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TVD, WENO and blended BDF discretizations for Asian options

C. W. Oosterlee[1], **J.C. Frisch**[2], **F.J. Gaspar**[3]

[1] Fac. Information, Technology and Systems, Delft University of Technology, Delft, the Netherlands
[2] Fraunhofer Institute for Algorithms and Scientific Computing (SCAI), D-53754 Sankt Augustin, Germany
[3] C.P.S. University of Zaragoza, 50008 Zaragoza, Spain

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Abstract. In this paper, we discuss topics for a fast and accurate solution of continuous American-style Asian option problems from computational finance. These problems lead to 2D time-dependent convection-dominated partial differential equations with a free boundary. As a pre-study for accurate discretization schemes in "asset price space" and in time, we solve numerically reference problems based on the Black-Scholes equation with small volatility and with discontinuous final conditions.

1 Introduction

Research in option pricing theory concerns, among other issues, the computation of the value of an option during the lifetime of an option contract. A famous equation to determine the value of standard European options is the Black-Scholes partial differential equation. We refer to the basic literature in the option pricing field [8,18] for a detailed discussion. Next to the standard options, whose values typically depend, among other parameters, on time t and asset price s, the class of *path-dependent options* exists, whose value also depends on the asset price *development* [18,20]. They can depend on whether or not the asset reaches some prescribed value before expiry (the digital and barrier options), on the maximum or the minimum of the asset price (lookback options), or on some form of average of the asset price, for example, over the whole option contract lifetime. These latter options are the Asian options. In this paper, we discuss the fast and accurate solution of the 2D time-dependent partial differential equations (pdes) arising in Asian option pricing.

American-style variants of the options listed above exist, indicating that an option can be exercised at any time during the lifetime of the option. In contrast, the European-style options can only be exercised at the expiration date, i.e., at the end of the option contract. American-style options lead to free boundary problems: A special s value exists, the optimal exercise price s_f, with the following properties: on one side of s_f it is beneficial to hold the option, on the other side, it is advantageous to exercise the option. $s_f(t)$ is time dependent and not known in advance. In [18], it has been shown that for American-style options the theory of linear complementarity, as it was developed in the 1970's, applies.

Two factors that determine the difficulty of pricing and hedging Asian options numerically are the path dependency and early exercise. If an option contract is without any path-dependency, there may be a solution in the form of a multiple integral. In this case numerical integration is the method of choice. For very high dimensional problems, in the case of multiple assets, Monte Carlo simulations are adequate. They can cope with path-dependent features, however, they are typically not very efficient for American-style early exercise. For 2D or 3D problems, the finite difference/volume methods are the obvious choice. They cope well with early exercise and path-dependent features can be incorporated at the cost of an extra dimension. Processes with path-history represent an important class of stochastic control problems. The solution satisfies an ultra-parabolic variational inequality [12]. For European-style Asian options it is possible to reduce the dimensionality of the pde by a similarity transformation. For Asian options with the early exercise possibility, this is not the case.

The purpose of this paper is to include knowledge on accurate discretizations and fast iterative solution methods from computational fluid dynamics into the numerical solution of option pricing pdes. Inspired by the numerical work in option pricing by the group of Forsyth and Vetzal (for example, in [14,20,21]), we merely offer some alternatives in discretization and solution methods. With respect to accurate discretization schemes, we will employ discretizations on Cartesian grids. The structure of these grids is beneficial for accuracy as well as for efficient solution by means of multigrid iterative methods [16]. Accurate discretizations in space, i.e. along the asset price axes, and in time are of importance, because financial engineers are highly interested in *derivatives* of the computed option value. Derivatives are in general especially sensitive with respect to even small oscillations in a solution.

In this paper, we compare the weighted essentially non-oscillatory (WENO) discretization schemes [10,15] with a total variation diminishing (TVD) limiter-based discretization scheme for the convective terms [11,20]. These schemes are known to be of a high accuracy in smooth solution regimes. Near steep gradients, solutions without oscillations can be obtained. With these discretization schemes, we solve a reference problem based on the Black-Scholes equation with a small volatility. As the discretization in the time direction, we advocate the blended BDF2 discretization by Hundsdorfer [9]. This second order accurate method has very satisfactory stability properties and works well, for example, for options pricing problems with discontinuous final conditions. Numerical results will be presented in Section 3.3.

For iterative solution we use a multigrid variant for linear complementarity problems, the projected full approximation scheme (PFAS) [2]. The algorithm is a generalization of the projected SOR method [4]. PFAS has already been successfully used in the financial community for American options with stochastic volatility in [3, 13]. We will adapt PFAS for American-style Asian options. We start with a discussion on option pricing, in particular of Asian options, in Section 2 and on the discretization in Section 3. PFAS is briefly described in Section 4. American-style Asian options are solved numerically in Section 5.

2 Option Pricing Preliminaries

2.1 Black-Scholes equation

The Black-Scholes equation is a simple model for the values of the a European option [8,18]. In the case of a European *call* option, the holder has the right to *purchase* an asset on a certain date in the future for a prescribed amount. The other party of the option contract, the writer, is then obliged to sell the asset. The value of the option u is influenced by the exercise price E, the expiration time T $(0 \leq t \leq T)$, the interest rate r, and the volatility σ. The Black-Scholes equation reads

$$\frac{\partial u}{\partial t} = \mathcal{L}u := -\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - rs\frac{\partial u}{\partial s} + ru \quad . \tag{1}$$

Two important assumptions in the derivation of the equation are that the asset price follows geometric Brownian motion, and that there are no arbitrage possibilities, meaning that risk free profits greater than placing money at a bank are not allowed in the model. So, the instantaneously risk free portfolio must earn a rate of return that equals the interest rate [8,18].

The holder of a *European call* option will exercise his option, if $s > E$ at $t = T$. If $s \leq E$ at $t = T$ he will not exercise the call option. Therefore, the payoff function for a call reads,

$$u(s,T) = \max(s - E, 0). \tag{2}$$

In its basic form, (1) is a parabolic convection-diffusion-reaction operator in one "space-like" dimension, the s-direction, with boundary and final conditions. For European options it is trivially solved; even closed form solutions often exist.

The value of a *digital option* is also typically governed by the Black-Scholes equation (1). Digital options are, however, characterized by a discontinuous payoff condition. The option is valueless, if the asset does not reach the prescribed exercise price E. Otherwise, the payoff equals one. The payoff function then reads

$$u(s,T) = \begin{cases} 0.0 \text{ for } s < E \\ 1.0 \text{ for } s \geq E \end{cases} . \tag{3}$$

Options with discontinuous payoff are treated in [14]. We will use a digital call option in Section 3.3 to examine the accuracy of the time discretization scheme used.

2.2 Hedging

Two important hedge parameters for financial engineers are Delta, Δ and Gamma, Γ:

$$\Delta = \frac{\partial u}{\partial s}, \quad \Gamma = \frac{\partial^2 u}{\partial s^2} \quad . \tag{4}$$

They represent the rate of change of the value of an option with respect to asset s and the change in Δ, respectively. Δ determines the number of shares, that should be kept with each option issued in a portfolio to cope with a loss in the case of exercise. It gives an important insight in maintaining the instantaneous risk-free portfolio. Γ is the derivative of Δ with respect to s. If Γ is high, a portfolio under consideration does result only for a very short period of time in a risk-less scenario. Δ, along with Γ give investors an idea of the sensitivity of their option positions as the price of the asset moves.

2.3 Asian options

For Asian options the payoff function depends on the average value of asset prices. Average values usually respond more strongly on price changes during the initial period of an option's lifetime than at the end. Due to their structure, Asian options are well-suited for securing, for example, cash flows over a longer time period. A typical range of application of Asian options is in securing cash flows with respect to changes in exchange rates. One can distinguish discrete and continuous Asian options. In the discrete case, asset prices at specific sampling times, like 12 AM daily, are used to update the average value of the asset prices. The average price a_k is then obtained by

$$a_k = \frac{\sum_{i=1}^{k} s(t_i)}{k} = \left(a_{k-1} + \frac{s(t_k) - a_{k-1}}{k} \right) . \tag{5}$$

The t_i values are the sampling times. In the valuation of discrete Asian options the sampling times and the period between them need to be handled separately: At the sampling times the average is updated, but the asset prices remain unchanged, whereas between sampling times the average remains constant but the asset prices develop. At sampling times (5) is used, whereas between sampling times u is determined by the Black-Scholes equation (1). The arbitrage concept prescribes that the average update (5) may only take an infinitesimal small time period. With $t^+ = t_k + \epsilon$, $t^- = t_k - \epsilon$, (5) can be reformulated as

$$a(s,t^+) = a(s,t^-) + \frac{s - a(s,t^-)}{k} \quad , \tag{6}$$

and

$$u(s(t^-), a_{k-1}, t^-) = u(s(t^+), a_k, t^+) \ . \tag{7}$$

Equations (6) and (7) are called the jump conditions.

The discrete case serves as a basis for the continuous Asian options. In the latter, the average is *continuously* updated. It is the limit case of the discrete options.

We denote the average a(T) by

$$a(T) := \int_0^T s(\tau) d\tau / T.$$

In one variant of the (final condition) payoff function for Asian options, the exercise price is determined by the average $a(T)$. The corresponding options are called floating strike options, as the strike (or exercise) price is not known before the end of its lifetime. In a second variant the value of the underlying asset is determined by the average of the asset prices, whereas the strike price is fixed. This contract is the fixed strike option. The corresponding payoff functions for the call option read

Floating strike call: $u = f_2 := \max\left(s(T) - a(T), 0\right)$, (8)

Fixed strike call: $u = f_2 := \max\left(a(T) - E, 0\right)$. (9)

In the continuous case, the problem dimension w.r.t. (1) increases due to the path dependency; the average $a(t)$ represents a second "space-like" direction. The operator for the continuous Asian option reads

$$\frac{\partial u}{\partial t} = \mathcal{L}u := -\frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} - rs\frac{\partial u}{\partial s} + ru - \frac{1}{t}(s-a)\frac{\partial u}{\partial a}. \tag{10}$$

The convective term $(s-a)/t \ \ \partial u/\partial a$ ensures the satisfaction of the jump condition in the continuous case. At $t = 0$, this convective term is not well-defined. However, since $a \equiv s$ at a first time step in this process, the term $(s-a)/t \ \ \partial u/\partial a$ naturally drops out of the problem formulation for $t = 0$. At $t = 0$, a Black-Scholes equation remains to be solved along the diagonal $s = a$.

The option pricing problems lead to the numerical solution of a time-dependent pde with an operator $\mathcal{L}$ of convection-diffusion-reaction type. After a transformation $t^* = T - t$, the equation which is backward in time with a final condition is transformed to an equation forward in time with an initial condition. This puts a minus sign in front of the $\partial/\partial t$ term. The transformed pde for Asian options reads,

$$\frac{\partial u}{\partial t^*} = \widehat{\mathcal{L}}u := \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 u}{\partial s^2} + rs\frac{\partial u}{\partial s} - ru + \frac{1}{T-t^*}(s-a)\frac{\partial u}{\partial a}. \tag{11}$$

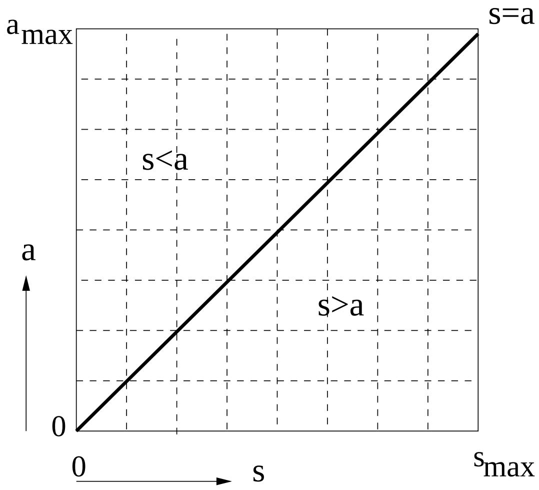

Fig. 1. Schematic picture of domain and grid.

The convection in a-direction is determined by the term $-(s-a)/(T-t^*) \ \ \partial u/\partial a$: For $s = a$ there is no convection of information in a-direction; for $s > a$ it is in negative and for $s < a$ in positive a-direction. Each time step information is convected from the diagonal $s = a$ outwards (see Figure 1).

It is not trivial to define boundary conditions at all boundaries of the computational domain from an economical point of view. Here, the ultra-parabolic nature of the problem gives some insight [12]. Artificial boundary conditions are necessary, because we need a computational domain of finite size. It is, of course, important that the numerical boundary conditions do not have any global un-economical influence on the option values. Often the computational domain is chosen large and squared for this purpose [19]. The diagonal $s = a$ serves, in a certain view, as a boundary. At the "outflow" boundaries $a = 0$ and $a = a_{max}$, boundary conditions are not explicitly prescribed. We merely discretize at these boundaries with appropriate upwind stencils from inside the domain. At the boundaries $s = 0$ and $s = s_{max}$ the boundary conditions are derived based on the jump condition. At $s = 0$ the reduced differential equation

$$\frac{\partial u}{\partial t^*} = -\frac{a}{T-t^*}\frac{\partial u}{\partial a} - ru$$

serves as the boundary condition. At $s = s_{max}$ the jump condition prescribes at the boundary,

$$\frac{\partial u}{\partial t^*} = \frac{s_{max} - a}{T - t^*}\frac{\partial u}{\partial a}.$$

Finally, another different scaling of the average $a(t)$ is also sometimes used (see [18,20]). This gives rise to a slightly different pde, which can be solved with similar methods.

2.4 American-style options

The exercise of American-style options is permitted at any time during the lifetime of an option, $0 < t \leq T$. When early exercise is permitted, a constraint

$$u(s,t) \geq f_2(s) \tag{12}$$

must be imposed in order to avoid an arbitrage possibility in the s range in which the value of the option is less than the pay-off function. Here, f_2 is the payoff function from (2), (3), (8) or (9), depending on the type of option. The 2D formulation of the American-style Asian option is used for discretization. We *can not transform* it to 1D equation (see, for example [18]). Because of the free boundary aspect, the discussion of the boundary conditions for Asian options becomes less relevant. The addition of the constraint (12) to the pde (10) gives rise to a so-called obstacle or free boundary problem. Based on the classical theory of linear complementarity, it is possible to reformulate this problem, so that the free boundary condition need not be handled explicitly.

Summarizing, for American-style Asian options, the following linear complementarity problem needs to be solved,

$$\frac{\partial u}{\partial t} - \mathcal{L}u(s,y,t) \leq 0 \tag{13}$$

$$u(s,y,t) \geq f_2(s) \tag{14}$$

$$(u(s,y,t) - f_2(s))(\frac{\partial u}{\partial t} - \mathcal{L}u(s,y,t)) = 0 \tag{15}$$

with f_2 as in (8), (9), $\mathcal{L}$ from (10), before the transformation, and boundary conditions.

3 Discretization

3.1 Discretization in time

The operator $\widehat{\mathcal{L}}$ (11) consists of a diffusion, reaction and a convection part, $\widehat{\mathcal{L}} = C$ onvection$+D$ iffusion$-R$ eaction. Correspondingly, L_h is the discrete version of $\widehat{\mathcal{L}}$: $L_h = C_h + D_h - R_h$.

For the time discretization, we prefer a backward difference formula (BDF) variant that is especially of interest for convection-dominated pdes, the "partially implicit BDF2 blend" [9] from the family of implicit/explicit (IMEX) θ-BDF schemes. The discrete diffusion D_h and reaction R_h terms are treated fully implicitly, whereas the convection C_h in (11) is discretized by a combination of implicit and explicit BDF2:

$$\frac{3u_h^{(m+1)} - 4u_h^{(m)} + u_h^{(m-1)}}{2\delta t} - D_h u_h^{(m+1)} + R_h u_h^{(m+1)} -$$
$$C_h(\theta u_h^{(m+1)} + (1-\theta)(2u_h^{(m)} - u_h^{(m-1)})) = 0 \tag{16}$$

The time discretization accuracy of this scheme is $O(\delta t^2)$ for $0 \leq \theta \leq 1$. For $\theta = 1$, we obtain the well-known BDF2 scheme [5].

We prefer BDF-type discretizations over the well-known Crank-Nicolson scheme, because of their stability characteristics, for example, for diffusion problems. The Crank-Nicolson scheme is not *L-stable* (see [6]), but BDF2 is. BDF2 has more favorable damping properties than the Crank-Nicolson scheme. This is especially beneficial for option pricing problems with discontinuous initial conditions. The reason for θ in (16) is that for convection-dominated problems with a time step restriction due to a CFL condition for monotonicity, a larger time step can be chosen for θ less than 1 (but ≥ 0.75). In fact, with BDF2 the numerical solution of convection-dominated problems is somewhat disappointing [9]. An optimal value for a Van Leer limited discretization of convection for model problems in [9] is $\theta = 0.75$, for which (16) is unconditionally stable. Furthermore, a very satisfactory time step depending on δs can be chosen with respect to monotonicity. The θ-BDF2 blended method combines the advantages of BDF2 for diffusion with those of explicit schemes for convection problems. The implicit part in (16) needs, however, be solved by an iterative solution method.

3.2 Discretization in space

Obviously, a second order derivative is not present in (10) in the a-direction. Accurate solutions without artificial oscillations can be obtained with the help of the discretizations that have been developed in computational fluid dynamics (CFD) [17]. These include "total variation diminishing" (TVD) schemes and "essentially non-oscillatory" (ENO) schemes for convection. Such sophisticated discretizations are especially relevant in option pricing for accurate solutions of the hedge parameters Δ and Γ (4). Careless discretization, by central, first or second order upwind schemes can result in undamped oscillations in these financial quantities [20].

Most "total variation diminishing" or "essentially non-oscillatory" theory to obtain oscillation-free solutions in CFD is based on flux conservation formulations of hyperbolic conservation laws and on explicit time discretizations [7,15]. The equations (1), (10) are, however, in non-conservative form. Diffusion plays an important role in the option pricing pdes and the time discretization employed is partly implicit. Furthermore, we are interested here in derivatives of the solution. Therefore, the outcome of a comparison of various discretization methods for the problems at hand may give different results than those known from CFD. Here, we compare a TVD discretization with van Leer limiter [11] with a $O(h_s^5)$ finite difference weighted ENO scheme (WENO) [10]. The van Leer limiter has already been used successfully for Asian options in [20]. Whereas the formula for the limited scheme may be well-known, the WENO scheme may not be and is therefore repeated here. First order derivatives are approximated as,

$$\frac{\partial u}{\partial s} \approx \frac{1}{h_s}(u_{i+1/2} - u_{i-1/2}) \ . \tag{17}$$

The u-quantities are made up of several neighbor values u_j by interpolation. In the case of a negative convection direction, as we encounter in option pricing after the transformation (11), u is defined by:

$$u_{i+1/2} = \omega_0 u^0_{i+1/2} + \omega_1 u^1_{i+1/2} + \omega_2 u^2_{i+1/2} \ ,$$

with the 3rd order quantities:

$$\begin{aligned} u^0_{i+1/2} &= (-u_{i-1} + 5u_i + 2u_{i+1})/6 \\ u^1_{i+1/2} &= (2u_i + 5u_{i+1} - u_{i+2})/6 \\ u^2_{i+1/2} &= (11u_{i+1} - 7u_{i+2} + 2u_{i+3})/6 \ . \end{aligned} \tag{18}$$

By means of *smoothness indicators*, the interpolation formula to determine a flux between two grid points from unknowns at several grid points is switched fully automatically near discontinuities. In this way, the smoothest interpolant (17) approximates the derivative at each grid point. This is reflected in the weights ω_i,

$$\omega_i = \alpha_i / \sum_{i=0}^{2} \alpha_i, \quad \alpha_i = d_i/(\beta_i + \varepsilon)^2$$

with $d_0 = 0.3,\ d_1 = 0.6,\ d_2 = 0.1$ and $\varepsilon = 10^{-6}$. With β_i, nonlinearity enters the WENO discretization, since

$$\begin{aligned} \beta_0 &= \frac{13}{12}(u_{i-1} - 2u_i + u_{i+1})^2 + \frac{1}{4}(u_{i-1} - 4u_i + 3u_{i+1})^2 \\ \beta_1 &= \frac{13}{12}(u_i - 2u_{i+1} + u_{i+2})^2 + \frac{1}{4}(u_i - 4u_i + 3u_{i+2})^2 \\ \beta_2 &= \frac{13}{12}(u_{i+1} - 2u_{i+2} + u_{i+3})^2 + \frac{1}{4}(u_{i+1} - 4u_{i+2} + 3u_{i+3})^2 . \end{aligned}$$

In theory, WENO schemes are to be preferred over TVD schemes. Whereas limiters in TVD schemes eliminate oscillations successfully, the accuracy necessarily reduces to first order near smooth extrema. The ENO approach is a successful attempt to obtain a self-adapting discretization scheme that is uniformly of high order, essentially non-oscillatory, for piecewise smooth functions.

We will choose from both schemes the one that gives the most accurate representation of the financial derivatives on a coarse grid in space and time.

3.3 Black-Scholes reference results

The numerical experiments presented here are for European call options based on equation (1). First of all, we present the computed Γ-values for a *digital call option* with a discontinuous final condition (3). We compare the θ-BDF2 time discretization ($\theta = 0.75$) with the Crank-Nicolson scheme. The problem parameters are: time to expiry $T = 0.5$, exercise price $E = 40$, interest rate $r = 0.05$, volatility $\sigma = 0.3$. We use step sizes $\delta t = 0.02, \delta s = 0.05$. Due to the discontinuous final condition, it is expected that Crank-Nicolson will suffer from oscillations, due to the lacking L-stability property. This feature can be seen clearly in Figure 2, which presents the computed Γ-values for both time discretizations. The θ-BDF scheme ($\theta = 0.75$) converges very satisfactory and gives a Γ-curve that compares very well with the reference in [14]. (In [14] a “Rannacher-type” stabilization of Crank-Nicolson is advocated.)

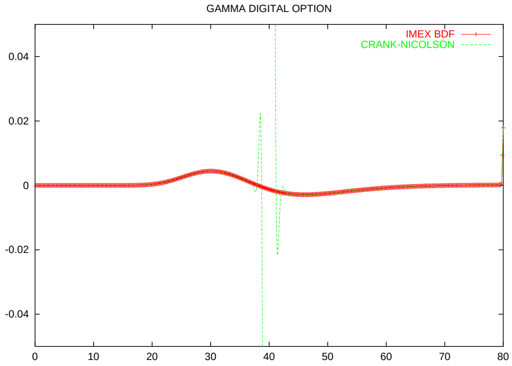

Fig. 2. Derivative Γ for a digital option with two time discretization schemes.

As a second Black-Scholes test case, an extremely small volatility, $\sigma = 0.01$, is chosen (as in Zvan et al. [20]) for a standard European call option with final condition (2). For this problem, we compute option value u and Γ at initial time $t = 0$. With such small volatility this reference problem serves as a pre-study for the Asian options (without any diffusion in the a-direction). and time discretizations. Problem parameters are $T = 1.0, E = 15, r = 0.15$. The computational domain is chosen as $0 \le s \le 30,\ \ 0 \le t \le 1$. Since an analytic solution is available for this European call option [18], we compare it to the numerical results.

Figure 3 shows both option price u and derivative Γ versus the asset price s in part of the domain with the well-known central and first order upwind discretization. The results are computed on an equidistant grid with step sizes $h_s = 0.05,\ \delta t = 0.02$. Whereas both discretizations perform reasonably well with respect to u, the accuracy in Γ is dramatically inaccurate for the central and first order upwind solution. Central discretization leads to oscillations in (Δ and) Γ, the first order upwind discretization results in a solution with too much artificial diffusion, see also [20].

The WENO5 and the van Leer limited discretization and the blended θ-BDF2 time discretization with $\theta = 0.75$ and $\theta = 1$ are compared for the same problem setting. The numerical solutions are much more accurate in Γ (see Figure 4). We have chosen step sizes in s and t^* such that the difference between BDF2 and θ-BDF2 is clearly visible. BDF2 gives a larger wiggle in Figure 4. Also a clear difference can be seen between the van Leer limited and the WENO5 discretization. The Γ-values in the left picture of Figure 4 are more accurate than those in the right picture, which is in favor of the TVD discretization over the WENO discretization for the computation of derivative Γ. The results in Figure 4 are representative for a variety of grid spacings in s- and in t^*-direction. Based on these numerical experiments, we choose the van Leer limited TVD discretization for convection and the θ-BDF2 time discretization with $\theta = 0.75$ for the Asian options. If the derivatives of the solution are not of primary interest, the grid can be chosen much coarser for an accurate solution u.

4 Solution method

For iterative solution we use multigrid [16]. In 1983, Brandt and Cryer [2] proposed a multigrid method for linear complementarity problems arising from free boundary problems. The method is based on Brandt's multigrid variant for nonlinear pdes, the full approximation scheme (FAS). It is called the projected full approximation scheme, PFAS in [2]. We adapt PFAS to the American-style Asian options and discuss the smoothing and coarse grid correction separately, starting with the smoother.

4.1 Projected relaxation

The classical PFAS was developed for a nicely elliptic operator L, for which the projected Gauss-Seidel method [4, 2] works well as a smoother. In a first step, a lexicographic pointwise Gauss-Seidel iteration is applied to (13) with the equality sign. Then, the solution is projected, so that constraint (14) is satisfied.

It is, however, well-known that a Gauss-Seidel iteration is not convergent for *convection-dominated* problems discretized with a limited discretization scheme. A classical technique, that we use here for the time-dependent Asian option pde is defect-correction. Within each time step, the first order upwind part of the convection discretization $\bar{C}_h$ is in the lefthand side of the defect-correction iteration. A few iterations of PFAS are then necessary to approximate the inverse of the first order operator. Second order accuracy is achieved by correcting the righthand side on the finest grid with the van Leer limited discretization C_h,

$$\bar{C}_h u_h^m = f_h - (C_h - \bar{C}_h) u_h^{m-1}$$

In summary, the iterative scheme with time index k and inner iteration m reads,

$$(T_h^k - \theta \bar{C}_h - D_h + R_h) u_h^{(k,m)} = T_h^{k-1} u_h^{(k-1)} - T_h^{k-2} u_h^{(k-2)} + (1-\theta)\bar{C}_h \bar{u}_h^{(k)} - (\bar{C}_h - C_h)\tilde{u}_h^{(k,m-1)}, \quad (19)$$

with $\bar{u}_h^{(k)} := 2u_h^{(k-1)} - u_h^{(k-2)}$ and $\tilde{u}_h^{(k,m-1)} = \theta u_h^{(k,m-1)} + (1-\theta)\bar{u}_h^{(k)}$. It can be seen that certain terms in (19) cancel out. As the smoothing method inside PFAS, we use a projected version of the four-direction lexicographic pointwise Gauss-Seidel method [16]. This method consists of four partial steps, in which iteration departs each time from another grid corner in a different direction. In this way, all convection directions are taken into account in a pointwise fashion. After these steps the unknowns are projected, so that constraint (14) is satisfied. This robust smoother is somewhat expensive.

4.2 Coarse grid correction

It should be mentioned that for the Asian option operator linewise relaxation also performs well. This cannot be used, however, without modification for linear complementarity problems. These problems are characterized by a solution that exhibits a free boundary. The solution consists of "active" points in which the first equation (13) is satisfied with equality sign and the constraint (14) with inequality sign, and "inactive" points with (13) with inequality and (14) with equality sign. Linewise relaxation methods must be combined with a detection mechanism to determine the position of the free boundary. Line smoothing is then performed only on the line segments of active points (see, for example [3]). Here, we adopt the pointwise relaxation instead. Fast convergence is, due to the character of the pde, achieved by using a grid coarsening based on *semi-coarsening in the s-direction.* Semicoarsening is used here in combination with the multigrid V(1,1)-cycle.

It is also important for coarse grid convergence acceleration to keep the regions with active and inactive points separated on the coarse grids. This fact determines the transfer operators between the grids [2, 13].

Suppose that the error $v_h^m := u_h - \overline{u}_h^m$ between exact solution u_h and approximation $\overline{u}_h^m$ is smooth after relaxation. The following linear complementarity problem holds for v_h^m:

$$\begin{aligned} L_h v_h^m &\le d_h^m \quad \mathbf{x} \in \Omega \\ v_h^m + \overline{u}_h^m &\ge f_{2,h} \quad \mathbf{x} \in \Omega \\ (v_h^m + \overline{u}_h^m - f_{2,h})(L_h v_h^m - d_h^m) &= 0 \quad \mathbf{x} \in \Omega \ . \end{aligned}$$

with defect: $d_h^m = -L_h \overline{u}_h^m$, L_h the discrete version of $\widehat{\mathcal{L}}$. A smooth error v_h^m can be approximated on a coarse H-grid without any essential loss of information. The coarse grid equation for the error $\widehat{v}_H^m$ is then defined in PFAS by:

$$\begin{aligned} L_H \widehat{v}_H^m &\le I_h^H d_h^m \\ \widehat{v}_H^m + I_h^H \overline{u}_h^m &\ge f_{2,H} \\ (\widehat{v}_H^m + I_h^H \overline{u}_h^m - f_{2,H})(L_H \widehat{v}_H^m - I_h^H d_h^m) &= 0 \ , \end{aligned}$$

with I_h^H the straight injection restriction operator. Straight injection [16] is used in order to keep defects from active and inactive fine grid points separated on coarse grids. The injection operator is in a certain sense "constraint preserving".

We solve for a full approximation $u_H^m := \widehat{v}_H^m + I_h^H \overline{u}_h^m$, but interpolate only $\widehat{v}_H^m$ back to the fine grid. For this reason, the bilinear interpolation prolongation operator I_H^h is applied only for unknowns on the "active" points, which resulted in the best convergence in [2]:

$$\begin{aligned} &u_h^m \Leftarrow \overline{u}_h^m + I_H^h \widehat{v}_H^m \quad \text{if} \quad \overline{u}_h^m > f_{2,h} \\ &u_h^m \Leftarrow \overline{u}_h^m \quad \text{elsewhere} \ \ (\overline{u}_h^m = f_{2,h}) \end{aligned}$$

5 Numerical Asian option results

We present the computation of American floating and fixed strike call and put options. The problem parameters are: $T = 1$, present asset price: $s_0 = 100$, $r = 0.1$. For the volatility two values, $\sigma = 0.1$ and $\sigma = 0.2$, are chosen. Discretization was done in a computational domain with $s_{max} = a_{max} = 250$. Choosing the domain 2.5

Table 1. Comparison of computed option values u at s_0 for $t = 0$ with reference values from [1] (denoted by B/P) and [20] (denoted by Z/F/V)

option type	σ :	$u_h(s_0)$	B/P	Z/F/V
floating strike call:	0.1	5.77	5.67	-
	0.2	8.56	8.46	-
fixed strike call:	0.1	5.39	5.40	5.39
	0.2	7.55	7.52	7.55
floating strike put:	0.1	1.93	1.8	1.95
	0.2	4.98	4.81	4.93
fixed strike put:	0.1	1.11	1.08	-
	0.2	3.20	3.18	-

Table 2. CPU times in seconds for the single grid and the multigrid solution method for American-style Asian options, $\delta s = \delta a = 1; \delta t = 0.01$.

option type	σ :	sg.	mg.
floating strike call:	0.1	351	281
	0.2	786	312
fixed strike call:	0.1	703	440
	0.2	1189	467
floating strike put:	0.1	553	282
	0.2	1012	324
fixed strike put:	0.1	370	270
	0.2	732	255

times the present asset value is considered sufficiently large for the volatilities under consideration. Mesh sizes are $\delta s = \delta a = 1$, and $\delta t = 0.01$. The reference value to be computed is the value of the option u at the present asset price s_0 at $t = 0$. Table 5 compares the results obtained with reference results from [1] and/or [20]. The results obtained compare very well with the reference results, especially with those in [20] (whose discretization is also very similar). The solution of the American-style Asian fixed strike call is presented graphically in Figures 5 and 6.

With respect to CPU time, we compare the single grid with the multigrid time results. Times are measured on a Pentium3, 700 Mhz PC for 100 time steps on a 250^2-grid. The stopping criterion within the defect-correction iteration is a reduction of the initial defects by 10^{-4}. Table 5 presents CPU times. As can be seen from Table 5, the difference in CPU time between single grid and multigrid CPU times is more pronounced for the larger volatility value. This is also to be expected, as multigrid is most efficient for diffusion-type equations. It can also be seen that for these 2D time-dependent problems the difference between the two is less pronounced as for stationary equations. This is also expected, because the time derivative term is beneficial for single grid solvers (see also [16]). In fact, a well-chosen overrelaxation parameter may further improve the single grid CPU time. It is, however, not trivial to chose such a parameter properly for this operator.

6 Conclusions, Outlook

In this paper, we solve continuous American-style Asian options numerically. Several choices with respect to the discretization in space and time, and to the iterative solution method are made. The accurate solution of the derivatives of an option value governed by the Black-Scholes equation is the reference test case. For a small volatility and for discontinuous final conditions, we compare TVD, WENO and θ-BDF discretizations. Based on this pre-study, we choose the convection discretization with van Leer limiter and the θ-BDF time discretization with $\theta = 0.75$ for the Asian options. A projected multigrid method based on semi-coarsening within a defect-correction technique is employed for solving the American-style Asian option problems.

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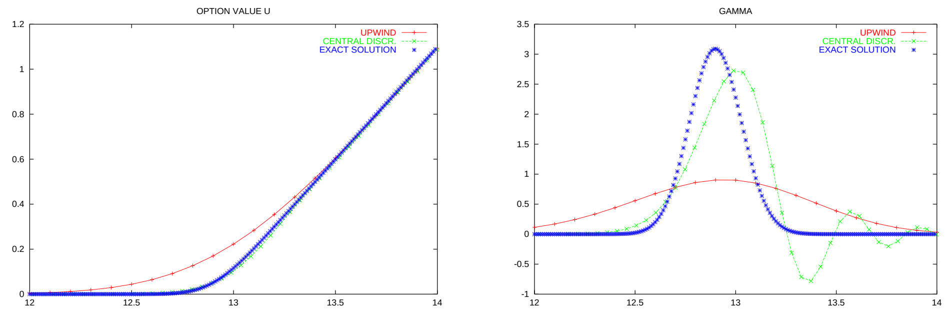

Fig. 3. Option price u and derivative Γ versus asset price s compared to the analytic solution for different discretizations in a part of the solution domain, $h = 1/300$.

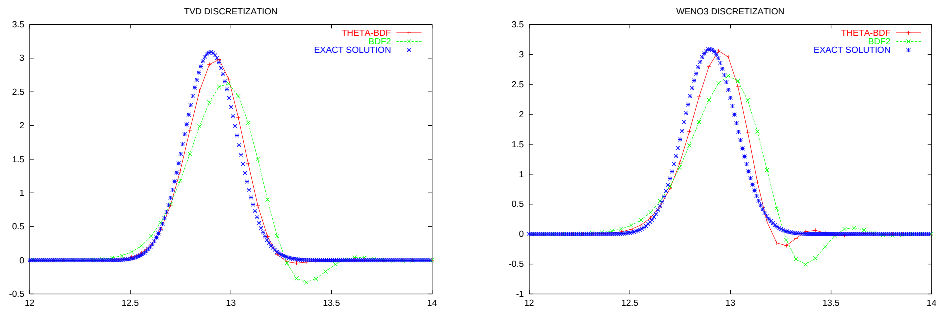

Fig. 4. Derivative Γ versus asset price s compared to the analytic solution for TVD (left picture) and WENO5 (right picture) discretizations in space and BDF variants in time, $\delta t = 0.02, h = 0.05$.

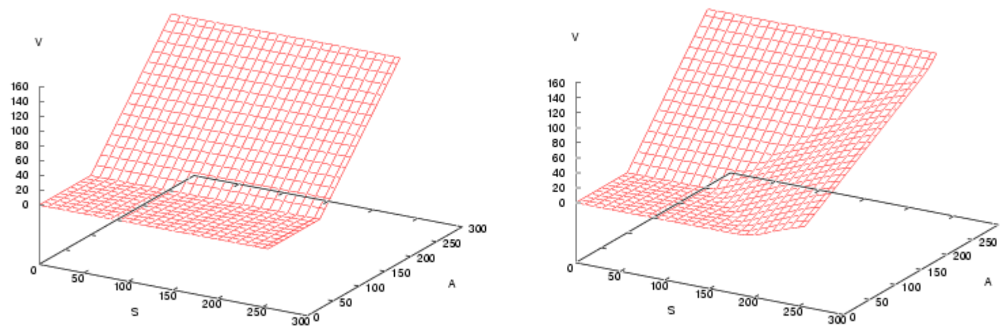

Fig. 5. Initial condition for American fixed strike call, and option value u at $t = 0.5$.

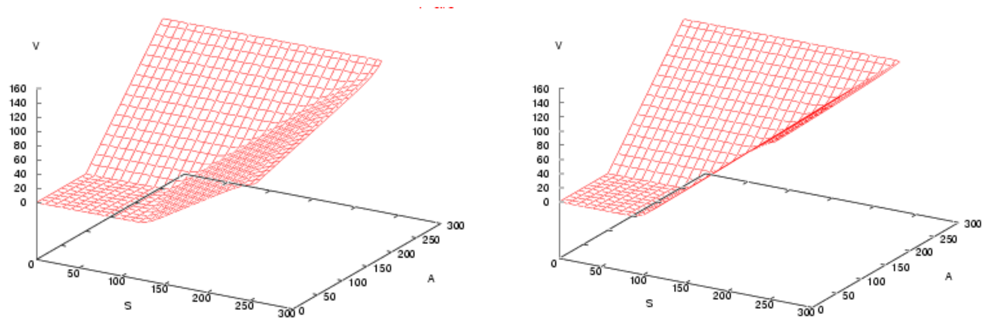

Fig. 6. American fixed strike call, option value u at $t = 0.75$ and at $t = 0.99$.