Highly Accurate Evaluation of European and American Options Under the Variance Gamma Process*

Ariel Almendral † Cornelis W. Oosterlee ‡

February 8, 2005

Abstract

European and American option prices under the VG process are computed with high accuracy by numerically solving the corresponding partial Integro-Differential equation (PIDE) on a stretched grid.

Keywords: Integro-differential equations, Variance Gamma, finite differences, grid stretching.

1 Introduction

The numerical computation of financial derivatives when the underlying process is a Lévy process has received significant attention in the recent years. Several useful methods have been proposed in the literature [3, 4, 5, 6, 14, 16, 18, 22]. Common theme in these citations is the numerical solution of a Partial Integro-Differential Equation (PIDE) satisfied by the option prices. The present paper also aims at numerically computing prices, but differs from the existing literature by the fact that we mimic the success of grid transformations applied to the Black-Scholes PDE [12, 25, 28], to obtain a high order accurate numerical method for Variance Gamma (VG) vanilla option prices.

In previous work [3, 4, 5], we computed the integral term by the trapezoidal rule, a choice that produces at best second order accurate convergence, but usually resulted in first order accuracy. The reason for this lies in the lack of differentiability of the initial condition. The main idea of this paper is to use an analytic stretching of the grid that renders the payoff function smooth. If we transform the convolution integral by a non-linear change of variables, it is not directly possible to apply the fast Fourier transform (FFT) algorithm to expedite the computation of the convolutions. On one hand we loose in the

*This research was supported by the Dutch government through the national program BSIK: knowledge and research capacity, in the ICT project BRICKS (http://www.bsik-bricks.nl), theme MSV1.

†Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, Mekelweg 4, 2628 CD Delft (ariel@math.uio.no).

‡Delft University of Technology, Faculty of Electrical Engineering, Mathematics and Computer Science, Mekelweg 4, 2628 CD Delft (c.w.oosterlee@math.tudelft.nl).

speed of computation, but on the other hand, a high order method is obtained and therefore fewer grid points will suffice to achieve a similar accuracy. That is, we show numerically how a simple transformation results in higher accuracy in the process of numerical integration. Of special interest for the VG process is the family of transformations having zero velocity exactly at the kink (or discontinuity in the case of a Binary option) of the payoff, since these are the mappings that produce a smoothed transformed payoff function.

The equation satisfied by options, when the underlying asset follows a geometric Lévy process has in general a non-zero convection term originating from the drift of the driving Lévy process. It is however possible under the VG process to transform the PIDE so that the convection term disappears. One later recovers the solution by a convenient shift and interpolation. This idea is fundamental to the method discussed in this paper.

The paper is organized as follows: In Section 2 we briefly summarize the Variance Gamma based option pricing, focusing on the PIDE interpretation. We give the main ideas of the stretching method in Section 3, along with a theoretical study of the effect of the proposed transformation on the integrand. A series of numerical experiments in Section 4 illustrates the numerical convergence improvements on examples of European and American options.

2 The put option under the VG model

The VG process is obtained by replacing the deterministic time in the classical Samuelson's model by a random business time given by a Gamma process. The resulting process turns out to be a Lévy process of infinite activity and finite variation; see the pioneering papers [11, 21] for further details on the VG process and [13, 26] for general information on Lévy processes in Finance.

Throughout this paper we work with the logarithmic prices $x = \log(S)$. All the ideas may be adapted to solve in terms of the asset price, if desired. By a generalization of Ito's Lemma, it is possible to prove that the price of a European put under the Variance Gamma model satisfies the following Partial Integro-Differential Equation (PIDE) [13]:

$$w_t + (q - r - \omega)w_x + rw = \int_{\mathbb{R}} (w(x + y, t) - w(x, t))k(y)dy, \tag{1}$$

with initial condition

$$\psi(x) := w(x, 0) = (K - e^x)^+. \tag{2}$$

Here

- q and r are the continuous dividend yield and the interest rate respectively;
- ω is a so-called "compensation constant", that is, a constant such that discounted prices become martingales. For ω we have the following expression

$$\omega := \int_{\mathbb{R}} (1 - e^y)k(y)dy = C \log[(1 + 1/G)(1 - 1/M)]. \tag{3}$$

  In order for ω to be defined, $M > 1$ must be satisfied in (3);

- t denotes the time to expiration, that is, the equation is written forward in time;
- $k(y)$ denotes the Lévy density associated to the Variance Gamma process:

$$k(y) = C\left[\frac{e^{-M|y|}}{|y|}\mathbf{1}_{[y>0]} + \frac{e^{-G|y|}}{|y|}\mathbf{1}_{[y<0]}\right], \tag{4}$$

  where C, G and M are positive quantities.

The constant C controls the intensity of all the jumps, G and M model the decay rate of large positive and negative jumps respectively. That is, the larger the value of C, the greater the activity of the jumps. From the exponential decay of the Lévy density we deduce that for large values of G, large positive jumps become less probable, increasing therefore the presence of small positive jumps. We assume throughout the paper that these constants are such that we are in a risk-neutral world. The notation employed here resembles that of [18] and note the use of the letters C, G and M to remind us that the VG process is a particular case of the so-called CGMY process [10].

Let $\mathcal{I}(u)$ denote the integral operator on the right-hand side of (1). The price of an American put option is given as the solution to the following Linear Complementarity Problem [13, 18]:

$$\begin{cases} w_t + \alpha w_x + rw \geq \mathcal{I}(w) & \text{in } (0,T]\times\mathbb{R}, \\ w \geq \psi & \text{in } [0,T]\times\mathbb{R}, \\ (w_t + \alpha w_x + rw - \mathcal{I}(w))\,(w-\psi) = 0 & \text{in } (0,T]\times\mathbb{R}, \\ w(x,0) = \psi(x), & \end{cases} \tag{5}$$

Here, and in the rest of the paper, we write

$$\alpha = q - r - \omega. \tag{6}$$

Existence and uniqueness of solutions to the Cauchy problem (1)-(2) and to the LCP (5) are beyond the scope of this paper. A discussion of this issue may be found in [13], where the concept of viscosity solution is employed to interpret the solution in absence of diffusion component. Our objective with this paper is to provide an accurate numerical scheme to solve these problems.

Two possible approaches to handle the convection term in (1) and (5) arise: keep the convection term, and discretize it by some upwind scheme, or simply eliminate it by transforming the equation. We opt for the second approach, since this gives accurate numerical results easily and enables us to use a particular form of grid stretching. The convection term may be removed using a change of variables. Indeed, note that the new function

$$u(x,t) := w(x+\alpha t, t), \tag{7}$$

is such that $u_t = w_t + \alpha w_x$. Therefore, u satisfies the equation

$$u_t + ru = \int_{\mathbb{R}} (u(x+y,t) - u(x,t))k(y)dy, \tag{8}$$

which does not contain the convection term.

In a similar way, if we apply the shift transformation (7) on problem (5), we obtain the following LCP formulation in terms of u:

$$
\begin{cases}
u_t + ru \geq \mathcal{I}(u) & \text{in } (0,T] \times \mathbb{R}, \\
u \geq \psi(x+\alpha t) & \text{in } [0,T] \times \mathbb{R}, \\
\left(u_t + ru - \mathcal{I}(u)\right)\left(u - \psi(x+\alpha t)\right) = 0 & \text{in } (0,T] \times \mathbb{R}, \\
u(x,0) = \psi(x).
\end{cases} \tag{9}
$$

So, (8) and (9) replace (1) and (5) respectively.

3 Stretched grid discretization

3.1 Smoothing the payoff

This paragraph contains the core of the proposed method. The methodology here is not new, it has been used with great success to treat integral equations with weakly singular kernels [9, 7, 24]. To our knowledge it has not yet been used in handling integro-differential equations in finance. Briefly, the method consists in applying a convenient non-linear change of variable to smooth the input function, either at points of singularity or at the end points of the interval. Since we are considering integrals over the whole real line, we need only apply the smoothing at one point, namely at $\kappa = \log K$. The advantage of carrying out such transformation is the increase in the order of accuracy when using either quadrature rules, or some other numerical integration procedure that normally demands on smoothness of the integrand.

Consider a one-to-one function $H(t) : \mathbb{R} \to \mathbb{R}$ belonging to $C^J(\mathbb{R})$ such that:

$$H(\kappa) = \kappa, \quad H^{(j)}(\kappa) = 0, \quad j = 1, \ldots, J. \tag{10}$$

This function will be used to define a new grid in the form of an analytic grid stretching. Define now

$$\varphi(a) = \psi(H(a)), \quad \text{for} \quad \psi(x) = (K - e^x)^+. \tag{11}$$

The following result describes the regularity of φ.

Proposition 1. *The transformed payoff φ belongs to $C^J(\mathbb{R})$.*

Proof. For $a > \kappa$ the function φ vanishes together with all its derivatives. At $a = \kappa$, the right derivative vanishes as well. It is thus sufficient to consider $a < \kappa$. The chain rule together with the product rule yields

$$\varphi^{(J)} = \sum_{j=1}^{J} \psi^{(j)}(H) p_j(H^{(1)}, \ldots, H^{(J)}) \tag{12}$$

for certain polynomials p_j without zero degree terms. From this expression and (10) follows the claim. $\square$

That is, the only possibility for the new payoff function to become smooth is that we match both the right derivative and the left derivative at κ. This is possible if the derivative is zero at κ. The degree of smoothness of the new payoff is determined by the number of derivatives vanishing at κ. The following straightforward example will be our preferred choice due to its simplicity.

Example: Consider the function

$$H(t) = \kappa + \operatorname{sgn}(t-\kappa)\,|t-\kappa|^R, \quad R \geq 1. \tag{13}$$

Here, the parameter R controls the degree of stretching. The transformed payoff function $\varphi(a)$ is displayed in Figure 1 for different degrees of stretching. The grid in the variable a is uniform.

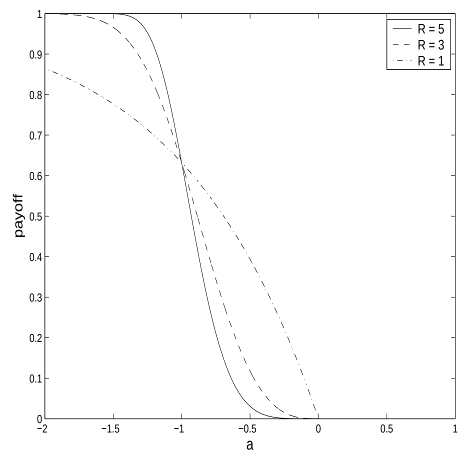

Figure 1: Transformed payoff function for different values of R.

The advantage of this transformation is two-fold: on one hand, more points are situated around the kink of the payoff, and on the other hand, fewer points describe the growth at infinity. As a consequence, the integral is now localized on a smaller interval.

Remark 1. *This kind of transformation is useful if no convection and diffusion terms are present in the equation, otherwise it would give rise to coefficients with singularities. For equations with convection and diffusion terms as in the Merton's model [23], we recommend the use of mappings with a non-vanishing derivative at the kink, such as the variable change employed in [12].*

3.2 Transforming the integral term

The convolution form of the integral term of (1), though convenient for FFT purposes, does not easily allow the change of variables from the grid stretching. An alternative form is preferable, so we first change to the variable $z = x + y$, and then we set $z = H(b)$, $x = H(a)$. Thus, the a-grid will be the computational grid and the b-grid is used for the integration. This yields

$$\begin{aligned}
\int_{-\infty}^{+\infty} & (u(x+y)-u(x))k(y)dy \\
&= \int_{-\infty}^{+\infty} (u(z)-u(x))k(z-x)dz \\
&= \int_{-\infty}^{+\infty} (v(b)-v(a))k(H(b)-H(a))H'(b)db \\
&= \int_{-\infty}^{a} G_a(b)k_1(a,b)db + \int_{a}^{\infty} G_a(b)k_1(a,b)db \qquad (14)
\end{aligned}$$

where $v(b) := u(H(b))$ and we have introduced the functions

$$G_a(b) := \frac{v(b)-v(a)}{b-a} \quad \text{and} \quad k_1(a,b) := k(H(b)-H(a))H'(b)(b-a). \qquad (15)$$

For ease of notation, we sometimes omit the variable t in $v(a,t)$. The reason for the splitting at $b=a$ is because the integrand changes sign at this point. Note that each term in (14) has now the form of a *Volterra integral operator* of the first kind, as the variable a appears in the integration limit. The corresponding integral equations have been studied numerically, see for example [9]. For the discretization of these operators we use a direct quadrature method, see §3.4 below.

In the following result we state that G_a loses one degree of smoothness, as compared to the transforming function H itself.

Proposition 2. *The derivatives $G_a^{(k)}$, for $k=0,\dots,J-1$, admit a continuous extension at $b=a$.*

Proof. We prove that

$$\lim_{b\to a} G^{(k)}(b) = \frac{v^{(k+1)}(a)}{k+1}, \quad k=0,\dots,J-1. \qquad (16)$$

The proof is by induction. Assume we have proved (16) for $k \le j < J-1$. For $j=0$ this statement is evident. It is easy to prove (also by induction, differentiating on both sides of the identity $G(b)(b-a)=v(b)-v(a)$) that

$$G^{(j+1)}(b)(b-a)+(j+1)G^{(j)}(b) = v^{(j+1)}(b). \qquad (17)$$

By the induction hypothesis

$$G^{(j+1)}(b)(b-a)+(j+1)(G^{(j)}(b)-G^{(j)}(a)) = v^{(j+1)}(b)-v^{(j+1)}(a). \qquad (18)$$

Dividing by $b-a$ and letting $b\to a$ yields

$$(j+2)G^{(j+1)}(a) = v^{(j+2)}(a), \qquad (19)$$

which proves our claim for $k=j+1$. ☐

On the other hand, we would like to draw a similar conclusion about the kernel k_1. This new kernel is obviously smooth for $b \ne a$, and may be differentiated

$J-1$ times. However it is not smooth at the point $b=a$. To see this, let for instance $b>a$ ($b<a$ goes similarly) and consider the family of functions $H(t)$. Observe first that it is sufficient to examine the smoothness of the function

$$F_a(b) := \frac{H'(b)(b-a)}{H(b)-H(a)}. \tag{20}$$

It is straightforward to check that $F_a(b)$ is continuous for all $a \neq \kappa$, with $F_a(a^+) = 1$. Let us verify that $F_a(b)$ ($F(b)$ for short) is not differentiable at $b=a$. Differentiating the identity

$$F(b)(H(b)-H(a)) = H'(b)(b-a) \tag{21}$$

with respect to b and using $F(a)=1$, produces the second identity

$$F'(b)(H(b)-H(a)) + H'(b)(F(b)-F(a)) = H''(b)(b-a). \tag{22}$$

Dividing by $b-a$ and letting $b \to a$ yields

$$F'(a) = \frac{H''(a)}{2H'(a)}, \quad a \neq \kappa. \tag{23}$$

We deduce from this expression that $F'(a^+) = \dfrac{R-1}{2(a-\kappa)}$, thus the first derivative does not exist at $a=\kappa$. Note also that $F_\kappa(b)$ is well defined also for $b=\kappa$, and therefore $F_a(b)$ is continuous for all a as a function of b.

We conclude that each of the integrands in (14) with the transformation (13) is a product of a $C^{[R]-1}$ function ($[\cdot]$ denotes the integer part) and a continuous function. However, this lack of smoothness of the integrand is only at the end point $b=a$, while for points inside $(-\infty, a) \cup (a, \infty)$, the integrand may be differentiated $[R]-1$ times.

3.3 Truncating the integral

In this section we determine a bound for the error when truncating the integrals (14) to a finite domain. To this goal, we restrict the domain of variable x to a bounded interval of the form $\Omega^*_\rho = [B^*_0 + \rho,\ B^*_1 - \rho]$, for a certain $\rho > 0$, and consider the error

$$E(x) := \int_{\mathbb{R}/\Omega^*} (u(z)-u(x))k(z-x)dz, \tag{24}$$

where Ω^* is the enlarged interval $\Omega^* := [B^*_0,\ B^*_1]$. Our objective is to show that this error may be bounded uniformly for x belonging to Ω^*_ρ.

We carry out the reasoning for the positive semiaxis, (for the negative semiaxis the reasoning goes similarly), using the fact that the VG European option in log-prices satisfies the inequality

$$|u(z)-u(x)| \leq Ke^{-rt}\,|z-x|\,, \quad \forall z, x \in \mathbb{R}. \tag{25}$$

This is a consequence of the mean value theorem combined with the fact that the absolute value of the Delta is bounded everywhere by Ke^{-rt}; see the appendix for a proof. This yields

$$
\begin{aligned}
|E(x)| &\leq C \int_{B_1^*}^{+\infty} \frac{|u(z)-u(x)|}{|z-x|} e^{-M(z-x)} dz \\
&\leq K e^{-rt} C e^{Mx} \int_{B_1^*}^{\infty} e^{-Mz} dz \\
&= K e^{-rt} C \frac{e^{M(x-B_1^*)}}{M} \\
&\leq K e^{-rt} C \frac{e^{-M\rho}}{M}, \quad \forall x \in \Omega_\rho^*.
\end{aligned}
$$

That is, we find a bound that holds uniformly in the x variable, and that depends on the parameter ρ: the larger ρ, the better the approximation by truncation. However, a better estimate may be obtained in the new variable a that exploits the fact that the transformed integrand is more 'localized'. Define the interval $\Omega_\delta := [B_0+\delta, B_1-\delta]$, for a given number $\delta > 0$ and write $x = H(a)$ and $B_1^* = H(B_1)$. By the mean value theorem, one obtains the following lower bound

$$
B_1^* - x = H(B_1) - H(a) \geq H(B_1) - H(B_1-\delta) \geq H'(B_1-\delta)\delta. \tag{26}
$$

provided H' is increasing (which is the case for H given in (13)). We arrive at the error estimate

$$
|E(a)| \leq K e^{-rt} C \frac{e^{-MH'(B_1-\delta)\delta}}{M}. \tag{27}
$$

Observe that this upper bound decreases rapidly for values of R between 2 and 4, which we typically use in our computations.

We may now summarize the criterion. In order to determine the truncation interval Ω_δ, one decides first a certain margin δ and a tolerance ϵ. Thereafter, one computes the integration limits B_0 and B_1 by solving the following two inequalities

$$
\begin{cases}
K e^{-rt} C \frac{e^{-MH'(B_1-\delta)\delta}}{M} < \epsilon, \\
\\
K e^{-rt} C \frac{e^{-GH'(B_0+\delta)\delta}}{G} < \epsilon.
\end{cases} \tag{28}
$$

3.4 Simpson's rule

We dedicate this part to explain how Simpson's rule may be applied to obtain a highly accurate evaluation of the integrals in (14). Let $f(x)$ be a smooth function in the interval $[x_0, x_{2n}]$ and define the nodes $x_i = x_0 + ih$, for $i = 0, 1, \ldots, 2n$ and mesh width $h = (x_{2n} - x_0)/2n$. The compound Simpson's rule states that

$$
\begin{aligned}
\int_{x_0}^{x_{2n}} f(x)dx =& \frac{h}{3}[f_0 + 4(f_1 + f_3 + \cdots + f_{2n-1}) \\
&+ 2(f_2 + f_4 + \cdots + f_{2n-2}) + f_{2n}] + E_n
\end{aligned} \tag{29}
$$

and the error term $E_n = O(h^4)$, provided $f \in C^4[x_0, x_{2n}]$; see for example [15]. Note that an odd number of points $x_0, \ldots, x_{2n}$ is necessary to use this rule.

We now apply Simpson's rule to (14). Let $a_{\min} := B_0$ and $a_{\max} := B_1$ according to the notation in §3.3. Next, consider a uniform grid for the variable a as follows: let N be some integer and define the nodes $a_i = a_{\min} + (i-2)h$ ($i = 1, \ldots, N+2$) where $h = (a_{\max} - a_{\min})/N$. In particular this means $a_1 = a_{\min} - h$ and $a_2 = a_{\min}$. Once we know the sampling points a_i, we approximate the truncated integrals

$$J_i = \int_{a_{\min}}^{a_i} \frac{v(b) - v(a_i)}{b - a_i} k_1(a_i, b) db. \tag{30}$$

We use for the integration variable b the same grid spacing, namely, $b_j = a_{\min} + jh$ ($j = 0, \ldots$). Hence, to be able to apply Simpson's rule, one has to choose even values of the index i in (30), since the number of nodes b_j in $[a_{\min}, a_i]$ has to be odd. In other words, we cannot approximate (30) for i odd. A remedy to fill these gaps is to apply Simpson's rule on the integrals

$$J_i = \int_{a_{\min}-h}^{a_i} \frac{v(b) - v(a_i)}{b - a_i} k_1(a_i, b) db, \quad i = 1, 3, \ldots. \tag{31}$$

Another strategy could be to use the composite Simpson's $3/8$ rule for i odd, see [9]. Special care should be taken at the end point $b = a_i$, for each i, since there is an avoidable singularity. To deal with this situation numerically, a second-order one-sided discretization has been chosen.

The integrand is a continuous function in the closed interval $[a_{\min}, a_i]$ for the family of functions (13); see Proposition 1 and the comment that follows. In the open interval $(a_{\min}, a_i)$ this integrand belongs to $C^{[R]-1}$, where $[R]$ is the degree of smoothness of the change of variable H. This is a crucial observation that confirms the accurate results in the next section.

An important remark is that the method outlined here is computationally intensive since it requires $O(N^2)$ operations to compute all the integrals J_i. It is the subject of current research to increase the speed of evaluation of these integral transforms, for which methods like Multilevel integration [8] or Hierarchical matrices [17] may be suitable.

3.5 Time integration

We evaluate the explicit BDF2 method to solve (8) and (9), with initial condition (2). It is observed that the equation is not stiff and therefore an explicit method is appropriate.

First, we rewrite (8) in terms of the new function $v(a,t) = u(H(a), t)$:

$$v_t + rv = \int_{-\infty}^{\infty} \frac{v(b,t) - v(a,t)}{b - a} k_1(a, b) db. \tag{32}$$

We denote now by $\mathcal{J}(v)$ the integral operator on the right hand side of (32). The approximation of the integral operator described in §3.4 is denoted by $J(v)$. Let $v_n^j \approx v(a_j, t_n)$ be an approximation to the solution at times $t_n = nk$ ($n = 0, \ldots, L$) and spatial points a_j. By v_n we mean the vector of spatial unknowns (v_n^j). Likewise, a stands for the vector (a_j).

For the European option under the VG process, the explicit BDF2 method reads

$$\frac{3}{2} v_n - 2v_{n-1} + \frac{1}{2} v_{n-2} + krv_n = kJ(\bar{v}_n), \quad \text{where} \quad \bar{v}_n = 2v_{n-1} - v_{n-2}. \tag{33}$$

This leads to a simple iteration of the form $v_n = \hat{J}(v_{n-1}, v_{n-2})$. The first input to this iteration is the payoff v_0 and the second, v_1, is the result of one explicit Euler iteration. Note that the update $\bar{v}_n$ is obtained by linear extrapolation.

The discrete solution to the American case is also straightforward, since for BDF2 the corresponding discrete LCP may be written as

$$\begin{cases} v_n \geq \hat{J}(v_{n-1}, v_{n-2}) \\ v_n \geq \psi(H(a) + \alpha t_n) \\ (v_n - \hat{J}(v_{n-1}, v_{n-2}), v_n - \psi(H(a) + \alpha t_n)) = 0 \\ v_0 = \psi(H(a)). \end{cases} \tag{34}$$

Here $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^d$. The solution of this problem is simply $v_n = \max(\hat{J}(v_{n-1}, v_{n-2}), \psi(H(a) + \alpha t_n))$.

3.6 Computation of the Greeks

In the Black-Scholes model the Greeks are important to determine a hedging portfolio. However, in the presence of jumps their importance is not as clear as in general a hedging portfolio does not exists. This paragraph is included mainly to illustrate the scope of the stretching method. We show that even with a discontinuity at the initial condition, it is possible to compute accurate first and second derivatives.

The Greeks may be obtained by numerically differentiating the solution w. However, this may produce oscillations if differentiation is applied without care. The remedy proposed in this section is to numerically solve an Integro-Differential equation for the derivative, on which we may apply the same numerical procedure.

If one formally differentiates equation (1) with respect to x, and defines the derivative in log-prices as $\bar{w} := w_x$, one arrives at the same equation, namely

$$\bar{w}_t + (q - r - \omega)\bar{w}_x + r\bar{w} = \int_{\mathbb{R}} (\bar{w}(x + y, t) - \bar{w}(x, t))k(y)dy, \tag{35}$$

but with initial condition

$$\bar{w}(x, 0) = -\exp(x)\mathbf{1}_{\{x<\log K\}}. \tag{36}$$

Here $\mathbf{1}_A(x)$ denotes the indicator function of the set A. All the ideas explained so far now apply to solve (35)-(36). When it comes to the second derivative, it is more convenient to numerically differentiate $\bar{w}$, rather than numerically differentiating w twice.

4 Numerical experiments

4.1 A first numerical test

A first numerical test is presented that illustrates the numerical improvements after the non-linear change of variables. We evaluate the following integral

$$\tau(x) = \int_{-\infty}^{+\infty} (\psi(x + y) - \psi(x))k(y)dy, \tag{37}$$

where $\psi(x) = (1 - e^x)^+$ and the constants defining the kernel $k(y)$ (cf. (4)) are chosen to be: $C = G = M = 1$. The analytic expression for $\tau(x)$ is given in the appendix. In Figure 2 we show the effect of transformation (13) with $R = 3.7$ on the analytic solution. Notice that the kink in the analytic solution has been smoothed out as well.

The results in Table 1 show the application of the compound Simpson's rule as described in §3.4. We observe a high rate of convergence as expected from the application of Simpson's rule on a smooth integrand, as compared with the results from a uniform grid ($R = 1$) that only produce linear convergence.

	$R = 3.7$		$R = 1$	
N	ℓ_∞-error	rate	ℓ_∞-error	rate
10	0.097223		0.3742	
20	0.004883	19.9	0.2060	1.8
40	0.000445	10.9	0.1091	1.8
80	0.000041	10.7	0.0573	1.9

Table 1: Stretching vs. uniform grid

The choice $R = 3.7$ is merely to show that it is possible to have a fractional stretching factor. For values of R between 2 and 4 one has a high order of convergence, and no significant differences are observed. By a trial and error procedure it is also possible to find some optimal value for this particular test problem.

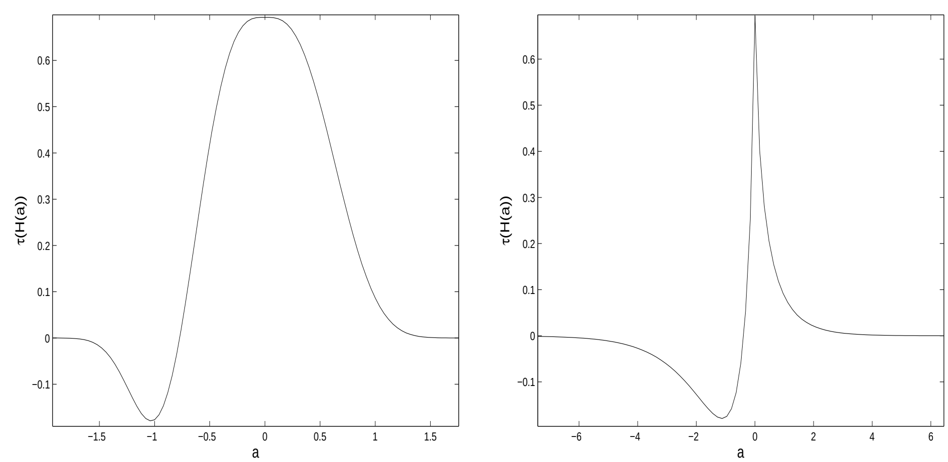

Figure 2: Left picture: Transformed solution $\tau(H(a))$ with $R = 3.7$. Right picture: Analytic solution without transformation, *i.e.* $R = 1$.

4.2 European and American option computation

We study first the order of convergence of the explicit BDF2 method, with the following parameters

$$K = 1, \quad r = 0.1, \quad q = 0, \quad C = 1, \quad G = 5, \quad M = 5, \quad T = 0.5. \tag{38}$$

The effect of the stretching factor is tested by comparing $R = 2$ with $R = 3$. For this set of parameters, criterion (28) for the truncation of the integral, with $\delta = 0.2$ and a tolerance of $\epsilon = 10^{-8}$, gives $-B_0 = B_1 \approx 2.5$. We compute the solution $u(z, T)$ of equation (8) (by solving on the stretched grid with (32)), and later apply the shift (7) to recover the solution, i.e., $w(x, T) = u(x - \alpha T, T)$. An interpolation is needed at this stage as the grid points $x_i - \alpha T$ will not necessarily coincide with points in the x-grid. Cubic splines are used to interpolate to this new grid. We present in tables 2 and 3 the errors and the empirical rates of convergence in the infinity norm, with truncation to the interval $B_0 = -2.5$, $B_1 = 2.5$. The composite Simpson's rule as in §3.4 is compared with the well-known composite Trapezoidal rule. The notation w^{ref} indicates the numerical solution computed with a large number of spatial points.

Remark 2. *The popular Fourier method to compute European prices requires tuning a certain parameter β. That is, if we define $x_0 = \log(S_0)$ and $S_t = e^{x_0 + L_t}$, the put price may be found with the aid of the formula [20]:*

$$C(x_0, \kappa, T) = \frac{e^{-rT + \beta x_0}}{2\pi} \int_{-\infty}^{\infty} e^{-iyx_0} e^{T\Phi(-z)} \hat{\psi}(z) dy, \tag{39}$$

where

$$z = y + \beta i, \qquad e^{T\Phi(z)} = E[e^{izL_T}], \quad \textit{and} \quad \hat{\psi}(z) = -\frac{K^{iz+1}}{z^2 - iz}.$$

It is not trivial which choice of β will produce a desired level of accuracy; see [19, 20, 27] for possible choices. The PIDE method with the stretching reaches a high level of accuracy with only a few grid points, making this method an alternative for pricing European options.

The numerical results for this example are displayed in Figure 3. The analytic solution in Figure 3 used for benchmarking was obtained by approximating formula (39) using an adaptive Simpson quadrature. It turns out that the analytic solution is prone to instabilities (visible on the derivatives) for small and large values of the asset, contrarily to the numerical solution in which one does not observe oscillations.

From tables 2 and 3, Simpson's rule is clearly preferable over the Trapezoidal rule, as already on coarse grids one digit in accuracy is gained. With moderate stretching ($R = 2$), we confirm BDF2's second order convergence in time. With a more severe stretching ($R = 3$) a higher order convergence is observed. By the combination of grid stretching ($R = 3$), explicit BDF2 and Simpson's rule, the tolerance set to 10^{-8} is reached already with 160×80 points. This latter combination is taken to evaluate an American option.

It is shown in Figure 5 that the stretching also works for a discontinuous initial condition. We use the parameters from (38), with $R = 3$, and, in addition, $q = r + \omega$ (i.e., $\alpha = 0$, generating a kink at $s = 1$). In this example the delta of a European put option (in log-prices) has been computed. This graph was generated by numerically differentiating the solution of problem (35)-(36), as explained in §3.6.

In Table 4 we display the pointwise rates for an *American put* option with parameters (38), $R = 3$ and explicit BDF2. With the chosen parameters we encounter the case where the "smooth pasting" condition is not valid (see e.g.

N	L	Trapezoidal	rate	Simpson	rate
20	10	0.027		0.0044	
40	20	0.014	2.0	0.0014	3.1
80	40	0.0038	3.6	0.00028	5.1
160	80	0.0011	3.5	0.000072	3.9

Table 2: Uniform errors $\|w^N - w^{ref}\|_\infty$ for the Trapezoidal rule and Simpson's rule, European case, $R = 2$.

N	L	Trapezoidal	rate	Simpson	rate
20	10	0.045		0.0072	
40	20	0.016	2.7	0.0011	6.3
80	40	0.0057	2.9	0.00011	10.0
160	80	0.0018	3.2	0.0000084	13.1

Table 3: Uniform errors $\|w^N - w^{ref}\|_\infty$ for the Trapezoidal rule and Simpson's rule, European case, $R = 3$.

[2, 3, 5]). The reference value used for comparison was obtained with $N = 1000$ points. As we may observe, the rate of convergence is still high, but the algorithm is more demanding on the time restriction. Therefore, we choose $N = L$ here, whereas $L = N/2$ could be chosen for the European case. The loss in accuracy with respect to the European counterpart is due to the lack of smooth fit, see Figure 4.

Remark 3. *The PIDE approach requires the knowledge of the Lévy measure. The explicit method suggested here gives however the freedom to choose Lévy measures coming from the market, as soon as the underlying follows a finite activity process.*

Remark 4. *This method does not however determine the free boundary accurately on coarse grids, given the sparsity of the mesh points around the free boundary. Thus, a large amount of spatial points are required, rendering this method rather slow. Again, improved fast integration methods [8, 17] will enable using a large number of points.*

N	L	error at $x = 0$	rate
20	20	7.060e-03	
40	40	7.129e-04	9.9
80	80	1.437e-04	4.9
160	160	1.295e-05	11.0
value at $x = 0$		0.044865	

Table 4: Pointwise convergence for BDF2, American case, $a_{\max} = -a_{min} = 3$.

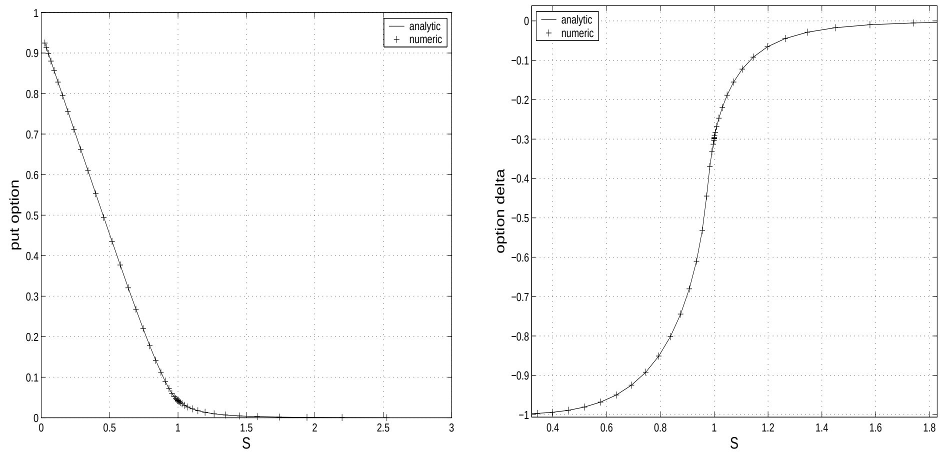

Figure 3: Left picture: European put option on the stretched grid and analytic solution. Right picture: Option Delta and numerical derivative of analytic solution.

5 Conclusions

We have computed European and American vanilla option prices when the underlying process is of VG type. It has been shown that working with a refined grid around the kink $x = \kappa$ (or discontinuity) of the payoff function helps increase the accuracy of the numerical integration, and in particular a higher precision of the option price is reached with only a few points on the grid. The method is easy to implement and can be extended to other types of infinite-activity, finite-variation processes.

A Estimate for the option Delta in log-prices

In §3.3 we made use of the following estimate for the Delta in log-prices

$$\left| \frac{\partial w}{\partial x}(x,t) \right| < K e^{-rt}. \tag{40}$$

In fact, we used this bound on the shifted function u instead, but we derive it here for the original function w. Recall that we are working with a put option under the VG process.

The solution w of (1) has the following stochastic interpretation

$$w(x,t) = e^{-rt} E_Q[\psi(x + L_{T-t})], \tag{41}$$

where $\{L_t\}_{t\geq 0}$ is the Lévy process defined in terms of the VG process $\{X_t\}_{t\geq 0}$, i.e., $L_t := -\alpha t + X_t$. The expectation has been taken with respect to a risk-neutral measure. Denoting by $q(y)$ the probability density function of the vari-

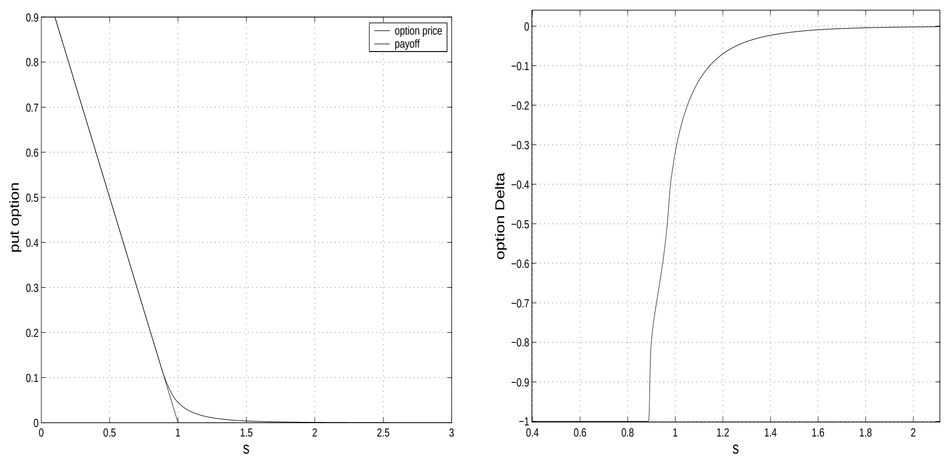

Figure 4: American put option on a uniform grid and option Delta. The non-smooth fit situation is clearly observed in the Delta.

able L_{T-t}, we may write

$$w(x,t) = e^{-rt}\int_{-\infty}^{\log K-x}(K-e^{x+y})q(y)dy \tag{42}$$

$$= e^{-rt}\{K\int_{-\infty}^{\log K-x}q(y)dy - e^{x}\int_{-\infty}^{\log K-x}e^{y}q(y)dy\}. \tag{43}$$

Differentiation with respect to x on the last equality gives

$$\frac{\partial w}{\partial x} = e^{-rt}\{-Kq(\log K-x)- \\ [e^{x}\int_{-\infty}^{\log K-x}e^{y}q(y)dy - e^{x}e^{\log K-x}q(\log K-x)]\}. \tag{44}$$

This expression simplifies into

$$\frac{\partial w}{\partial x} = -e^{-rt}e^{x}\int_{-\infty}^{\log K-x}e^{y}q(y)dy. \tag{45}$$

The upper bound (40) now follows since e^{y} is less than Ke^{-x} on the interval $(-\infty, \log K - x)$, and the integral of $q(y)$ over this interval is less than one.

B Analytical formula for a particular case

The aim of this short appendix is to provide the derivation of the analytic expression used to compare the numerical results from §4.

Let the parameters G, M and C be equal 1, i.e., $k(y) = e^{-|y|}/\,|y|$. Recall that the payoff function for $K = 1$ is by definition $\psi(x) = (1-e^{x})^{+}$.

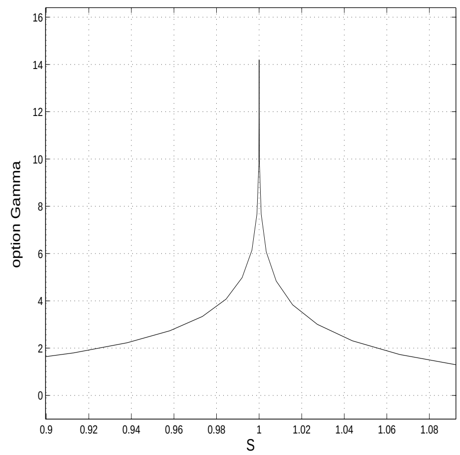

Figure 5: Option Gamma for $q = r + \omega$.

Proposition 3.

$$\tau(x) = \int_{-\infty}^{+\infty} (\psi(x+y) - \psi(x))k(y)dy = \begin{cases} -E_1(-x) - e^x(\gamma + \log(x/2)) & \textit{if } x < 0 \\ \log(2) & \textit{if } x = 0 \\ E_1(x) - e^x E_1(2x) & \textit{if } x > 0 \end{cases} \tag{46}$$

where $\gamma = 0.577216...$ *is the Euler constant and* E_1 *is the special function*

$$E_1(x) = \int_x^{\infty} \frac{e^{-t}}{t} dt.$$

Proof. Start by defining the following function

$$\tau^+(x) := \int_0^{\infty} (\psi(x+y) - \psi(x)) \frac{e^{-y}}{y} dy.$$

Observe that $\tau^+(x) = 0$ for $x \geq 0$. For $x < 0$ we have

$$\begin{aligned}
\tau^+(x) &= \int_0^{-x} \left[(1 - e^{x+y}) - (1 - e^x) \frac{e^{-y}}{y} dy \right] + \int_{-x}^{\infty} -(1 - e^x) \frac{e^{-y}}{y} dy, \\
&= e^x \left[\int_0^{-x} \frac{e^{-y} - 1}{y} dy \right] + (e^x - 1) E_1(-x) \\
&= \lim_{\varepsilon \to 0} e^x \left[E_1(\varepsilon) - E_1(-x) - (\log(-x) - \log(\varepsilon)) \right] + (e^x - 1) E_1(-x) \\
&= \lim_{\varepsilon \to 0} e^x \left[E_1(\varepsilon) + \log(\varepsilon) \right] - E_1(-x) - e^x \log(-x) \\
&= -e^x (\gamma + \log(-x)) - E_1(-x).
\end{aligned}$$

We made use here of the asymptotic formula $E_1(x) = -\gamma - \log x + o(x)$, see [1] Chapter 5.

Analogously, for the negative semiaxis one defines

$$\tau^-(x) := \int_{-\infty}^{0} (\psi(x+y) - \psi(x))k(y)dy = \int_0^{\infty} (\psi(x-y) - \psi(x)) \frac{e^{-y}}{y} dy.$$

For $x \leq 0$ one finds

$$\begin{aligned}
\tau^-(x) &= e^x \int_0^\infty (1 - e^{-y}) \frac{e^{-y}}{y} dy \\
&= e^x \lim_{\varepsilon \to 0} \left[\int_\varepsilon^\infty - \int_\varepsilon^\infty \right] \\
&= e^x \lim_{\varepsilon \to 0} [E_1(\varepsilon) - E_1(2\varepsilon)] \\
&= e^x \log(2).
\end{aligned}$$

And for $x \geq 0$, $\psi(x)$ vanishes, so that

$$\begin{aligned}
\tau^-(x) &= \int_x^\infty \frac{e^{-y}}{y} dy - e^x \int_x^\infty \frac{e^{-2y}}{y} dy \\
&= E_1(x) - e^x E_1(2x).
\end{aligned}$$

Now the analytic expression (46) follows from adding τ^+ and τ^-. $\square$

References

[1] M. Abramowitz and I. A. Stegun, eds., *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications Inc., New York, 1992. Reprint of the 1972 edition.

[2] L. Alili and A. E. Kyprianou, *Some remarks on first passage of Lévy processes, the American put and pasting principles.* Submitted, 2004.

[3] A. Almendral, *Numerical valuation of American options under the CGMY process*, in Exotic Option Pricing under Advanced Lévy Models, W. S. Andreas Kyprianou and P. Wilmott, eds., 2004.

[4] A. Almendral and C. W. Oosterlee, *Numerical valuation of options with jumps in the underlying*, tech. rep., Delft University of Technology, 2003. To appear in App. Num. Math.

[5] ——, *On American options under the Variance Gamma process*, tech. rep., Delft University of Technology, 2004.

[6] L. Andersen and J. Andreasen, *Jump-diffusion processes: Volatility smile fitting and numerical methods for option pricing*, Review of Derivatives Research, 4 (2000), pp. 231–262.

[7] K. E. Atkinson, *The numerical solution of integral equations of the second kind*, vol. 4 of Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, Cambridge, 1997.

[8] A. Brandt and C. H. Venner, *Multilevel evaluation of integral transforms with asymptotically smooth kernels*, SIAM J. Sci. Comput., 19 (1998), pp. 468–492.

[9] H. Brunner and P. J. van der Houwen, *The numerical solution of Volterra equations*, vol. 3 of CWI Monographs, North-Holland Publishing Co., Amsterdam, 1986.

[10] P. P. Carr, H. Geman, D. B. Madan, and M. Yor, *The fine structure of asset returns: An empirical investigation*, Journal of Business, 75 (2002), pp. 305–332.

[11] P. P. Carr, D. B. Madan, and E. C. Chang, *The Variance Gamma process and option pricing*, European Finance Review, 2 (1998), pp. 79–105.

[12] N. Clarke and K. Parrot, *Multigrid for American option pricing with stochastic volatility*, Appl. math. finance, 6 (1999), pp. 177–179.

[13] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall, Boca Raton, FL, 2004.

[14] R. Cont and E. Voltchkova, *A finite difference scheme for option pricing in jump diffusion and exponential Lévy models*, Tech. Rep. 513, CMAP, Palaiseau, France, 2003.

[15] P. J. Davis and P. Rabinowitz, *Methods of numerical integration*, Computer Science and Applied Mathematics, Academic Press Inc., Orlando, FL, second ed., 1984.

[16] Y. d'Halluin, P. A. Forsyth, and K. R. Vetzal, *Robust numerical methods for contigent claims under jump diffusion processes.* Submitted, 2004.

[17] W. Hackbusch, *A sparse matrix arithmetic based on $\mathcal{H}$-matrices. I. Introduction to $\mathcal{H}$-matrices*, Computing, 62 (1999), pp. 89–108.

[18] A. Hirsa and D. B. Madan, *Pricing American options under Variance Gamma*, Journal of Computational Finance, 7 (2004).

[19] R. Lee, *Option pricing by transform methods: Extensions, unification, and error control*, Journal of Computational Finance, 7 (2004), pp. 51–86.

[20] A. L. Lewis, *A simple option formula for general jump-diffusion and other exponential Lévy processes.* 8th. Annual CAP Workshop on Derivative Securities and Risk Management, November 2001.

[21] D. B. Madan and E. Seneta, *The variance gamma (V.G.) model for share market returns*, Journal of Business, 63 (1990), pp. 511–524.

[22] A. M. Matache, P. A. Nitsche, and C. Schwab, *Wavelet Galerkin pricing of American options on Lévy driven assets*, working paper, ETH, Zürich, 2003.

[23] R. C. Merton, *Option pricing when the underlying stocks are discontinuous*, J. Financ. Econ., 5 (1976), pp. 125–144.

[24] G. Monegato and L. Scuderi, *High order methods for weakly singular integral equations with nonsmooth input functions*, Math. Comp., 67 (1998), pp. 1493–1515.

[25] C. W. Oosterlee, C. C. Leentvaar, and A. Almendral, *Pricing options with discrete dividends by high order finite differences and grid stretching*, in ECCOMAS 2004, P. Neittaanmäki, T. Rossi, S. Korotov, E. Oñate, J. Périaux, and D. Knörzer, eds. Proc. Jyväskylä, Finland, 24–28 July 2004.

[26] W. Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, John Wiley & Sons Ltd., 2003.

[27] W. Schoutens, E. Simons, and J. Tistaert, *A perfect calibration ! now what ?*, tech. rep., K.U.Leuven, 2003.

[28] D. Tavella and C. Randall, *Pricing financial instruments, the finite difference method*, Wiley, New York,, 2000.