On Multigrid for Linear Complementarity Problems with Application to American-style Options

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Outline

- Brief introduction to option pricing
- Discuss different types of options:
  the impact on differential operators, boundary conditions
- American option –> Free boundary problem –>
  Linear complementarity problem (LCP)
- Multigrid for LCPs
- accelerate by a recombination technique
- Numerical result

Application: Option pricing

Types of options

- European Put option: at a certain time in the future (the expiry date), the holder of the option may sell an asset (shares, stocks) for a prescribed amount (the exercise price). The other party of this contract (the writer) must buy the asset, if the holder decides to sell it.
- European Call option: gives the right to purchase an asset on a certain date at a prescribed amount.
  The writer is then obliged to sell the asset.
- American options (exercise permitted at any time during the life of an option)
- Path-dependent options (whose values does not only depend on the stock price, but also on average values of the price development).
- Here we concentrate on the American put option.

Notation

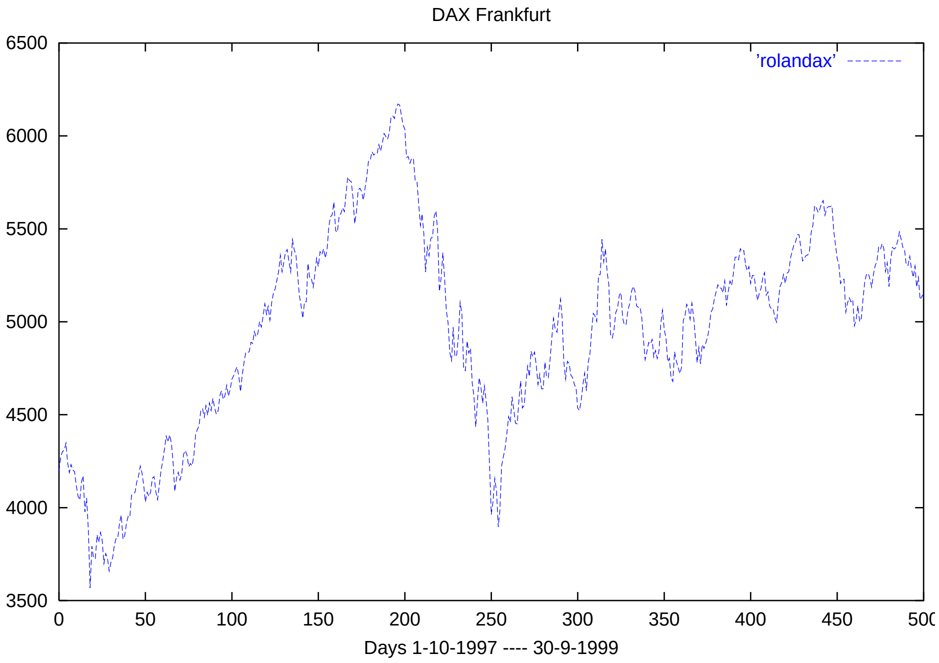
- The price of an asset (stock, share) is $s(t)$
- The value of an option is $u(s(t), t)$
- Its exercise price is E
- The expiry time is T, so $0 \leq t \leq T$
- The interest rate r, (here assumed to be constant), $r = 0.05$ means 5 % interest per year
- The volatility σ, indicates the randomness of the asset price, $\sigma = 0.2$ means 20 % volatility

With a constant interest rate r, one can receive a guaranteed amount E at the time T and pay now an amount M. I.e. money in the bank, M, grows exponentially:

$$\begin{aligned} \frac{dM}{dt} &= rM \quad \text{with solution} \quad M = ce^{rt}, \\ \text{boundary condition} \quad M &= E \quad \text{at} \quad t = T: \\ \Rightarrow M &= Ee^{-r(T-t)} \end{aligned}$$

Randomness of stock prices

- We do not know and cannot predict tomorrow's values of asset prices.



- Model for generating asset prices is geometric Brownian motion:

$$\frac{ds}{s} = \sigma dw + \mu dt$$

with μdt a deterministic return similar to placing money in a bank, and σdw the random change with dw a sample from a normal distribution with mean 0.

- The volatility σ measures the standard deviation of returns.

Main Assumptions

There are different approaches for obtaining information on the prices of options

- Binary trees
- Stochastic approaches, like Monte Carlo simulations
- Numerical integration methods
- Eliminate the largest contributions of randomness.
  Set up a partial differential equation (with the help of Ito's lemma) and solve it by numerical methods, if necessary.

- The asset price follows the lognormal random walk.
- Interest rate r and volatility σ are constant.
- Transaction costs for hedging are not included in the model.
- No divident is paid during the life of the option.
- There are no arbitrage possibilities.

$\Rightarrow$ Black-Scholes partial differential equation:
(for a European option)

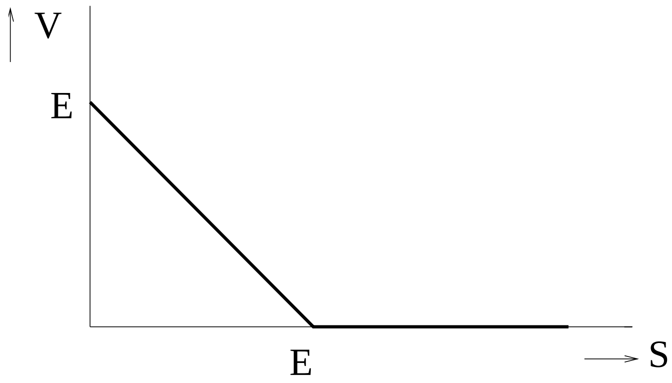
$$\frac{\partial u}{\partial t} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru = 0$$

- The Black-Scholes equation is a parabolic PDE

Final and boundary conditions

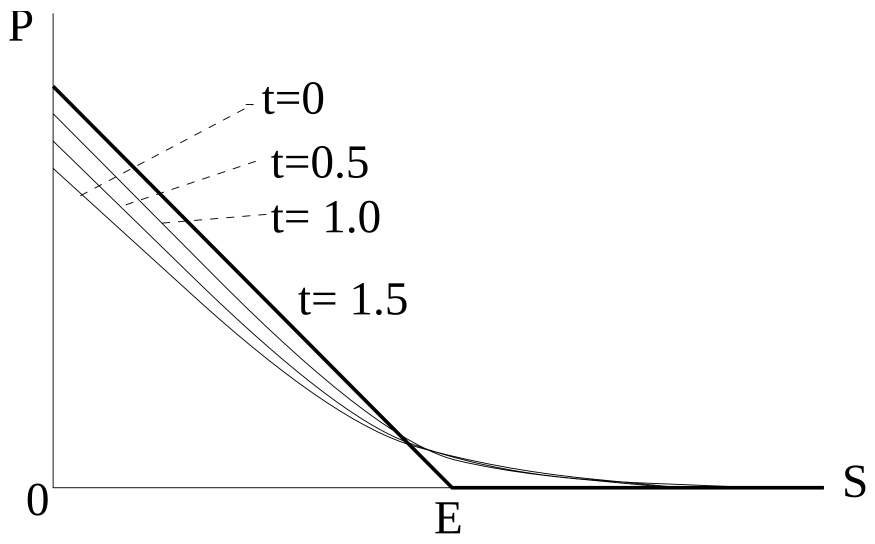
• The holder of a European put option will logically exercise his option, if $s < E$ at $t = T$, receiving an amount E and sell an asset worth s. The profit is $E - s$.
• If $s > E$ at $t = T$ one will typically not exercise the European option. The option is in this case valueless. So,

$$u(s, T) = \max(E - s, 0).$$



- Boundary condition for a put at $s = 0$: $u(0, t) = Ee^{-r(T-t)}$.
- Boundary condition for a put at $s \to \infty$: $u(s, t) \to 0$.
- With such conditions, the European option can often be solved analytically !
- A transformation $t^* = T - t$ transforms the equation backward in time to a forward equation with initial condition

American Options



- European put option is in a certain s range less than the pay-off function.
- For an American option exercise is permitted at any time during the life of an option. In this s range, $u(s,t) < \max(E-s,0)$.
- Buying the option for u, exercise it immediately, sell the asset for E, and immediately buy the asset in the market for s would lead to a risk-free profit of $E-u-s$

$\Rightarrow$ When early exercise is permitted, a constraint

$$u(s,t) \geq \max(E-s,0)$$

must be imposed.

$\Rightarrow$ Free boundary problem: A special s value exists, s_f, to one side of s_f one should hold the option, to the other side, one should exercise the option.

- s_f is not known in advance.

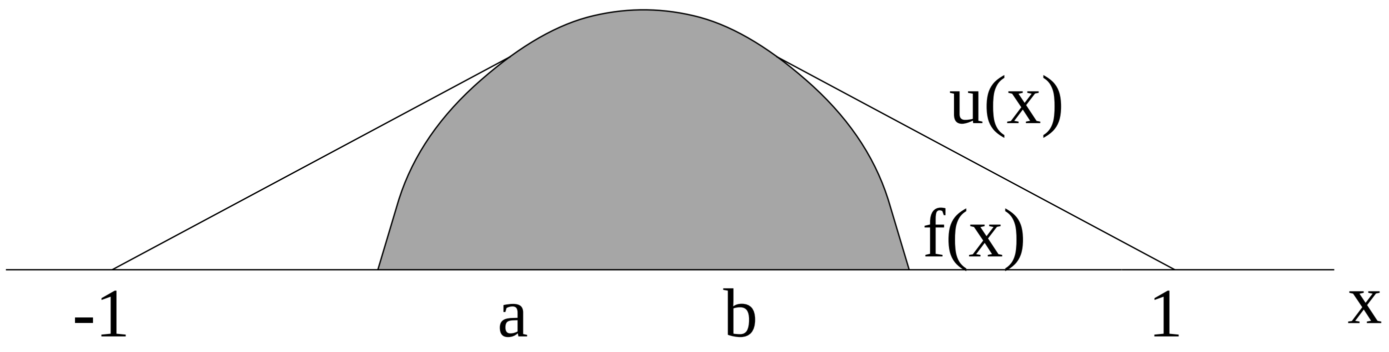
Intermezzo: Obstacle problems

(as in Wilmott et al. 1993)

- Given: an "obstacle" $f(x)$, with $f(x) > 0$ for $a < x < b, f \in C^2,\ f'' < 0$ and $f(-1) < 0, f(1) < 0$.
- Over the obstacle, one spans a function u of minimal length. This curve u lies on the obstacle between a and b.
- This obstacle problem is a simple free boundary problem:

$$\begin{aligned} -1 < x < a: &\quad u'' = 0 \quad (u > f) \\ a < x < b: &\quad u = f \quad (u'' = f'' < 0) \\ b < x < 1: &\quad u'' = 0 \quad (u > f) \end{aligned}$$

- Reformulate the obstacle problem, so that the free boundary conditions need not be handled explicitly.



Linear complementarity

- $u > f$, then $u'' = 0$
  $u = f$, then $u'' < 0$.
- Reformulation of the obstacle problem:
  Find $u(x)$, so that

$$u''(u-f) = 0, \quad -u'' \geq 0, \quad u - f \geq 0,$$
$$u(-1) = u(1) = 0, \quad u \in C^1[-1,1]$$

- This formulation is beneficial for iterative numerical solution processes.
- A similar complementarity follows for the American options:
  For a put: $u > \max(E-s,0) \quad (s > s_f(t))$,
  then Black-Scholes equation,
  $u = E - s \quad (s \leq s_f(t))$,
  then Black-Scholes inequality
- For a call: $u = s - E \quad (s \geq s_f(t))$,
  then Black-Scholes inequality
  $u > \max(s-E,0) \quad (s < s_f(t))$,
  then Black-Scholes equation

Higher dimensional problems:

Stochastic volatility

- A generalization of the Black-Scholes equation w.r.t. the volatility is based on the assumption of a stochastic volatility $\sqrt{y}$, as follows:

$$\begin{aligned} ds &= \mu s dt + \sqrt{y} s dw_1 \\ dy &= \alpha(\beta - y)dt + \gamma\sqrt{y} dw_2 \end{aligned}$$

  with $w_1,\ w_2$ standard Brownian motion with correlation coefficient $\overline{\rho} \in [-1, 1]$. γ is the volatility of the variance process. $\alpha,\ \beta > 0$ determine the mean reversion (so that the variance will drift back to some mean value β at a rate governed by α).
  For details see Hull 1997, Ball and Roma J.F.Q.A. 1994, or Clarke and Parrot Appl. Math. Finance 1999.

- The volatility can be shown to be positive valued provided $\gamma^2 \leq 2\alpha\beta$.

- The assumption of these stochastic processes leads to problems with a 2D differential operator in which y is, next to s and t, a third variable (degree of freedom).

Stochastic volatility

the 2D operator

- The LC Problem for an American put option:

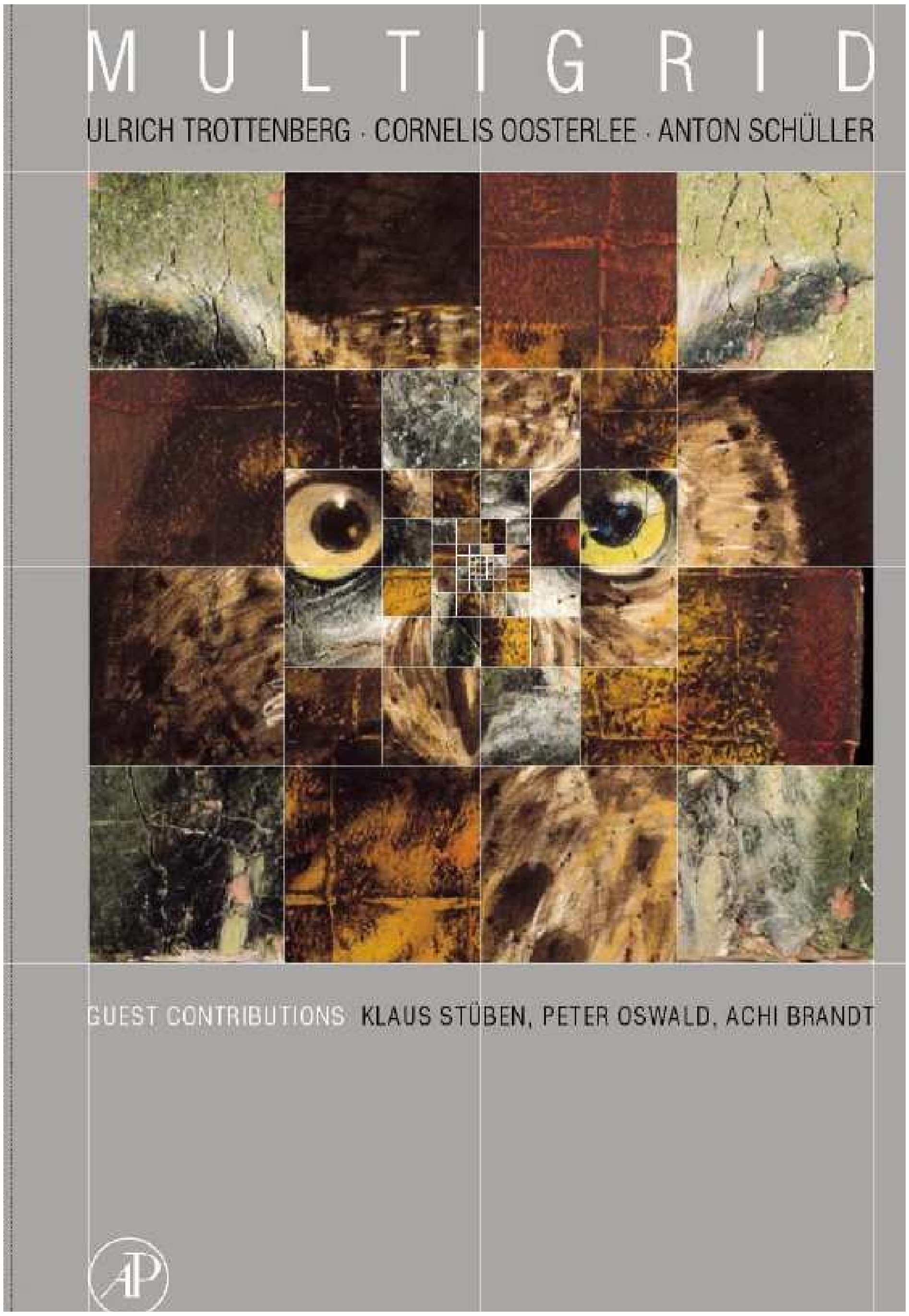
$$\begin{aligned} Au(s,y,t) &\le 0 \quad \forall (s,y)\in\Omega, \quad t\in[0,T) \\ u(s,y,t) &\ge g(s,t) := \max(s-E,0) \\ (u(s,y,t)-g(s,t))Au(s,y,t) &= 0 \end{aligned}$$

  with $\Omega = \{(s,y)|s\ge 0, y\ge 0\}$ and with

$$\begin{aligned} A &= \frac{\partial}{\partial t} + \frac{1}{2}[s^2 y\frac{\partial^2}{\partial s^2} + 2\rho\gamma y s\frac{\partial^2}{\partial s\partial y} + \gamma^2 y\frac{\partial^2}{\partial y^2}] + \\ & rs\frac{\partial}{\partial s} + [\alpha(\beta-y) - \lambda\gamma\sqrt{y}]\frac{\partial}{\partial y} - r \end{aligned}$$

- λ is the so-called "market price of the risk function" (for foreign currencies a nonzero constant is parameter of choice). Details can be found in the financial literature.
- Multigrid has been successfully applied by Clarke and Parrot with a line version of P-GS for a first order accurate discretization of first derivatives.

Small Advertisement for a new book “Multigrid” by Trottenberg, Oosterlee and Schüller



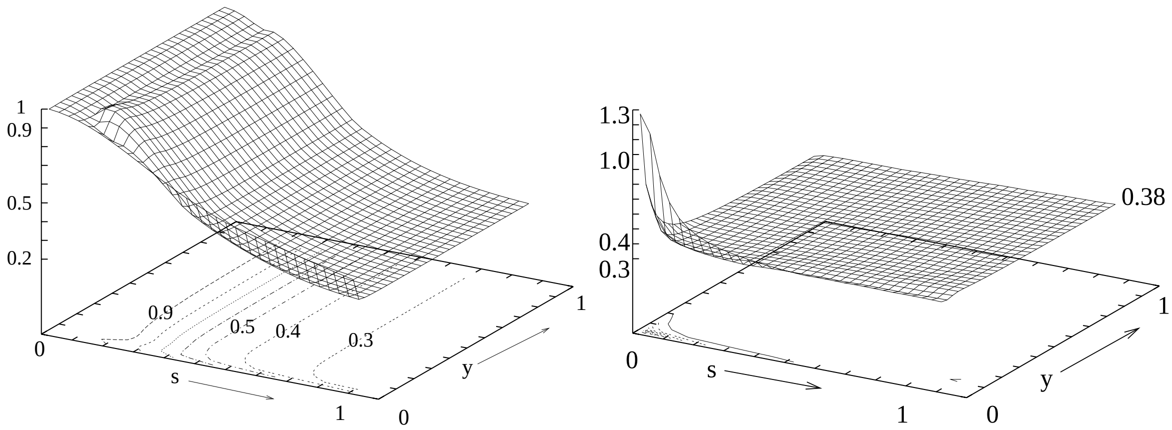
(see: http://www.gmd.de/SCAI/multigrid_book/multigrid.html)

Two-grid Fourier analysis

- We analyze the steady convection-diffusion operator

$$\frac{1}{2}[s^2 y \frac{\partial^2 u}{\partial s^2} \;+\; 2\overline{\rho}\gamma y s \frac{\partial^2 u}{\partial s \partial y} + \gamma^2 y \frac{\partial^2 u}{\partial y^2}] + rs\frac{\partial u}{\partial s}$$
$$+\; [\alpha(\beta - y) - \lambda\gamma\sqrt{y}]\frac{\partial u}{\partial y} - ru, \quad \Omega = \{(s,y)|s \geq 0, y \geq 0\}$$

- Coefficients $s_i = i\Delta s$ and $y_j = j\Delta y$ are locally frozen
- $\alpha = 5,\; \beta = 0.16,\; \gamma = 0.9, \rho = 0.1, \lambda = 0, r = 0.1$
- Discretization with second order upwind (kappa) discretization for convection
- Lexicographic point versus s-line Gauss-Seidel smoother



- F(1,1) multigrid cycle, 256^2-grid, different domains.

Smoother:	lex. point Gauss-Seidel				s-line GS
Domain Ω:	$[0,1]^2$	$[0.2,1]^2$	$[0.4,1]^2$	$[0.6,1]^2$	$[0,1]^2$
	0.994	0.85	0.55	0.30	0.35

- Due to the free boundary, that will cover the left part of the domain, we concentrate interest in the point smoother.

Multigrid for linear complementarity problems

Arising from free boundary problems

Problem of interest:

$$\begin{array}{rcll} Au & \le & f_1 & x \in \Omega \qquad (*) \\ u & \ge & f_2 & x \in \Omega \\ (u-f_2)(Au-f_1) = 0 & & & x \in \Omega \quad \text{(plus boundary conditions)} \end{array}$$

- Nonlinear multigrid based method: Projected Full Approximation Scheme, PFAS by Brandt and Cryer (SIAM, 1983)
- PFAS is based on projected SOR, Cryer, 1970's, or projected GS which is used as a smoother.

  (Related method: monotone multigrid by Kornhuber (1994)
  Theory exists for the convergence of this method.
  $\Rightarrow$ Monotone multigrid is claimed to be more robust.)
- PGS: Just apply a Gauss-Seidel relaxation to $(*)$ with the equality sign:

$$u_i^* = \frac{1}{a_{ii}}\left(f_{1,i} - \sum_{k=1}^{i-1} a_{ik} u_k^{(m+1)} - \sum_{k=i+1}^{N} a_{ik} u_k^{m}\right).$$

  Then "project" the solution:

$$u_i^{(m+1)} = \max\{f_{2,i}, u_i^*\} \quad i = 1, \ldots, N$$

The coarse grid LC Problem

- Suppose we applied smoothing; the error $v_h^m := u_h - u_h^m$ is smooth. v_h^m satisfies the LCP:

$$\begin{aligned} A_h v_h^m &\le d_h^m \\ v_h^m + u_h^m &\ge f_{2,h} \\ (v_h^m + u_h^m - f_{2,h})(A_h v_h^m - d_h^m) &= 0 \\ \text{+ boundary conditions} & \end{aligned}$$

with the residual: $d_h^m = f_{1,h} - A_h u_h^m,\ x \in G_h$.

$\Rightarrow$ We can represent the error v_h^m well on a coarse grid without any essential loss of information.

The LCP coarse grid equation for the coarse grid approximation of the error $\hat{v}_H^m$ is defined in PFAS by:

$$\begin{aligned} A_H \hat{v}_H^m &\le I_h^H d_h^m \\ \hat{v}_H^m + \hat{I}_h^H u_h^m &\ge f_{2,h} \\ (\hat{v}_H^m + \hat{I}_h^H u_h^m - f_{2,h})(A_H \hat{v}_H^m - I_h^H d_h^m) &= 0 \\ \hat{v}_H^m = 0 & \quad (\partial G_H) \end{aligned}$$

- Since the problem is nonlinear (and we are solving inequalities), we solve for a full approximation $u_H^m = \hat{v}_H^m + I_h^H u_h^m$ on G_H and interpolate $\hat{v}_H^m$ back to the fine grid.

- If we assume that the fine grid LCP is solved and consider the coarse grid problem, this leads to the following requirements on restriction operators (Brandt/Cryer, 1983):

$$\begin{aligned} I_h^H(f_{1,h} - A_h u_h^m) &\geq 0 \\ \hat{I}_h^H u_h^m &\geq f_{2,H} \\ (\hat{I}_h^H u_h^m - f_{2,H}) I_h^H (f_h - A_h u_h^m) &= 0 \end{aligned}$$

- (For equations, with $f_{1,h} - A_h u_h^m = 0$, these requirements are satisfied for any reasonable choice of I_h^H and $\hat{I}_h^H$.)

⇒ Here, we choose for both restriction operators Injection. This operator maps nonnegative vectors into nonnegative vectors. Therefore it is "constraint preserving".

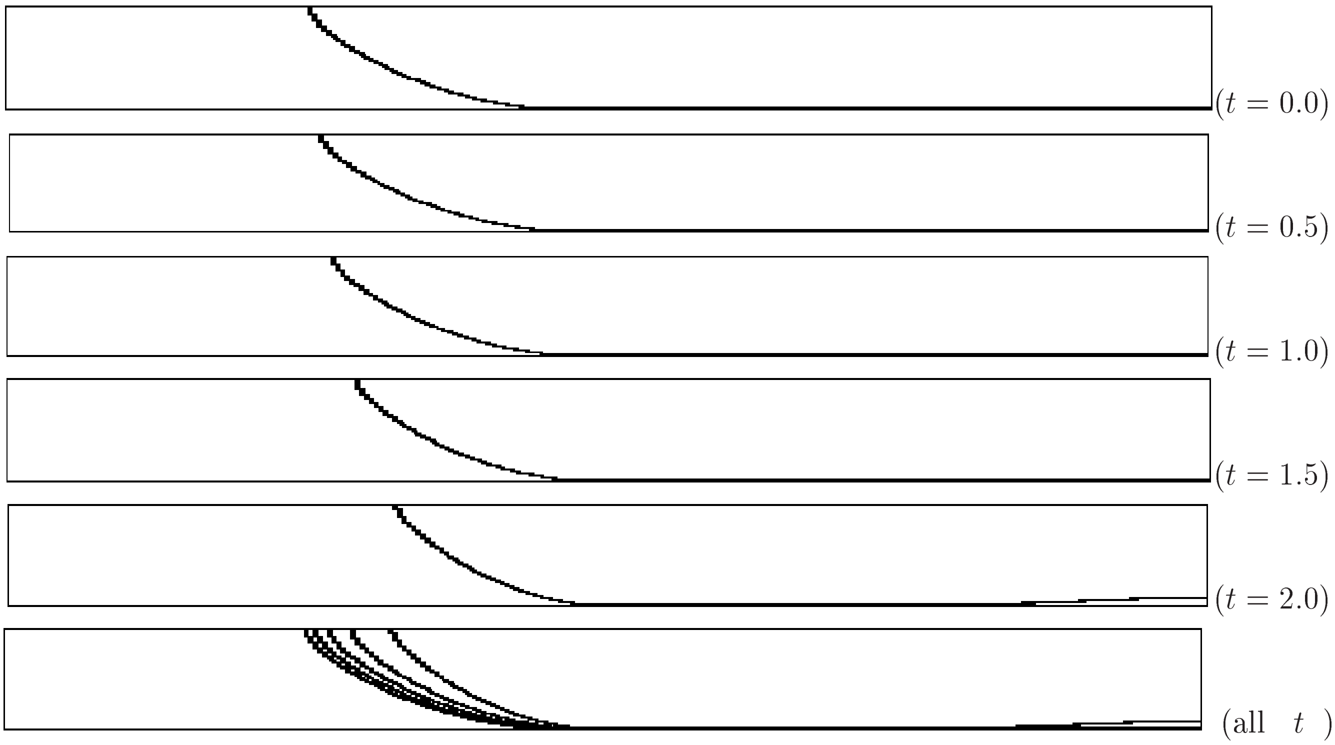
- Prolongation operator is only applied for unknowns on "active" points

$$\begin{aligned} u_h^{(m+1/2)} &= u_h^m + I_H^h \hat{v}_H \quad \text{if} \quad u_h^m > f_2 \\ u_h^{(m+1/2)} &= u_h^m \quad \text{elsewhere} \end{aligned}$$

Multigrid result

American Put

- $E = 10$, $\Omega = [0, 20] \times [0, 1] \times [0, 0.25)$, Grid: 256^2, $\Delta t = 0.025$
- Sharp gradients in the numerical solution are not expected: BDF2 time discretization, second order kappa discretization scheme for convection.
- lex. point Gauss-Seidel has been used as the smoother, F-cycles, 9 multigrid levels.
- On average: 5.3 multigrid F(2,2)-cycles per time step for a 10^{-5} reduction of the initial defect
- Accuracy results agree very well with reference !



Recombination

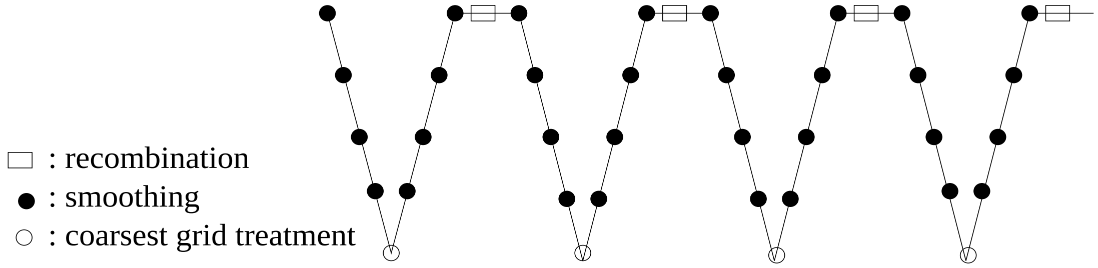
- As an acceleration of the PFAS convergence, we apply a recombination technique

$$u^m_{h,\mathbf{acc}} = u^m_h + \sum_{i=1}^{\widetilde{m}} \overline{\alpha}_i (u^{m-i}_h - u^m_h) \ .$$

- Parameters $\overline{\alpha}_i$ are determined so that the defect $d^m_{h,\mathbf{acc}}$ is minimized w.r.t $||\cdot||_2$.
- For linear equations, the corresponding defect, $d^m_{h,\mathbf{acc}} = f_h - A_h u^m_{h,\mathbf{acc}}$, is given by $d^m_{h,\mathbf{acc}} = d^m_h + \Sigma_{i=1}^{\widetilde{m}} \overline{\alpha}_i (d^{m-i}_h - d^m_h)$
- In the LCP context, we use the equation in the LCP system for the defect minimization
- We compute the corresponding defect,

$$d^m_{h,\mathbf{acc}} = (u^m_{h,\mathbf{acc}} - f_{2,h})(Au^m_{h,\mathbf{acc}} - f_{1,h}) \ .$$

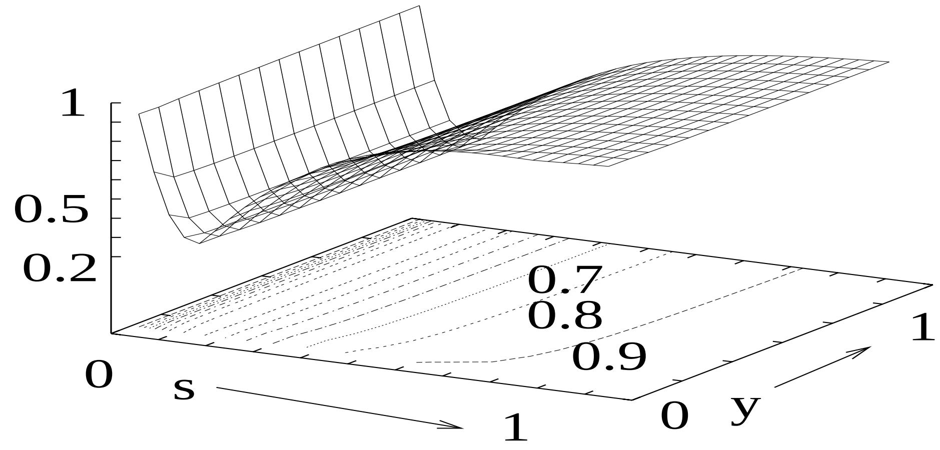
- Criteria are needed to select u^m_h or $u^m_{h,\mathbf{acc}}$.



Result with recombination

American Put

- Fourier two-grid factors for a grid with $h_s = 1/256$, $h_y = 1/32$



- Multigrid convergence factors with the point smoother increase for $h_y >> h_s$
- By iterant recombination (only 3 iterants) still a satisfactory convergence is achieved for the previous problem

- On average 46.5 multigrid iterations with the point smoother for a 10^{-5} reduction of the initial defect
- Combined with recombination: 12.2 iterations

Conclusions and Outlook

- We deal with PDEs for option pricing.
- Typically, European options are simple and can often be solved analytically.
- American options lead to obstacle problems.
- Multigrid methods are well-suited for the linear complementarity problems based on the Black-Scholes type equation.

- With more complicated situations (different shares in a portfolio, including transaction costs, divident payment, realistic models for future volatility values, . . .), interesting numerical issues will be encountered (high dimensional problems, systems of equations, jumps in coefficients, . . .).
- By retaining a point smoother for the 2D problem here, we constructed a fast solver that can serve as a basis for higher dimensional problems.