ onvection $+ \mathcal{D}$ iffusion $+ \mathcal{R}$ eaction

- Diffusion and reaction are treated fully implicitly. Convection is done by a combination of implicit and explicit BDF2:

$$\begin{aligned}\frac{3u_h^{(m+1)} - 4u_h^{(m)} + u_h^{(m-1)}}{2\delta t} &+ C_h(\theta u_h^{(m+1)} + (1-\theta)(2u_h^{(m)} - u_h^{(m-1)})) \\ &+ D_h u_h^{(m+1)} + R_h u_h^{(m+1)} = 0\end{aligned}$$

- $\theta = 0.75$: unconditionally stable for convection-diffusion problems.

Discretization of convection



Suppose $b < 0$: $\quad b\dfrac{\partial u}{\partial s}$

- WENO5: $\dfrac{\partial u}{\partial s} = 1/h_s(\tilde{u}_{i+1/2} - \tilde{u}_{i-1/2})$
- For $b < 0$, $\quad \tilde{u}_{i+1/2} = \omega_0 u^0 + \omega_1 u^1 + \omega_2 u^2$ with

$$\begin{aligned} u^0 &= (-u_{i-1} + 5u_i + 2u_{i+1})/6 \\ u^1 &= (2u_i + 5u_{i+1} - 2u_{i+2})/6 \\ u^2 &= (11u_{i+1} - 7u_{i+2} + 2u_{i+3})/6 \end{aligned}$$

with weights: $\omega_i = \alpha_i / \Sigma_{i=0}^{2} \alpha_i, \quad \alpha_i = d_i/(\beta_i + \varepsilon)^2$.

- β_i are the WENO smoothness indicators, for example,

$$\beta_0 = \frac{13}{12}(u_{i-1} - 2u_i + u_{i+1})^2 + \frac{1}{4}(u_{i-1} - 4u_i + 3u_{i+1})^2, etc.$$

- Parameters $d_0 = 0.3,\ d_1 = 0.6,\ d_2 = 0.1,\ \varepsilon = 10^{-6}$.

$\Rightarrow$ Summarizing, we blend discretizations in time and for convection in space.

Important Quantities

Sensitivity Analysis

• The important quantities to be calculated are the so-called hedge parameters.

• Delta: the rate of change of the value of the option (or portfolio) with respect to s. The largest random component of a portfolio is eliminated.
It indicates the number of shares, that should be kept with each option issued in order to cope with a loss in the case of exercise.

$$\Delta = \frac{\partial u}{\partial s}$$

• Gamma: indicates the change in Delta

$$\Gamma = \frac{\partial^2 u}{\partial s^2}$$

• If Gamma is low, it is only necessary to sometimes change the portfolio. If it is high, the portfolio under consideration results only for a very short period of time in a risk-less scenario.

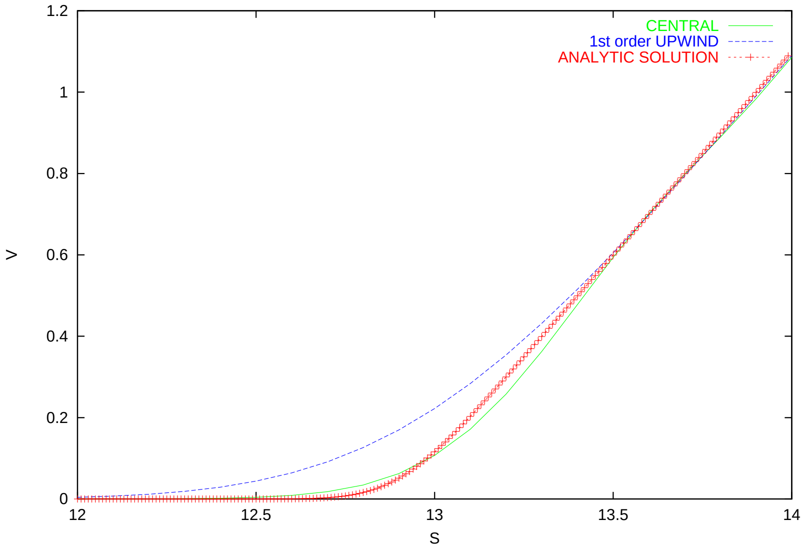
• There are several other important hedging parameters.

Experiment: European Call Option

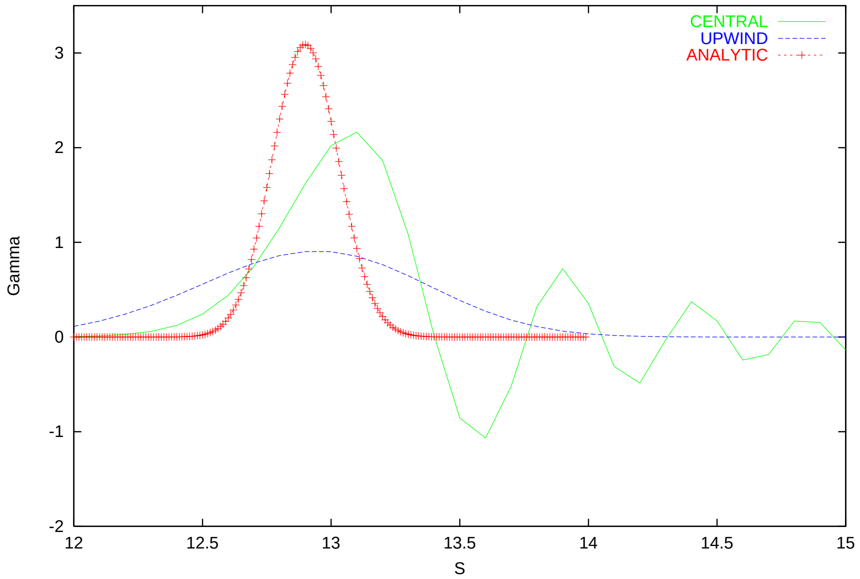
- with small volatility σ (as in Zvan et al. J. Comp. Fin. 1998): $r = 0.15, \sigma = 0.01, T = 1.0, E = 15$

$\Rightarrow$ Convection-diffusion with extremely small diffusion term.

- Central discretization leads to oscillations (in Δ and in Γ).
- 1st order upwind discretization leads to a solution with too much artificial diffusion.



u versus s at $t = 0$

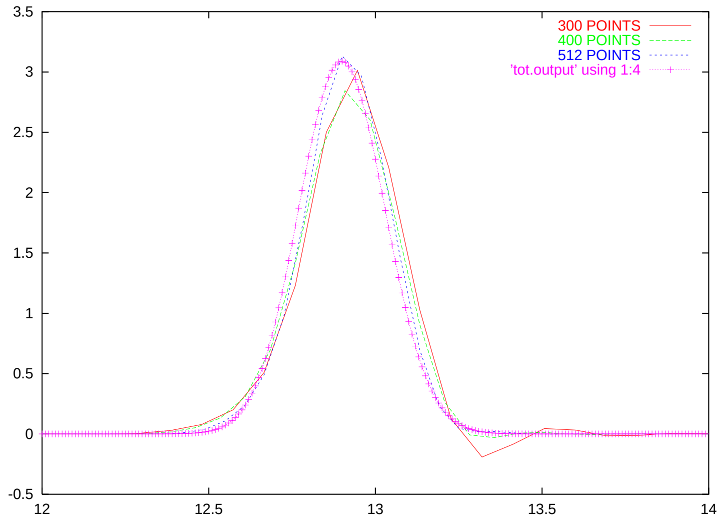


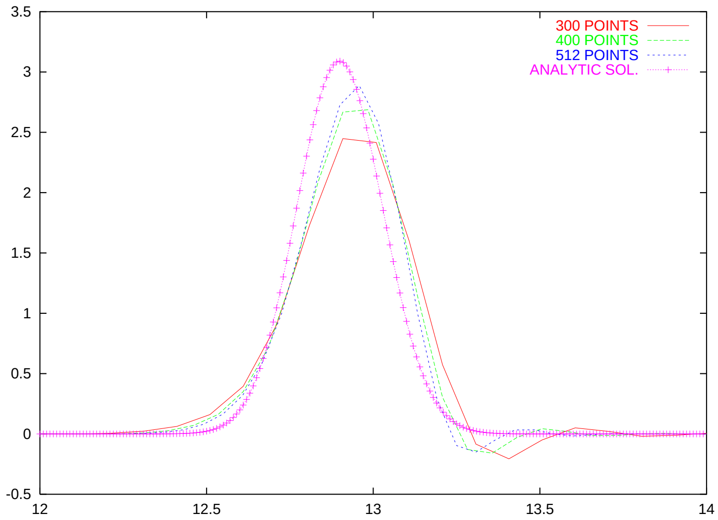
Γ versus s at $t = 0$

Modern Discretizations

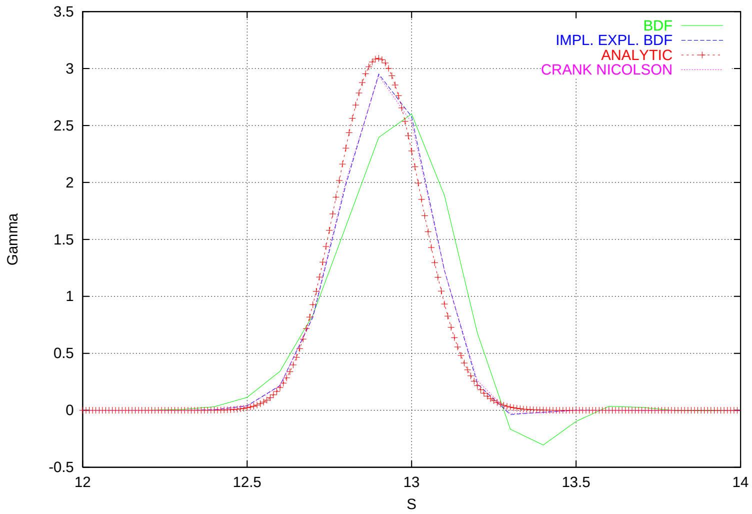
from CFD

- Γ versus s
- WENO5 for convection,
- BDF3 versus Implicit-explicit BDF(2) scheme in time.
- Mesh sizes: $\delta t = 0.01$ for BDF3, $\delta t = 0.02$ for θ-BDF2 with $\theta = 0.75$,
- h_s varies from 0.05 to 0.1.

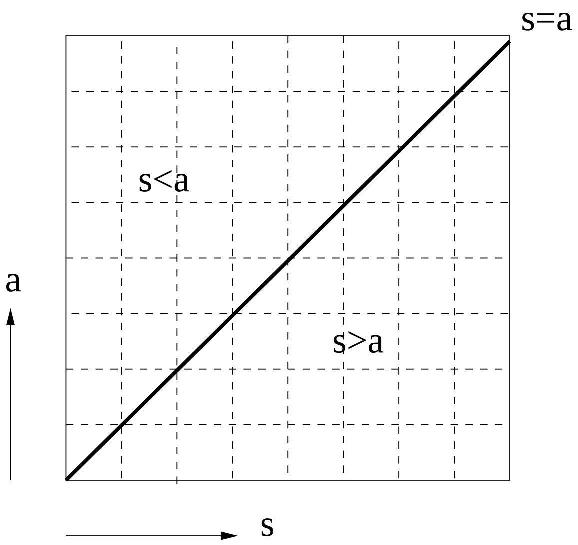




- $h_s = 0.05,\ \delta t = 0.02$, van Leer limiter.



Boundary and initial conditions



- Transformed equation:

$$u_t = \frac{1}{2}\sigma^2 s^2 u_{ss} + rsu_s - \frac{a-s}{T-t}u_a - ru$$

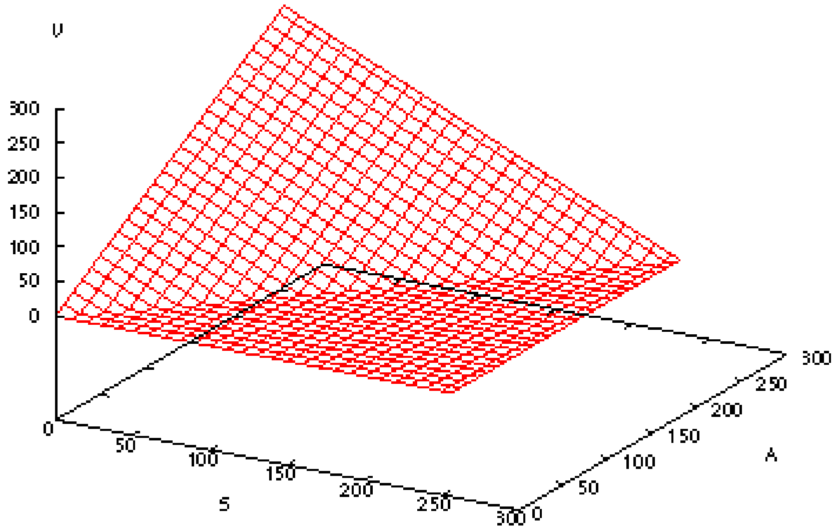
- Initial condition: $u(s,a,0) = g^*(s,a,0)$
- Boundary condition at $s = 0$ and $s = s_{max}$ arise from a jump condition based on arbitrage arguments:
  - $s = 0: \quad u_t = -\frac{a}{T-t}u_a - ru$
  - $s = s_{max}: \quad u_t = -\frac{(a-s_{max})}{T-t}u_a$
- The line $s = a$ acts as a sort of boundary, too.

Asian Example

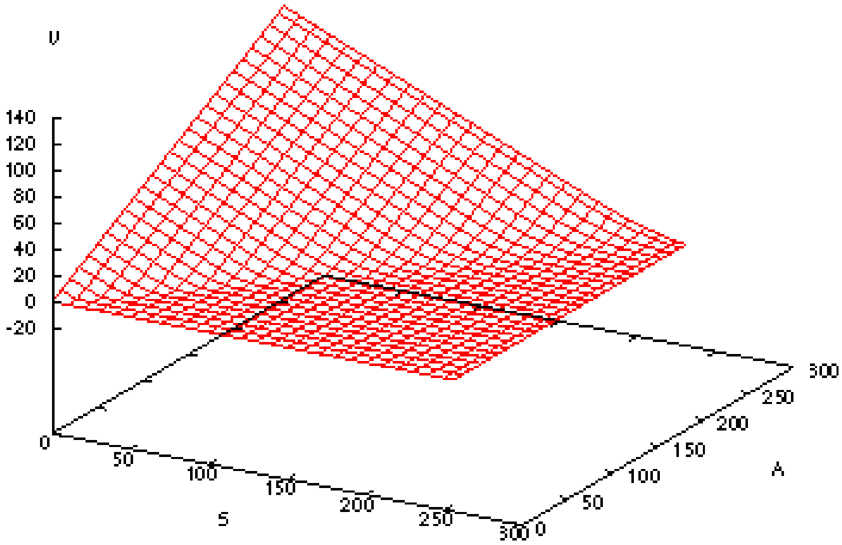
Floating strike put option

- Final condition: $u(s,T) := g(s,T) = \max(s-a,0)$
- $E = 100$, $\sigma = 0.1$, $T = 1$, $\delta S = \delta A = 1, \delta t = 0.01$
- IMEX BDF ($\theta = 0.75$) and TVD discretization with van Leer limiter.

- Initial solution and solution at $t = 0.5$



$t = 0$

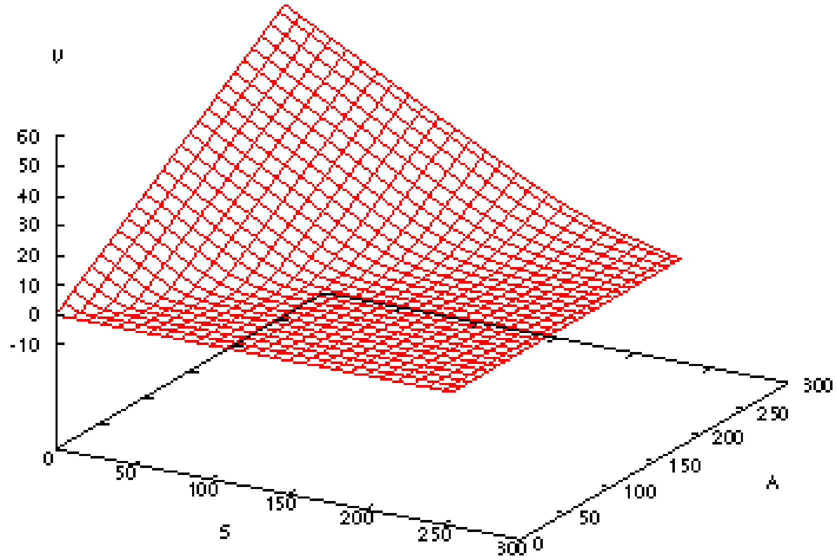


$t = 0.5$

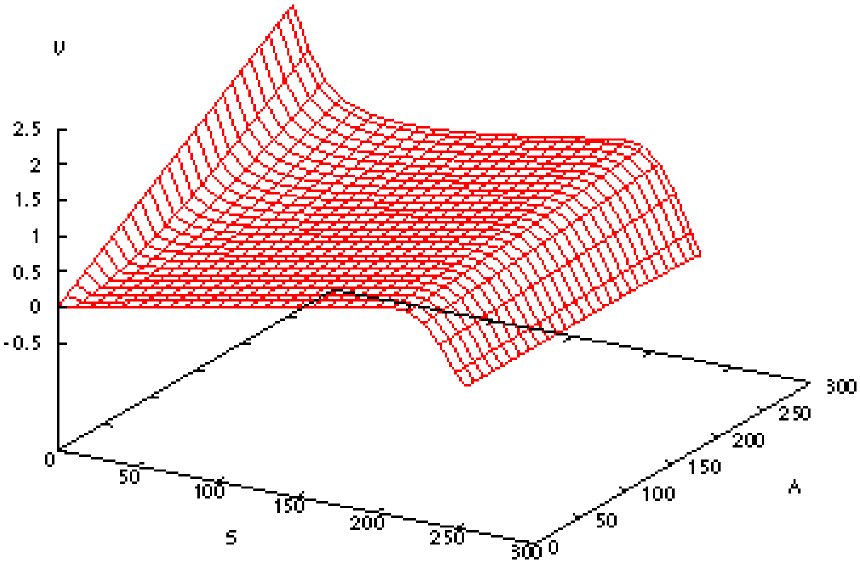
Asian Example

Floating strike put option

- Solution at $t = 0.75$ and $t = 0.99$



$t = 0.75$

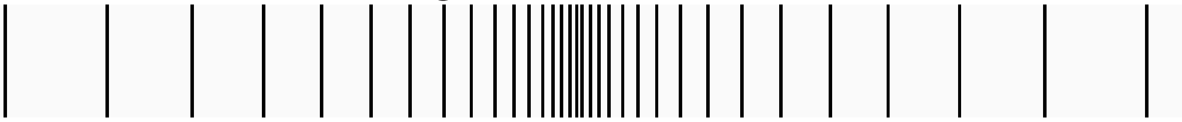


$t = 0.99$

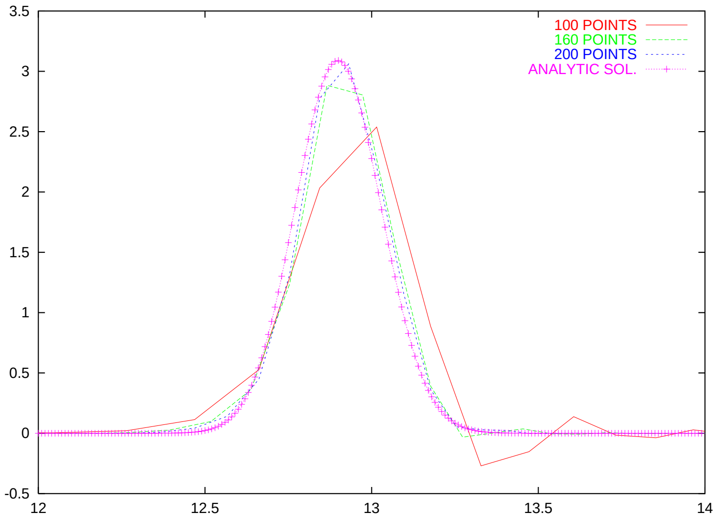
- Computed u at $t = 0$ and $s = a = 100$, $\sigma^2 = 0.01$:
  0.59 (Refs. 0.59 ZVF (1998), 0.61 BP (1996))
- Computed u at $t = 0$ and $s = a = 100$, $\sigma^2 = 0.04$:
  2.44 (Refs. 2.42 ZVF (1998), 2.49 BP (1996))

Stretched grid

Exponentially stretched grid:



- BDF3 time discretization



Conclusions and Outlook

- We deal with PDEs for option pricing.
- Typically, European options are simple and can often be solved analytically.
- Hedge parameters are important for financial engineers.
- With small volatility, or exotic options (no diffusion type operator in one dimension) discretizations developed in CFD are helpful for accurate prediction of option prices and hedge parameters.
- We use high order accurate discretizations in space and time.
- Further reduction of grid points is possible by local grid stretching in ‘space’ near the exercise price and in time.