Geometric Multigrid with Applications to Computational Fluid Dynamics

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Abstract:

An overview is given of the development of the geometric multigrid method, with emphasis on applications in computational fluid dynamics over the last ten years. Both compressible and incompressible flow problems and their corresponding multigrid solution methods are discussed. The state of the art is described with respect to methods employed in industry as well as the multigrid efficiency obtained in academic applications.

Keywords: multigrid, nonlinear problems, computational fluid dynamics

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1 Introduction

The multigrid or multilevel approach has the unique potential of solving many kinds of mathematical problems with N unknowns with $\mathcal{O}(N)$ work. As discussed in [15], this applies to diverse areas such as integral equations, or optimization methods in various scientific disciplines. Complexity of $\mathcal{O}(N)$ has been shown theoretically for discretizations of a large class of elliptic linear partial differential equations ([38, 4, 43, 44, 45, 148]).

We can distinguish between algebraic multigrid (AMG) [121] and geometric multigrid. In algebraic multigrid no information is used concerning the grid on which the governing partial differential equations are discretized. Therefore it might be better to speak of algebraic multilevel methods. In geometric multigrid, coarse grids are constructed from the given fine grid, and coarse grid corrections are computed using discrete systems constructed on the coarse grids. Constructing coarse grids from fine grids by agglomeration of fine grid cells is easy when the fine grid is structured, but not if the fine grid is unstructured. That is where algebraic multigrid becomes useful. Unfortunately, AMG is less developed than geometric multigrid for the applications considered here. Algebraic multigrid is covered by K. Stüben in another paper in this issue. The present paper is about geometric multigrid.

For remarks on early multigrid history, see [149, 151]. In the early seventies, there were methods already available with low computational complexity, such as solution methods based on fast Fourier transforms, resulting in $\mathcal{O}(N \log N)$ work. But these methods are restricted to special classes of problems, such as separable partial differential equations on cubic domains. Multigrid, however, is much more robust: it is efficient for a much wider class of problems. The interest of practitioners of large scale scientific computing in multigrid was particularly stimulated by the 1977 paper [11] of Achi Brandt, generally regarded as a landmark in the field. Two series of conferences dedicated to multigrid were set up: the European Multigrid Conferences (EMG): Cologne (1981, 1985), Bonn (1991), Amsterdam (1993) , Stuttgart (1996), Ghent (1999), and in the US the Copper Mountain Conferences on multigrid, held bi-annually from 1983 until the present. Proceedings of the European meetings have appeared in [46, 48, 49, 54, 50] and of the Copper Mountain Conferences in special issues of journals: Applied Numerical Mathematics (Vol. 13, 1983, Vol. 19, 1986), Communications in Applied Numerical Methods (Vol. 8, 1992), SIAM Journal of Numerical Analysis (Vol. 30, 1993), Electronic Transactions on Numerical Analysis (Vol. 6, 1996). Another rich source of information on multigrid is the MGNet website maintained by C.C. Douglas: `http://www.mgnet.org`.

Introductions to multigrid methods can be found in [11, 17, 47, 134, 151], to be collectively referred to in the sequel as the basic literature. A thorough introduction to multigrid methods is given in [47]. The introduction in [151] requires less mathematical background and is more oriented towards applications. In Chapt. 9 of [151] the basic principles and the state of the art around 1990 of multigrid for computational fluid dynamics is described. Therefore here we will not dwell much on the basic principles, and confine ourselves mainly to developments that have taken place during the last decade.

Computational Fluid Dynamics (CFD) gives rise to very large systems requiring efficient solution methods. Not surprisingly, multigrid found applications in CFD at an early stage. The compressible potential equation was solved with multigrid in 1976 [131], the incompressible Navier-Stokes equations shortly after [12, 147]. Over the years, multigrid has become closely intertwined with CFD, and has become an ingredient in major CFD codes. The viscous flow around a complete aircraft configuration can now be computed

thanks to the availability of multigrid solvers [60], and also complex industrial flows in machinery are computed successfully with multigrid.

However, as remarked in [15], full textbook multigrid efficiency has not yet been achieved in realistic engineering applications in CFD in general. An important reason for this is that in CFD we often have to deal with singular perturbation problems. This gives rise to grids with cells with high aspect ratios. Another reason is that the governing equations may show elliptic or parabolic behavior in one part of the domain and hyperbolic behavior in another part of the domain. This requires careful design of both the discretization and the solver, putting a premium on robustness. With the increasing complexity of CFD applications (most of all due to the grid structures on which the equations are discretized), the demand for robustness of solution methods is increasing even more. Industrial practice is heading towards unstructured grids, more complex flow modeling, time-dependent problems and multi-disciplinary applications. These developments pose new challenges for multigrid research. The potential for large further gains is there, and consequently multigrid remains an active research topic in CFD. Ways for improvement are pointed out in [15, 16]. The current state of the art is surveyed below.

The field of computational fluid dynamics is too diverse and multigrid can be implemented in too many ways to make a self-contained synopsis possible within the confines of a journal article. Therefore our descriptions will be eclectic, global and fragmentary, but we will cast our net wide in referring to the literature after 1990 for further details.

2 Multigrid Basics

First, we present a brief description of multigrid principles. For further information one may consult the basic literature. To establish notation and terminology, we start by formulating the basic two-grid algorithm. Let us have a system of m partial differential equations on a domain Ω, discretized on a grid $G \subset \Omega$. The resulting nonlinear algebraic system is denoted as

$$N(u) = b, \quad u \in U = \{u : G \to \mathbb{R}^m\} \ , \tag{1}$$

where U is the space of grid functions on G. For the moment, the problem is assumed to be independent of time. Let there also be a coarse grid $\overline{G} \subset \Omega$ with fewer nodes than G. Overbars denote coarse grid quantities.

2.1 The Nonlinear Multigrid Method

The basic two-grid algorithm is given by:

Choose u^0
Repeat until convergence:
begin

(1) $S_1(u^0, u^{1/3}, b)$; $r = b - N(u^{1/3})$;

(2) Choose $\overline{u}_\alpha, \sigma$; $\overline{b} = \overline{N}(\overline{u}_\alpha) + \sigma R r$;

(3) $\overline{S}(\overline{u}_\alpha, \overline{u}^{2/3}, \overline{b})$;

(4) $u^{2/3} = u^{1/3} + \frac{1}{\sigma} P(\overline{u}^{2/3} - \overline{u}_\alpha)$;

(5) $S_2(u^{2/3}, u^1, b)$;

(6) $u^0 = u^1$;

end

Step (1) (pre-smoothing) consists of a few iterations with some iterative method for the fine grid problem (1) with initial iterate u^0 and result $u^{1/3}$. In step (2), $\overline{u}_\alpha$ is an approximation on the coarse grid of the exact solution, used to remain on the correct solution branch and/or to linearize the coarse grid correction problem. One may take, for example, $\overline{u}_\alpha = \widetilde{R} u^{1/3}$, with $\widetilde{R} : U \to \overline{U}$ a restriction operator from the fine to the coarse grid. But it may be more economical to keep $\overline{u}_\alpha$ fixed during two-grid iterations, if $\overline{N}(\overline{u}_\alpha)$ is expensive, or if $\overline{u}_\alpha$ is used to construct an expensive Jacobian. In step (2), σ is a parameter which, if chosen small enough, ensures solvability of the coarse grid problem, given by

$$\overline{N}(\overline{u}) = \overline{b} \equiv \overline{N}(\overline{u}_\alpha) + \sigma R r \quad , \tag{2}$$

where $\overline{N}$ is a coarse grid approximation to N, obtained, for example, by discretization of the underlying system of partial differential equations on $\overline{G}$. Furthermore, $R : U \to \overline{U}$ is a fine to coarse grid restriction operator, that need not be the same as $\widetilde{R}$. In step (3), $\overline{S}$ stands for solving the coarse grid problem (2) approximately by some iteration method with initial guess $\overline{u}_\alpha$ and result $\overline{u}^{2/3}$. In step (4), the coarse grid correction is added to the current fine grid iterate. Here, P is a prolongation or interpolation operator $P : \overline{U} \to U$. In step (5) post-smoothing takes place.

With $\sigma = 1$, we obtain the well-known Full Approximation Scheme (FAS) [11], which is most commonly used in CFD. The nonlinearity of the problems enters in the smoothing operators S_1, S_2 and $\overline{S}$. Global linearization is not necessary, so that there is no need to store a global Jacobian, which is, moreover, frequently very ill-conditioned in CFD. For many problems, however, the nonlinearity can also be handled globally, resulting in a sequence of linear problems that can be solved efficiently with linear multigrid. The multigrid method is obtained if solution of the coarse grid problem in step (3) is replaced by γ iterations with the two-grid method, employing a still coarser grid, and so on, until the coarsest grid is reached, where one solves more or less exactly. With $\gamma = 1$ or $\gamma = 2$, the V- or W-cycle is obtained, respectively. If N is the number of unknowns on G and N/β is the number of nodes on $\overline{G}$, then the above multigrid algorithm requires $\mathcal{O}(N)$ storage and, for accuracy commensurable with discretization accuracy, $\mathcal{O}(N \log N)$ work, if $\gamma < \beta$. To get $\mathcal{O}(N)$ work, the multigrid cycles must be preceded by nested iteration,

also called full multigrid; see the basic literature. Starting on a coarse grid and refining this grid successively leads to a well-defined grid hierarchy also on unstructured grids (see, for example [7]). For a given unstructured grid it is usually not difficult to define a sequence of finer grids and a corresponding multigrid method. However, it might be difficult to define a sequence of coarser grids starting from an irregular fine grid [21]. This is where AMG [121] comes into play.

A natural generalization of multigrid is to combine it with locally refined grids. This leads to the Multilevel Adaptive Technique (MLAT) [11, 5] or to the Fast Adaptive Composite Method (FAC) [90]. The locally refined regions are incorporated as extra additional levels in multigrid where all multigrid components like smoothing are defined with minor modifications, see [11, 90] for details.

The favorable $\mathcal{O}(N)$ convergence behavior depends on satisfying the *smoothing property* and the *approximation property* [47]. The smoothing property requires that the smoothing processes S_1 and S_2 make the error between discrete solution and current approximation smooth. In not too difficult cases this can be checked by frozen coefficients Fourier analysis [11, 134, 151], leading to the determination of the smoothing factor. The approximation property says something about the accuracy of the coarse grid correction applied in step (4). Extrapolating from what is known from simple cases, this implies roughly, for example, that P and R, when applied to a given variable and its corresponding residual, should satisfy

$$m_P + m_R > M \quad . \tag{3}$$

Here, orders m_P, m_R of P and R are defined as the highest degree plus one of polynomials that are interpolated exactly by P or $\widetilde{\sigma} R^T$, respectively, with $\widetilde{\sigma}$ a scaling factor, R^T the transpose of R, and M the order of the highest derivative of the unknown concerned that occurs in the partial differential equation. For instance, for the Euler equations, we have $M = 1$, whereas for the Navier-Stokes equations we have $M = 1$ for the pressure but $M = 2$ for velocity components. No requirements are known for $\widetilde{R}$.

2.2 The Smoothing Method

An important issue is the *robustness* of the smoother. This implies that the smoother should be efficient for a sufficiently large class of problems. Two model problems that represent circumstances encountered in CFD practice are the convection-diffusion equation and the rotated anisotropic diffusion equation, given by, respectively

$$c u_x + s u_y - \varepsilon (u_{xx} + u_{yy}) = 0 \quad , \tag{4}$$

$$-(\varepsilon c^2 + s^2) u_{xx} - 2(\varepsilon - 1) cs u_{xy} - (\varepsilon s^2 + c^2) u_{yy} = 0 \quad , \tag{5}$$

where $c = \cos\phi$, $s = \sin\phi$, and ϕ and ε are parameters. The first equation contains the effect of strong convection (if $\varepsilon << 1$) and is related to hyperbolic systems; a method that does not work for a scalar hyperbolic equation ($\varepsilon = 0$) is likely to fail for a system. The second equation models the effect of large mesh aspect ratios, equivalent to $\varepsilon << 1$ on an isotropic grid, and grid non-orthogonality ($\phi \neq 0$), giving rise to mixed derivatives in boundary-fitted coordinates. A smoothing method may be called robust if it works for all ϕ and ε. For a multigrid method that is robust for the test problems (4) and (5) one may have some hope of success in application to the governing equations of CFD. Fourier

smoothing analysis for the simple test problems above is easy, and smoothing factors for many methods are reported in [150, 151, 104, 168], and for some three-dimensional cases in [67, 136]. For the two-dimensional case, robust smoothing methods exist. They are basically of line-Gauss-Seidel type and ILU type; a list is given in Section 7.12 of [151]. For the simple test problems (4) and (5), it can be said that efficient and robust multigrid smoothers and corresponding multigrid methods are available, even of black-box type for problems on structured grids, see [161]. What has kept CFD applications away from textbook multigrid efficiency is the generalization from the scalar case to the case of systems of partial differential equations, the difficulty of making the coarse grid correction sufficiently accurate, and finding efficient smoothers for higher order upwind discretizations with limiters. Furthermore, application of multigrid to unstructured grids has started only recently, and further development is necessary for achieving textbook multigrid efficiency in this context.

The historical development of multigrid in CFD has been such that smoothing methods come in two flavors: basic iterative methods (BIMs) and multistage (MS) (Runge-Kutta) methods. Assuming instead of (1) a linear problem $Au = b$, a BIM is given by

$$Mu^{n+1} = Nu^n + b, \quad A = M - N \ . \tag{6}$$

Typical examples of BIMs are methods of Gauss-Seidel type and of incomplete LU factorization (ILU) type. MS smoothers for the nonlinear system (1) are obtained by artificially creating a system of ordinary differential equations:

$$\frac{du}{dt} = N(u) - b \ . \tag{7}$$

MS methods of the following type are used:

$$\begin{aligned} u^{(0)} &= u^n, \\ u^{(k)} &= u^{(0)} - c_k \tau N(u^{(k-1)}) + c_k \tau b, \quad k = 1, ..., p \ , \\ u^{n+1} &= u^{(p)} \ , \end{aligned} \tag{8}$$

where τ is a time step. The appearance of time in our algebraic problem comes as a surprise, but in fact this application of MS to solve an algebraic system is old wine in a new bag. In the linear case, where $N(u) = Au$, we obtain by elimination of $u^{(k)}$:

$$u^{n+1} = P_p(-\tau A)u^n + Q_{p-1}(-\tau A)b \ , \tag{9}$$

with the amplification polynomial P_p of degree p given by

$$P_p(z) = 1 + z(c_p + c_{p-1}z(1 + c_{p-2}z(...(1 + c_1 z)...) \ , \tag{10}$$

and Q_{p-1} a polynomial that we will not write down. It follows that MS is an iterative method of the type

$$u^{n+1} = Su^n + Tb, \quad S = P_p(-\tau A) \ . \tag{11}$$

Methods for which the iteration matrix S is a polynomial of the matrix of the system to be solved are well-known in numerical linear algebra, and are called semi-iterative methods; see [142] for the theory of such methods. In the multigrid context, the MS coefficients c_k are not chosen to optimize time accuracy or stability, but the smoothing

performance. Because for stationary problems time is an artifact, τ need not be the same for all grid cells, but can vary, such that $c_k\tau$ is optimal for smoothing. Grids with large aspect ratios are generic for Navier-Stokes applications, due to the need to resolve thin boundary layers. Smoothing analysis reveals that when high mesh aspect ratios occur, modeled by test problem (5) by choosing $\phi = 0$, $\varepsilon << 1$, then point-wise smoothers such as Gauss-Seidel and MS, are not satisfactory. The unknowns in certain subsets of the grid points should be updated together, leading to line-Gauss-Seidel or ILU [161, 162].

In the MS case, implicit stages could be included, but this avenue has not been followed systematically. Instead, implicit residual averaging is applied. After a MS stage, the residual

$$r^{(k)} = b - Au^{(k)}$$

is replaced by $\widetilde{r}^{(k)}$ satisfying

$$B\widetilde{r}^{(k)} = r^{(k)} \ , \tag{12}$$

with B such that B^{-1} has a smoothing effect, and such that (12) is cheap to solve. For details, see [83]; tuning parameters are involved.

These ways to obtain a robust smoother clearly have drawbacks. Extensions to systems of differential equations leads to the need to solve subsystems that are more involved and costly than tri-diagonal systems encountered in implicit residual averaging or line-Gauss-Seidel for the scalar case. Parallelizability is likely to be impaired. An alternative is to strengthen the coarse grid correction instead of the smoother. This can be done by constructing the coarse grid by semi-coarsening, i.e, the mesh-size is doubled only in selected directions, such that the coarse grid becomes more isotropic than its parent fine grid. This approach, although not novel, has only recently started to have an impact of practical CFD; we will return to semi-coarsening below.

In three dimensions, the situation is less satisfactory, see [67]. Typical reference model problems in 3D can be

$$s\widetilde{c}\, u_x + c\, u_y + s\widetilde{s}\, u_z - \varepsilon(u_{xx} + u_{yy} + u_{zz}) = 0 \ , \tag{13}$$

where $c = \cos\phi$, $s = \sin\phi$, $\widetilde{s} = \sin\theta$, $\widetilde{c} = \cos\theta$ and ϕ, θ and ε are parameters, and

$$-\varepsilon_1 u_{xx} - \varepsilon_2 u_{yy} - \varepsilon_3 u_{zz} = 0 \ , \tag{14}$$

If a strong coupling of unknowns in two space directions in (14) exists, it is not possible to use cheap basic (collective) point or line Gauss-Seidel type smoothers and still obtain a fast and efficient multigrid method. The strong coupling of unknowns should be taken into account. This means that plane smoothers, that update all unknowns in a plane simultaneously, have to be applied. However, the coupled system in a plane, even for scalar equations, does not result in a tri-diagonal system as with line smoothers, but in a more general sparse system. It is found that often, an inexact (iterative) solution method, like one multigrid V-cycle in a 2D plane, is sufficient for an overall efficient 3D solution method [136, 103]. In this way a relatively cheap plane solver can be obtained for certain problems.

It helps considerably in finding efficient smoothers if the Jacobian of $N(u)$ is close to an M-matrix, as explained in Section 4.2 of [151]. Unfortunately, Godunov's order barrier theorem ([42]) implies that this can be the case only for first order accurate schemes (in the linear case), so that we are restricted to first order upwind schemes. This is unsatisfactory. Generally speaking, engineering practice requires second order accuracy. One

way around this is to use defect correction. Suppose, we have a first order discretization $N_1(u)$ for which we have a good smoother, and a second order scheme $N_2(u)$ that we would prefer to use. Defect correction works as follows:

begin Solve $N_1(\widetilde{y}) = b$;
  for $i = 1(1)n$ do
    Solve $N_1(y) = b - N_2(\widetilde{y}) + N_1(\widetilde{y})$;
    $\widetilde{y} = y$;
  od
end

It suffices to take $n = 1$ or 2 to achieve second order accuracy. But there are smoothers that work for second order schemes, so that defect correction is not necessary; see [34], [1] and [104].

2.3 The Coarse Grid Correction

In this section, we discuss several issues related to the coarse grid correction , that are relevant for CFD problems. In linear multigrid, there are two ways to approximate the fine grid operator on coarse grids, namely coarse grid discretization approximation and coarse grid Galerkin approximation. With the discretization approximation the coarse grid operator is obtained by rediscretizing the governing differential equations on the coarse grids. With the Galerkin approximation one puts

$$\overline{A} = RAP.$$

For more details on coarse grid Galerkin approximation, see Sect. 6.2 of [151]. Coarse grid construction by agglomeration of fine grid cells may lead to coarse grids that conform insufficiently to strongly curved boundaries, leading to inaccurate coarse grid corrections and degradation of multigrid efficiency. This issue is addressed in [52] and [128], where it is shown how to overcome this difficulty by deriving geometric information not from the coarse grid nodes but from the fine grid, effectively employing grid cells with curved boundaries. This difficulty is taken care of automatically when Galerkin coarse grid approximation is used.

Suppose the convection-diffusion equation is discretized with the first order upwind scheme or with a second order upwind biased scheme. In the case of dominating advection, this means that artificial viscosity is added. Standard coarse grid correction suffers from the difficulty that the artificial viscosity is implicitly multiplied by a factor 2 or more. This leads to inaccurate coarse grid correction, as explained in [13, 14, 157, 160]. This difficulty is overcome in two ways. In the first place, it is often the case for convection dominated problems that the smoother reduces both rough and smooth error components so effectively, that weakness of the coarse grid correction has no consequences. Some smoothers with this property are symmetric point Gauss-Seidel, alternating symmetric line Gauss-Seidel and ILU type smoothers, (provided there is no recirculation); but MS smoothers do not have this pleasant property.

In the second place, one may replace standard coarse grid approximation by something more accurate, namely by multiple semi-coarsening or by directional semi-coarsening. Instead of increasing the cost of the smoother, the coarsening strategy is changed so that cheaper smoothers (point smoothers like MS smoothers or line smoothers instead of plane

smoothers) can be applied. Although in multiple semi-coarsening methods [97, 98, 69, 146] many coarse grids exist on coarser grid levels, the work for solving a problem with N unknowns is still $\mathcal{O}(N)$, if an F-cycle [47, 151, 134] is applied. Directional coarsening is applied in all directions, so that a fine grid gives rise to d coarse grids, with d the number of dimensions, making it possible to work with very simple and cheap smoothers. Some coarse grids from different finer grids coincide. In 2D a diamond-like sequence of grids is obtained (see Figure 1). There are several options for the transfer operators between the

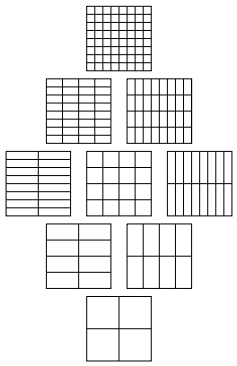

Figure 1: Multiple semi-coarsened grids in 2D with a finest grid consisting of 9×9 points.

grids [97], [102], [99].

The total number of cells in all grids combined is $8N$ (in three dimensions). In 3D, however, there are more possibilities for the grid coarsening. Depending on the number of coarse grids chosen, a point or a line smoother must be used to obtain a robust 3D multiple semi-coarsening based solution method. A multiple semi-coarsening variant that uses a coarsening strategy such that line smoothers guarantee robustness for 3D reference problems is presented in [146]. A reduction of the computational overhead can be achieved by *flexible multiple semi-coarsening*: only if strong couplings in certain directions exist [146], the semi-coarsening process is continued in that direction.

The second variant is *directional coarsening*: fine grid cells are coarsened in one direction only, by removing only the largest cell faces, so that coarsening is done only in a direction roughly perpendicular to refinement zones. When the coarse grid has become sufficiently isotropic, one can switch to full coarsening. This approach is commonly used for compressible Navier-Stokes equations, to be discussed in Section 4.2. Directional semi-coarsening combined with line relaxation is presented in [27].

2.4 Structured and Unstructured Grids

Grid generation is one of the most time consuming activities of the CFD practitioner. It is usually the main bottleneck for project turn-around time. In comparison, solver CPU time plays a minor role. This achievement is for a large part due to multigrid. In the past, much effort has gone into development of tools for the generation of structured grids. In such grids, the number of cells that share a common vertex is uniform in the interior of the domain. The approach is to divide the domain in subdomains, each of which is mapped by a boundary-fitted coordinate mapping to a cube, in which a uniform grid is generated, the image of which gives a boundary-fitted curvilinear grid in the subdomain. In order to cope with the geometric complexity of engineering applications, the subdomain decomposition must be unstructured, leading to multiblock block-structured grids. On structured grids

algorithms can be formulated that run fast on vector computers, less computer memory is required, and coarse grid generation for multigrid and the implementation of transfer operators between grids is straightforward. These are the main advantages of structured grids. Despite intensive research efforts, however, it has turned out to be not possible to automate the generation of structured grids to a sufficient extent to reduce the amount of human labor involved to a low level. In particular, the generation of the domain decomposition requires much time from the user.

As a consequence, unstructured grids are now getting more and more attention. Not only are these grids easier to generate than structured grids, they also lend themselves better to adaptive discretization, since it is relatively easy to insert and remove grid points. The development of accurate discretizations and efficient solution methods is more difficult than for structured grids. This is now an active field of research, and much remains to be done. We will give some pointers to recent work, but it is too early for a review. We will concentrate mostly on structured grids, which are still mostly used today.

A third type of grid consists of the union of locally overlapping grids that together cover the domain. The local grids are usually structured. The flexibility offered by this kind of grid is especially useful for flows in which boundary parts move relatively to each other. Furthermore, this is a way to include adaptivity in the structured grid context. The multigrid principles are the same as for structured grids. We will not discuss overlapping grids. Examples may be found in [3, 55, 22, 138].

3 Incompressible Flow

Unified methods that treat compressible and incompressible flows in a uniform way are emerging (see [9] and references quoted there). Unified methods departing from incompressible Navier-Stokes adopt corresponding solution methods, whereas low Mach equations arising from preconditioning compressible formulations are solved by compressible solution methods. Standard computing methods for compressible and incompressible flow differ substantially, and will therefore be discussed separately. We start with the incompressible case.

We will assume that discretization takes place on boundary-fitted structured grids. The flow is governed by the incompressible Navier-Stokes equations. For the high Reynolds numbers occurring in most applications thin boundary layers occur, necessitating locally refined grids, giving rise to high mesh aspect ratios (10^4 or even more). The primitive variables are commonly used. The solution of d momentum equations (d is the number of dimensions) and the continuity equation is required. The discretized continuity equation serves as an algebraic constraint, see any textbook on the incompressible Navier-Stokes equations, so that after discretization in space the incompressible Navier-Stokes equations constitute a differential-algebraic system (of index two), and the pressure acts as a Lagrangean multiplier, mathematically speaking.

Often, a semi-heuristic turbulence model is used to predict time-averaged flow variables based on a compromise between accuracy, memory requirements and computing time. More complete models such as large-eddy simulation and direct numerical simulation are time-dependent and can be efficiently implemented with explicit time-stepping schemes. In these models an algebraic pressure correction equation appears that can easily be handled by classical multigrid methods, that do not need further attention here. In fact, for time-accurate solutions the main use of multigrid, for example in many engineering

codes, is for some form of the elliptic equation for the pressure. It is, however, possible to benefit more from multigrid, as we will discuss below.

The spatial discretization should be such that spurious pressure modes are avoided. This has given rise to two different approaches. If grid-oriented velocity components with a staggered placement of unknowns are employed, no special measures are required, but the scheme is somewhat complicated, and must be formulated carefully to maintain accuracy on rough grids; see [152]. The alternative is to use a colocated (non-staggered) placement of the unknowns, for which the Cartesian velocity components and pressure can be used. This makes it easier to formulate accurate schemes on curvilinear grids, but artificial stabilization measures are required. For a discussion of the relative merits of staggered and colocated schemes, see, for example, [119, 152]. The two approaches will be discussed in separate subsections.

3.1 Multigrid for Staggered Schemes

Discretization of the stationary incompressible Navier-Stokes equations on a staggered grid results in a nonlinear algebraic system of the following form:

$$\begin{pmatrix} Q & G \\ D & 0 \end{pmatrix} \begin{pmatrix} U \\ p \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} . \tag{15}$$

Here, U contains the velocity components, p contains the pressure values in the cell centers, b_1 and b_2 are forcing terms resulting from the boundary conditions, Q is a nonlinear algebraic operator arising from the discretization of the inertia and viscous term, and G and D are the linear discrete gradient and divergence operator, respectively.

Because of the staggered placement of the unknowns, different prolongation (P) and restriction (R) operators are required for the velocity components and pressure. This is not a big problem; see [141, 151, 163] for examples. Because no higher than first order derivatives of the pressure occur, P and R can be less accurate than for the velocity, according to the accuracy rule (3). As discussed in Section 2, coarse grid approximation can be done either by discretization on the coarse grid, which is most commonly done, or by Galerkin approximation of the fine grid operators on the coarse grids.

The main issue is the design of good smoothers. Classical relaxation methods are inapplicable, due to the occurrence of a zero block on the main diagonal. Two approaches may be distinguished: box relaxation [141] and distributive iteration [12, 153]. For an introduction to these methods, see Section 9.7 of [151]. A brief synopsis is given below, after which recent developments are reviewed.

In box iteration, the unknowns belonging to a cell are updated simultaneously in Gauss-Seidel fashion. These unknowns are the normal velocity components in the cell faces and the pressure in the cell center. A $(2d+1) \times (2d+1)$ system, with d the number of dimensions, has to be solved for each cell, which can be done using an explicit formula. This is the symmetric coupled Gauss-Seidel (SCGS) method, introduced in [141]. Roughly speaking, the method performs similar to the Gauss-Seidel method for a convection-diffusion equation. This implies that a symmetric version is required, in which the cells are processed both in a forward and backward (reversed) order, so as to obtain a smoother that is robust with respect to flow direction. Furthermore, on highly stretched grids a line-version is required, with lines perpendicular to the refinement zone. Line versions of SCGS are described and applied in [101, 137]. Moreover, [137] presents the line smoother in an adaptive grid framework. When the grid is non-orthogonal, line-Gauss-Seidel might

become less effective, as predicted by analysis of scalar model problems; see Chapter 7 of [151]. But ILU remains effective as a smoother. Further applications of multigrid with point or line SCGS are described in [66, 155, 167]. In [62, 100] a $k-\varepsilon$ turbulence model is included.

Distributed iteration methods are best described as follows. The system (15), denoted as $Ay = b$, is replaced by

$$AB\overline{y} = b, \quad y = B\overline{y} \ , \tag{16}$$

where the distribution matrix B is chosen such that AB lends itself easily to iterative solution; for example, because AB is close to a triangular M-matrix the zero block in A disappears. Let C be some approximation to AB. Then we have the following iteration method for (16):

$$\overline{y}^{m+1} = \overline{y}^m + C^{-1}(b - AB\overline{y}^m) \ ,$$

or

$$y^{m+1} = y^m + BC^{-1}(b - Ay^m) \ . \tag{17}$$

Of course, C is chosen such that the action of C^{-1} is easily determined. It is made plain in [151], that depending on the choice of B and C, various well-known methods are obtained, the classical examples being the SIMPLE method of [109] and the distributed Gauss-Seidel method of [12]. Distributive iteration methods can be designed such that the individual velocity components and pressure are updated separately, so that for the velocity components one has essentially to deal with convection-diffusion equations, for which smoothing is relatively easy. But box variants are also possible. The box ILU distributive smoother employed in [164] is robust with respect to mesh aspect ratio and grid non-orthogonality. For accuracy, second order schemes are required for the inertia terms; this may necessitate the use of defect correction [28].

It is found that these methods lend themselves well as smoothers in multigrid. Interesting new insights in distributive smoothers are presented in [10]. Smoothing analysis is presented in [12, 123, 127] (elementary) and [153, 154] (advanced). The analysis in [14] is especially interesting, because the influence of boundaries is taken into account. Unfortunately, some distributed smoothing methods require problem-dependent underrelaxation parameters. In [127], the SCGS method is compared with distributed smoothers. It is found that SCGS is more efficient than distributed iteration for high Reynolds numbers, and that it is less sensitive to the value of the underrelaxation parameter. In [129], a SIMPLE type smoother is compared with point and line versions of SCGS on the basis of numerical experiments. SCGS come out best, with a line version being superior in the presence of large mesh aspect ratios. But in [107] it is found, that for stratified flow problems SIMPLE is to be preferred. Both approaches are compared as single grid solvers in [51]. Recent applications using distributive smoothers are presented in [24, 106, 114, 126, 165], for unstructured grids in [64, 65, 57] and in combination with adaptivity in [156]. Multigrid for divergence free finite elements is described in [140].

3.2 Multigrid for Colocated Schemes

In order to rule out spurious pressure modes, usually the continuity equation is perturbed by terms involving pressure. This can be done by introducing artificial compressibility ([23]), or by the pressure-weighted interpolation method (PWI) of [115], or by using a

Roe-type flux difference splitting ([32]). The second approach is most widespread in engineering practice.

We start with methods using the PWI scheme. The stabilizing terms replace the zero block in (15) by an operator that is not given explicitly in the literature, but that is easily deduced from equation (49) of [92]. Nevertheless, distributive methods dominate the field, that are quite similar to SIMPLE and its sisters for staggered schemes, and carry the same names. Smoothing analysis for smoothers of SIMPLE type is given in [93, 41]. Some recent publications discussing the application of multigrid methods to computation of laminar flows are [41, 75, 128]. Inclusion of two-equation turbulence models is discussed in [63, 74]. Turbulence is included on unstructured grids in [2, 57]. In [57] ILU is used as the smoothing method, and starting from a coarsest grid, adaptive unstructured grids are defined.

The numerics for artificial compressibility methods resemble closely the numerics for compressible flow methods, and will be discussed below. Laminar flow computations with the artificial compressibility method are reported in [36, 76, 130, 144]; turbulence modeling is included in [77, 124, 166], adaptivity in [76]. A staggered scheme with a multigrid acceleration of a so-called fraction step solver is presented in [120], the fractional step method is employed in a finite element context in [139].

In the flux-difference splitting of [32], a stabilizing pressure term is introduced in the discrete continuity equation in a natural way. This scheme does not give an M-matrix, but a so-called vector-positive discretization. This makes Gauss-Seidel type smoothing possible in a collective or a decoupled formulation. The flux-difference scheme is first-order accurate, but second order accuracy can be obtained in the standard way using the MUSCL approach. For details, see [32].

4 Compressible Flow

One of the first applications of multigrid in CFD may be found in [131]. This work concerns the compressible potential equation. This equation is now efficiently solved routinely with multigrid in the aerospace industry, and will not be discussed here. For a survey of past multigrid developments, see Chapter 9 of [151]. A general survey of discretization schemes for compressible flow equations is given in [56].

4.1 The Euler equations

The prevalent smoothing method for multigrid solution of the Euler equations is the MS method. Since its introduction in [59], the MS smoothing method has evolved quite a bit and has become steadily more efficient. An overview is given in [61]. To begin with, efforts have been made to optimize the MS coefficients c_k to enhance smoothing. This has been done in [72], [116] for the one-dimensional scalar case:

$$\frac{\partial u}{\partial t} + \lambda \frac{\partial u}{\partial x} = 0 \; .$$

Since in a system there are different eigenvalues λ, straightforward application of the optimal coefficients to the systems case is not optimal. This is shown in [73], where the straightforward approach is compared with what is called characteristic time-stepping (requiring the use of a flux splitting discretization scheme), for which only one effective wavespeed comes into play, so that more benefit is derived from optimization for the scalar

case. The difference in efficiency between the two methods is found to be considerable. Optimal MS smoothing for central schemes with artificial dissipation, i.e. the very popular Jameson-Schmidt-Turkel scheme ([59]), requires the use of preconditioning. That is, the discrete scheme to be solved is given by

$$\frac{dU}{dt} + P(U)R(U) = 0 \, ,$$

where $P(U)$ is a preconditioner. Of course, time accuracy is lost. The purpose of preconditioning is to cluster the eigenvalues, so that the coefficients obtained from scalar optimization can be more profitably applied. Preconditioning is usually done by collective Jacobi iteration ([1], [33], [34], [116], [19]); this is called the multistage Jacobi method. In [34] it is shown by experiments that it is more efficient to optimize c_k for increasing the time step rather than for smoothing. We think the disparity between optimality as derived in [72] by Fourier analysis and optimality in practice is due to the influence of boundary conditions, which is accounted for faster with larger time steps.

The optimal multistage coefficients c_k as determined for the one-dimensional case can also be used in the multi-dimensional case, because, as remarked in [135], a two-dimensional optimization leads to optimal coefficients that are not significantly different from those obtained in the one-dimensional case. Only the optimal CFL number (which has to do with the choice of the local time step τ) differs somewhat; in [135] a recipe is given for choosing a good CFL number in more dimensions.

Implicit smoothing methods work differently. The equations are discretized in time with the implicit Euler method, giving rise to

$$\frac{1}{\Delta t}(U^{n+1} - U^n) + R(U^{n+1}) = 0 \ .$$

If steady solutions are envisaged, the time difference can be deleted. A relaxation scheme is chosen for this nonlinear system for U^{n+1}, for example the collective Gauss-Seidel (CGS) smoothing method in a FAS method. All unknowns in a cell are updated simultaneously, keeping values in neighboring cells fixed. This requires the (approximate) solution of a small nonlinear system: usually one Newton iteration suffices. This approach is followed in [53], [68], [96], [132]. If applied in symmetric fashion (forward followed by backward ordering), CGS is a very efficient smoother for first order schemes for hyperbolic systems, more efficient than MS, as predicted by the model problem smoothing analysis in [151]. Furthermore, CGS does not require tuning of coefficients. But, unlike MS, GS does not work for second order schemes directly, but must be combined with defect correction, as done in [30], [31], [53], [68], [132] or line versions must be chosen [104]. According to [33], [34], [116] it is best to apply defect correction only on the finest grid, and to use the first order scheme on the coarse grids; this is called mixed discretization. In [34], [116] it is found that without the latest enhancements MS is less efficient than CGS, but with Jacobi preconditioning and mixed discretization it can compete with CGS and defect correction for Total Variation Diminishing (TVD) schemes. Because parallelization is easier, MS is probably to be preferred on multi-processor platforms. Adaptive grids are incorporated in compressible solvers in [8, 108, 145, 84, 39]

The multiple semi-coarsening method has been pioneered in [97], [98] for the Euler equations. The combination of multistage smoothing without frills, low Mach number preconditioning and semi-coarsening is shown to be quite robust for the two-dimensional

Euler equations in [25]. The efficiency is better than standard multigrid based methods; typically 500 to 1500 work units are required, with a work unit the work for a residual evaluation on the finest grid. We expect that if nested iteration (full multigrid) would have been incorporated, then optimal efficiency (100 work units, say) would not be far away. Similar performance in three dimensions and for Navier-Stokes still remains to be demonstrated.

Standard multigrid has been very successful for the Euler equations. For the Navier-Stokes equations the situation is less satisfactory, as we shall see in the next section.

4.2 The Navier-Stokes equations

As far as multigrid is concerned, the main difference with the Euler equations is the generic occurrence of highly stretched cells (with aspect ratios of up to 10^4), in order to resolve thin boundary layers. This leads to a widely observed deterioration of the standard multigrid convergence rate with MS smoothers, causing a wide gap between actual and textbook multigrid efficiency. From Fourier smoothing analysis of simple model problems (see e.g. [151]) it is clear that, if coarse grid correction is left alone, then unknowns in columns of cells sharing large faces (more or less perpendicular to refinement zones) must be updated simultaneously, giving rise to methods such as line-Jacobi, line-Gauss-Seidel [70], ADI [20] and ILU. In the scalar case this gives rise to solving tridiagonal or similar simple systems, but in the systems case the smoother becomes more involved and computing-intensive. As discussed in Section 2.3, the alternative is to leave the smoother alone, but to do something about the coarse grid correction. We will return to this shortly, but first we discuss robust smoothers for grids containing refinements zones.

The obvious extension of CGS to a line version (LU-SGS) has been undertaken in [158], [159]. In [1, 143] the MS scheme is made suitable for Navier-Stokes by choosing for the preconditioner $P(U)$ something similar to LU-SGS; [1] also provides Fourier smoothing analysis results, as do [133, 168]. In order to take care specifically of stretched cells in a boundary layer, one may choose $P(U)$ corresponding to collective line-Jacobi iteration, with lines chosen perpendicular to the boundary layer. This is described and analyzed in [111]. In [144] the MS smoother is used with line-implicit stages to take care of high aspect ratio cells, and Fourier smoothing analysis is presented; good efficiency is obtained. More reliable than smoothing analysis for prediction of actual multigrid performance is two-grid analysis, since this gives a better representation of the influence of coarse grid correction. Two-grid analysis for compressible flow problems is presented in [58].

A semi-coarsening variant for the Navier-Stokes equations is presented in [113]. The large number of coarse grids generated makes this method impractical for industrial purposes. For Navier-Stokes, however, directional coarsening is starting to be accepted. Smoothing analysis in the presence of directional coarsening and results are given in [1], [110], [112]; and applications are described in [77, 130]. Significant gains in efficiency over older methods are obtained by combining the MS method, point-Jacobi preconditioning and directional coarsening.

Closer towards the AMG approach is the approach followed in [85, 86, 87, 95, 88]. There the coarse grid operator for the compressible Navier-Stokes equations is constructed by the Galerkin coarse grid approximation. The coarsening strategy is also AMG based [95]. An efficient solver is presented in [88] with Krylov subspace acceleration of a multigrid method with directional AMG-like coarsening and "directional implicit smoothing", i.e.

combining the coarsening with simultaneous smoothing of coupled unknowns. Related to the previously mentioned methods, departing from an unstructured fine mesh, is the approach presented in [94, 29] and the references quoted therein.

Turbulence modeling brings in additional difficulties, that are typical for reactive flows as well. The case of the k-ε model can be taken as a typical example. Stiff source terms appear, some positive, some negative. For physical as well as numerical reasons, k and ε must remain positive. It turns out to be profitable to compute the flow variables and the turbulence quantities in a coupled manner, i.e. to include the turbulence model with the flow model in the multigrid method, as discussed in [77]. This requires some delicacy in the treatment of the turbulence model. The negative part of the source terms must be treated implicitly; this can be incorporated in the preconditioner. To ensure positivity, the coarse grid corrections for k and ε must be damped or limited. For details, see [35], [40], [80], [125],[6]. A multigrid application to compressible Direct Numerical Simulation (DNS) is described in [18].

5 Multigrid and Parallel Computing

Selection of a good bottle of wine is trivial if the price plays no role. Similarly, design of numerical algorithms is trivial if computer resources are unlimited. In reality they are scarce. Therefore it is not of much benefit to parallelize inefficient numerical algorithms, i.e. algorithms with computational complexity $\mathcal{O}(N^\alpha)$, $\alpha > 1$. The demands of applications in engineering and physics are such that the relevant problem size increases as much as the computer size allows. Let us ponder briefly the consequences of this fact. Assume that if p is the number of processors, then the problem size is $N = mp$. That is, we assume that the problem size is proportional to the number of processors. Let the parallel efficiency be perfect, so that the computing speed is sp flops. Then the turn-around time will be $T = \mathcal{O}(p^{\alpha-1})$. Hence, if $\alpha > 1$ the turn-around time gets worse when more processors are installed. We conclude that it makes sense if parallel computing goes hand in hand with $\mathcal{O}(N)$ numerical algorithms. Hence the special significance of multigrid for parallel computing.

An approach for the parallelization of grid-oriented problems, which is suitable for many solution methods for PDEs, is grid partitioning [89, 78, 118]. In grid partitioning, parallelization is achieved by splitting the grid into subgrids and by mapping the subgrids to different processors. This parallelization concept is reasonable for problems wherein all operations are sufficiently local. By providing "overlap" regions along all internal subgrid boundaries, local operations (for example, the operations that make up the multigrid algorithm) can be performed in parallel. The overlap regions contain the latest values of the unknowns at points belonging to neighboring blocks, thus allowing normal execution of all operations at all points, including points on internal subgrid boundaries. The latest values in the overlap regions are provided by communication between the processors. Since communication takes place on a lower dimensional subset (boundary data) than the computation (on volumes of data) and since the number of arithmetic operations per grid point is relatively large in CFD problems, the grid partitioning strategy results in a good computation/communication ratio, and hence in efficient parallel solvers, including multigrid.

As the special type of the PDE has no great influence on the parallelization, many of the following considerations carry over directly to the incompressible and the compressible

equations but also to other PDE systems. A block-structured grid provides a natural basis for the parallelization of PDE solvers. If these blocks are of completely different size and if each block is handled by one processor, problems with the load balance can be expected, since the work done in a processor essentially depends on the number of grid points owned by the processor. It is therefore important to split blocks in such a way that a satisfactory load balancing is achieved. It makes no sense to spend too much time at this stage: it is harmless if one (or a few) processors have less work than the average.

In parallel multigrid algorithms based on grid partitioning, the main part of the communication time is spent in updating the overlap regions. Locality in smoothers is beneficial for parallel processing. With the explicit MS smoothers, for example, it is possible to obtain a parallel multigrid algorithm which is identical to the single processor version. The situation is somewhat different with the implicit BIM smoothers, especially if unknowns are updated in a sequential order. The easiest way to parallelize such smoothers is by adapting the partitioning such that all unknowns that need to be updated simultaneously lie within one block. In the situation that an artificial block boundary cuts a strong coupling, parallel versions of implicit smoothers, like parallel line solvers [91, 71], are necessary for satisfactory convergence. Modifications in which lines within a block are handled are, for example, described in [81].

Of course, the ratio between communication and arithmetic costs on coarser grids becomes worse. An immediate response to the degradation in efficiency caused by coarse grid levels would be to use multigrid cycle types which minimize the amount of time spent on coarse grids. From this point of view the use of V-or F-cycles, which provide optimal multigrid convergence for many applications, should be preferred to W-cycles.

It might happen that on the coarse grids there are more processors available than there are grid points. An approach for treating such a coarse grid problem in parallel multigrid is found in the strategy of not going to the possible coarsest grid, but keeping all the processors busy. In this case, the parallel algorithm is different from the sequential algorithm. The efficiency of this approach depends particularly on the solution procedure on the coarsest grid.

As indicated above, an advantage of an $\mathcal{O}(N)$ method like multigrid is that the method scales well, i.e. for increasing problem sizes and for an increasing number of processors the scalability of the application is very satisfactory, if the number of grid points remains fixed per processor. It has been found in [79] that the multigrid solution method scales well for problems from CFD applications. (The number of processors varied in that study from 1 up to 192.) In [26], a staggered incompressible Navier-Stokes solver with the SCGS smoother is parallelized with grid partitioning, a colocated incompressible solver with distributed ILU smoothing in [37] and, for example, a projection method based solver with a parallel multigrid kernel in [82]. Adaptive multigrid on parallel computers for the Euler equations with implicit smoothing methods is presented in [117]. Three-dimensional industrial codes are parallelized by a communications library CLIC (Communications Library for Industrial Codes), which also supports adaptivity in [122]. Further 3D examples are found in [105].

Also parallel multiple semi-coarsening variants are most commonly based on the grid partitioning technique [102]. This has implications for the processing of the (multiple) fine and coarse grids. A sequential processing of certain fine and coarse grids seems natural, since these parts of the semi-coarsened grids are in the same processor. It does not make much sense to solve these parts in parallel because additional wall-clock time is not gained.

6 Conclusions

We have presented an overview of the developments in geometric multigrid methods for problems from computational fluid dynamics. With many pointers to the literature of the last ten years for the compressible and the incompressible case, we hope to have given a survey helpful for many practitioners. It is also clear that the desired textbook multigrid efficiency is not yet achieved for all relevant CFD problems and that the demands of engineering applications are orienting research in interesting new directions. With the strongly anisotropic grids that are currently used, advanced multigrid features, such as semi-coarsening techniques, adaptivity and generalization to unstructured grids are becoming more important. The same holds for parallel computing. We think that there is good reason to regard the multigrid approach to be one of the most significant developments in numerical analysis in the second half of the century that now lies behind us.

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