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Local mode analysis – an introduction in smoothing analysis and two-level analysis

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(Original lecture by J. Linden, GMD)

The most general tool for analyzing a multigrid process is the so–called "local mode analysis" or "local Fourier analysis". This type of analysis, introduced by A. Brandt, has been used right from the very early days of multigrid development. Though it is based on certain assumptions and simplifications, it has proved to be a very useful tool for developing, analyzing and debugging multigrid algorithms and programs and it helps to understand what multigrid really is and why it works.
Furthermore, in many typical situations it also yields quantitatively precise predictions of multigrid performance.

In its simplest form, the local mode analysis is used to analyze quantitatively the smoothing properties of relaxation schemes. This smoothing analysis (or one–level analysis) allows judgement to be made as to whether or not a given relaxation scheme is suitable as smoothing procedure within a multigrid algorithm.

The first part of this lecture gives an introduction to smoothing analysis.

The basic idea of any multigrid algorithm is to annihilate the high frequency error components efficiently on a fine grid by relaxation (smoothing) while the low frequencies are approximated from coarser grids (with considerably less work than on the fine grid). Smoothing analysis is used to analyze quantitatively this reduction of the high frequency error components on the fine grid. Assuming the ideal situation in which

- the relaxation does not affect the low frequencies
- all low frequencies are approximated more or less exactly on the coarse grid
- there is no interaction between high and low frequency error components

it already leads to an "estimate" or prediction of the 2–level convergence factor.

The local mode analysis can, however, also be used to analyze a **complete 2–level multigrid algorithm** more precisely. In this so–called 2–level analysis not only are the smoothing properties of the relaxation scheme on the fine grid considered but also the transfers between fine and coarse grid, the coarse grid operator and the coupling between high and low frequencies.

The second part of this lecture gives an introduction into this 2–level local mode analysis.

Local mode analysis – an introduction

Contents:

Smoothing analysis:

- Fourier components $e^{i\theta x/h}$
- Difference operators and Fourier symbols
- Smoothing analysis for scalar equations
  - relaxations based on regular splitting
    - Examples: Gauss–Seidel relaxation
      for the Poisson operator,
      for the anisotropic operator,
      point– and line–relaxations
- Prediction of multigrid performance
- Extension to systems of equations

Two-level analysis:

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- The two–level operator
- Fourier components, harmonics
- The two–level convergence factor
- Example: Poisson's equation
- Simplified two–level reasonings: an example
  The convection–diffusion operator and the role of
  characteristic low frequencies

In this lecture, the local mode analysis is introduced

- For a discretized PDE:

$$L^h U^h = F^h$$

With the basic assumptions of any local mode analysis:

– linearize operator L^h and/or freeze coefficients to local values

– neglect boundaries and boundary conditions

– consider the frozen, linearized operator on an infinite grid G^h

– apply the linear constant coefficient local mode analysis $\Longrightarrow$ **this lecture**

The 1D case:

infinite uniform mesh, meshsize h

$$G^h = \{k \cdot h : k \in \mathbb{Z}\}$$

1D Fourier components: $e^{i\omega x} = \cos \omega x + i \sin \omega x$

– On G^h only Fourier components $e^{i\theta x/h}$ with

$$\theta \in (-\pi, \pi]$$

are **"visible"**.

– **Definition:** A component $e^{i\theta x/h}$ is **"visible"** on G^h if there is no frequency θ_0 with $|\theta_0| < |\theta|$ such that

$$e^{i\theta_0 x/h} = e^{i\theta x/h} \text{ for all } x \in G^h.$$

i.e. A visible component $e^{i\theta x/h}$ does not coincide with any lower frequency on grid G^h.

– On G^h only components $e^{i\theta x/h}$ with $\theta \in (-\pi, \pi]$ must be considered.

– θ "frequency", $|\frac{2\pi h}{\theta}|$ "period"

– $\theta = 0$: constant component

$\theta = \pi$: +/- 1 oscillations on the gridpoints, the highest oscillating component.

The 2D case:

infinite uniform mesh, meshsize $h = (h_x, h_y)$ in x– and y–direction:

$$G^h = \{(kh_x, lh_y) : k, l \in \mathbb{Z}\}$$

– $\theta = (\theta_1, \theta_2)$ "frequency"

– On G^h, only Fourier components

$$\boxed{e^{i\theta\underline{x}/h} := e^{i\theta_1 x/h_x} \cdot e^{i\theta_2 y/h_y}}$$

with

$$\boxed{|\theta| := \max\left(|\theta_1|, |\theta_2|\right) \leq \pi}$$

are **"visible"**.

– For higher frequencies $|\theta| > \pi$, the Fourier components coincide with a lower frequency $\in (-\pi, \pi]$.

For higher dimensions: analogously:

– $\theta = (\theta_1, ..., \theta_d)\,, \quad \underline{x} = (x_1, ..., x_d)\,, \quad h = (h_1, ..., h_d)$

– $e^{i\theta\underline{x}/h} := e^{i\theta_1 x_1/h_1} \cdot ..... \cdot e^{i\theta_d x_d/h_d}$

– $|\theta| := \max\left(|\theta_1|, ..., |\theta_d|\right) \leq \pi$ for the "visible" components on the infinite grid:

$$G^h = \{(k_1 h_1, ..., k_d h_d) : k_i \in \mathbb{Z}\}$$

Scalar difference operators with constant coefficients

– Let G^h be an infinite uniform mesh, meshsize h ;

– u^h: grid function on G^h ;

– $I = \{-\nu, ..., 0, ..., \nu\}$: set of indices.

Difference operators L^h:

$\boxed{\textbf{2D}}$ $\quad \underline{x} = (x, y) \in G^h$

$$L^h u^h(\underline{x}) = \sum_{\mu_1 \in I} \sum_{\mu_2 \in I} a_{\mu_1 \mu_2} u^h(x + \mu_1 h_x, y + \mu_2 h_y)$$

with coefficients $a_\mu, a_{\mu_1 \mu_2} \in R$.

The stencil notation, examples:

$\boxed{\textbf{2D}}$ finite differences for Δu, $\quad I = \{-1, 0, 1\}$

$$L^h u^h(\underline{x}) = \begin{bmatrix} a_{-11} & a_{01} & a_{11} \\ a_{-10} & a_{00} & a_{10} \\ a_{-1-1} & a_{0-1} & a_{1-1} \end{bmatrix} u^h(\underline{x})$$

$$= \underbrace{\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}}_{\Delta^h} u^h(\underline{x}) \quad 5-\text{point} \;\; \text{stencil}$$

or for the **biharmonic operator** Δ^2, $\quad I = \{-2, -1, 0, 1, 2\}$

$$\Delta^h \Delta^h u^h(\underline{x}) = \frac{1}{h^4}\begin{bmatrix} & & 1 & & \\ & 2 & -8 & 2 & \\ 1 & -8 & 20 & -8 & 1 \\ & 2 & -8 & 2 & \\ & & 1 & & \end{bmatrix} u^h(\underline{x})$$

13-point stencil

Here: $\underline{x} = (x, y) \in G^h$, $\quad h_x = h_y = h$.

Difference operators applied to Fourier components

– On an infinite grid G^h, the Fourier components

$$e^{i\theta \underline{x}/h}$$

are **eigenfunctions** of any scalar difference operator L^h with constant coefficients.

$\boxed{\textbf{1D:}}$

$$L^h e^{i\theta x/h} = \underbrace{\left(\sum_{\mu \in I} a_\mu e^{i\theta\mu}\right)}_{\widetilde{L}^h(\theta)} \cdot e^{i\theta x/h}$$

$\boxed{\textbf{2D:}}$

$$L^h u^h e^{i\theta \underline{x}/h} = \underbrace{\left(\sum_{\mu_1 \in I} \sum_{\mu_2 \in I} a_{\mu_1\mu_2} e^{i\theta_1\mu_1} e^{i\theta_2\mu_2}\right)}_{\widetilde{L}^h(\theta)} \cdot e^{i\theta \underline{x}/h}$$

– $\widetilde{L}^h(\theta)$ is the **eigenvalue**, sometimes also called the **Fourier symbol** of L^h.

Examples:

$\boxed{\textbf{1D:}}$ $L^h = \frac{1}{h^2}[1 \quad -2 \quad 1]$

$$\Longrightarrow \quad L^h e^{i\theta x/h} = \frac{1}{h^2}\left(e^{-i\theta} - 2 + e^{i\theta}\right) \cdot e^{i\theta x/h}$$

$$= \underbrace{\frac{2}{h^2}(\cos\theta - 1)}_{\widetilde{L}^h(\theta)} \cdot e^{i\theta x/h}$$

$\boxed{\textbf{2D:}}$ $\theta = (\theta_1, \theta_2)$

$$L^h = \Delta^h = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}$$

$$\Longrightarrow \quad \widetilde{L}^h(\theta) = \frac{2}{h^2}(\cos\theta_1 + \cos\theta_2 - 2)$$

or for the **biharmonic operator** $L^h = \Delta^h \Delta^h$ (13-point stencil):

$$\widetilde{L}^h(\theta) = \frac{4}{h^4}(\cos\theta_1 + \cos\theta_2 - 2)^2$$

Smoothing analysis; a general class of relaxations:

– Consider a linear problem

$$L^h U^h(x) = F^h \quad \text{on grid} \quad G^h$$

where L^h is a difference operator.

– **Then:** Many (not all !) relaxation schemes for this problem are based on an operator splitting

$$L^h = A^h + B^h$$

where A^h and B^h are again difference operators.

– **Then:** A relaxation sweep starting from the initial approximation u^h ("old values") produces a new approximation $\overline{u}^h$ ("new values") by solving

$$\boxed{A^h u^h(\underline{x}) + B^h \overline{u}^h(\underline{x}) = F^h(x)}$$

at each gridpoint $\underline{x} \in G^h$.

– What does that mean for the errors before and after the relaxation sweep ?

– Let $v^h = U^h - u^h$ be the error **before**
and $\overline{v}^h = U^h - \overline{u}^h$ **after** the sweep.
Then:

$$\boxed{A^h v^h(\underline{x}) + B^h \overline{v}^h(\underline{x}) = 0}$$

at each gridpoint $\underline{x} \in G^h$.

How do these relaxations act on Fourier components ?

$$\begin{array}{llcll} - & \text{Let} \quad v^h & = & \mathcal{A}e^{i\theta x/h}: & \text{error } \textbf{before} \text{ relaxation;} \\ & \text{then} \quad \overline{v}^h(x) & = & \overline{\mathcal{A}}e^{i\theta x/h} & \text{error } \textbf{after} \text{ relaxation.} \end{array}$$

– $\mathcal{A}, \overline{\mathcal{A}} \in R$ are the error amplitudes before and after relaxation.

From

$$\boxed{A^h v^h(\underline{x}) + B^h \overline{v}^h(\underline{x}) = 0}$$

follows

$$\widetilde{A}^h(\theta)\mathcal{A} + \widetilde{B}^h(\theta)\overline{\mathcal{A}} = 0$$

and therefore

$$\overline{\mathcal{A}} \;=\; -\left(\frac{\widetilde{A}^h(\theta)}{\widetilde{B}^h(\theta)}\right)\mathcal{A}$$

where $\widetilde{A}^h(\theta), \widetilde{B}^h(\theta) \in C$ are the Fourier symbols of the difference operators A^h, B^h.

$$\mu(\theta) := \left|\frac{\widetilde{A}^h(\theta)}{\widetilde{B}^h(\theta)}\right|$$

is the **amplification factor** (or reduction factor) of the component θ.

- **What is smoothing ?**

- **How can the smoothing property of a given relaxation scheme be measured quantitatively ?**

- **How does the smoothing property (or its quantitative measure) influence the performance of multigrid cycling ?**

- **Can the performance of a multigrid cycle be predicted in terms of the smoothing measures ?**

- **Note:** In a multigrid cycle, relaxations are used to "smooth" the highly oscillating error components which cannot be approximated on coarser grids.

  Therefore, smoothing roughly means "convergence of high frequency error components".

- **What are these "high frequency components" ?**

What are high and low frequencies ?

– Let G^h be a fine grid and G^H a coarse grid.

– **Then**: A Fourier component $e^{i\theta x/h}$ on grid G^h is called a **high frequency** (with respect to G^H) if its restriction (injection) to the coarse grid G^H is not "visible" there. Otherwise it is called a **low frequency**.

In 1D:

– Take one–dimensional standard coarsening:

$$G^h = \{kh : k \in \mathbb{Z}\}, \; G^H := G^{2h} = \{2kh : k \in \mathbb{Z}\}$$

– Let $|\theta| \le \pi$; then by injection from G^h to G^{2h}:

$$\boxed{e^{i\theta x/h} \overset{inject}{\Longrightarrow} e^{i2\theta x/2h}}$$

– This component $e^{i2\theta x/2h}$ is "visible" on G^{2h} only if

$$|2\theta| \le \pi, \quad i.e. \; |\theta| \le \frac{\pi}{2}$$

$\Longrightarrow$ The **high frequencies** on G^h (with respect to G^{2h}) are those with $\pi \ge |\theta| \ge \frac{\pi}{2}$.

High and low frequencies in 2D:

– Fourier components in 2D: $e^{i\theta\underline{x}/h} = e^{i\theta_1 x/h} e^{i\theta_2 y/h}$, $\theta = (\theta_1, \theta_2)$ with

$$|\theta| := \max(|\theta_1|, |\theta_2|) \leq \pi$$

– $\boxed{\textbf{Standard coarsening}}$

$$\begin{aligned} G^h &= \{(kh, lh) : k, l \in \mathbb{Z}\} \\ G^H = G^{2h} &= \{(2kh, 2lh) : k, l \in \mathbb{Z}\} \end{aligned}$$

$\Longrightarrow$ **High frequencies:** $\frac{\pi}{2} \leq |\theta| \leq \pi$

Low frequencies: $|\theta| < \frac{\pi}{2}$

θ_2

π

$\frac{\pi}{2}$

$-\pi$ $-\frac{\pi}{2}$ $\frac{\pi}{2}$ π θ_1

$-\frac{\pi}{2}$

$-\pi$

Note: The definition of high and low frequencies
depends on the coarsening.

– $\boxed{\textbf{Semi coarsening}}$ (in x–direction)

$$
\begin{aligned}
G^h &= \{(kh, lh) : k, l \in \mathbb{Z}\} \\
G^H &= \{(2kh, lh) : k, l \in \mathbb{Z}\}
\end{aligned}
$$

– $\quad e^{i\theta x/h} \quad \overset{inject}{\Longrightarrow} \quad e^{i2\theta_1 x/2h} e^{i\theta_2 y/h}$

$\Longrightarrow$ **High frequencies:** $\frac{\pi}{2} \leq |\theta_1| \leq \pi, \quad |\theta_2| \leq \pi$
Low frequencies: $|\theta_1| < \frac{\pi}{2}, \quad |\theta_2| \leq \pi$

θ_2

π

$\frac{\pi}{2}$

$-\pi \quad -\frac{\pi}{2} \qquad \frac{\pi}{2} \quad \pi \qquad \theta_1$

$-\frac{\pi}{2}$

$-\pi$

What is smoothing ?

– "Smoothing" stands for convergence of high frequency error components which cannot be approximated from the coarse grids in a multigrid cycle.

The smoothing rate:

For a relaxation scheme with amplification factors $\mu(\theta)$ of Fourier components $e^{i\theta \underline{x}/h}$ the **smoothing rate** μ is defined by:

$$\boxed{\mu := \max\{|\mu(\theta)| : \theta = \text{high frequencies}\}}$$

Note:
- The above definition of μ depends on the coarsening.
- The amplification or reduction factors $\mu(\theta)$ are also called "smoothing factors".

Smoothing analysis: scalar standard examples:

- Pointwise Gauss–Seidel relaxation for Poisson's equation; the isotropic case.
- Pointwise Gauss–Seidel relaxation for the anisotropic model case.
- Linewise Gauss–Seidel relaxation for the anisotropic model case.

Gauss–Seidel relaxation for Poisson's equation:

$$\Delta^h U^h = \frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} U^h = F^h$$

- **Relaxation in error terms:**

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} & 0 & \\ 1 & -4 & 0 \\ & 1 & \end{bmatrix}}_{B^h} \overline{v}^h(x) + \underbrace{\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 0 & 0 & 1 \\ & 0 & \end{bmatrix}}_{A^h} v^h(x) = 0$$

Symbols: $\theta = (\theta_1, \theta_2)$

$$B^h e^{i\theta\underline{x}/h} = \underbrace{\frac{1}{h^2}\left(e^{-i\theta_1} + e^{-i\theta_2} - 4\right)}_{\widetilde{B}^h(\theta)} \cdot e^{i\theta\underline{x}/h}$$

$$A^h e^{i\theta\underline{x}/h} = \underbrace{\frac{1}{h^2}\left(e^{i\theta_1} + e^{i\theta_2}\right)}_{\widetilde{A}^h(\theta)} \cdot e^{i\theta\underline{x}/h}$$

$\Longrightarrow$ **Amplification factors:**

$$\mu(\theta) = -\frac{\widetilde{A}^h(\theta)}{\widetilde{B}^h(\theta)} = -\frac{e^{i\theta_1} + e^{i\theta_2}}{e^{-i\theta_1} + e^{-i\theta_2} - 4}$$

$\Longrightarrow$ **Smoothing rate:**

$$\mu = \max\{|\mu(\theta)| : \frac{\pi}{2} \leq |\theta| \leq \pi\} = 0.5$$

$\Longrightarrow$ The high frequency error components are reduced by a factor of (at least) 0.5 per relaxation sweep.

– **Note:** For low frequencies, the reduction per relaxation sweep is much worse:

$$|\mu(\theta)| \longrightarrow 1 \quad \text{if} \;\; \theta \longrightarrow 0$$

Anisotropic operators: an example

Differential equation: $\boxed{0 < \varepsilon < 1}$

$$LU = \varepsilon U_{xx} + U_{yy} = F$$

Central difference approximation on uniform mesh:

$$L^h U^h = \frac{1}{h^2} \begin{bmatrix} & 1 & \\ \varepsilon & -2-2\varepsilon & \varepsilon \\ & 1 & \end{bmatrix} U^h = F^h$$

– **Remark:** Anisotropy is not only induced by the differential operators but often also by different meshsizes in different coordinate directions (so–called anisotropic grids) or grid stretching. Example: Central difference approximation of $\Delta U = F$ on a rectangular grid with grid sizes

$$h_x \gg h_y$$

$\Longrightarrow$

$$L^h U^h = \frac{1}{h_y^2} \begin{bmatrix} & 1 & \\ \varepsilon & -2-2\varepsilon & \varepsilon \\ & 1 & \end{bmatrix} U^h = F^h$$

with

$$\varepsilon = \frac{h_y^2}{h_x^2} \ll 1$$

1. The pointwise Gauss–Seidel relaxation:

in error terms:

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} & 0 & \\ \varepsilon & -2-2\varepsilon & 0 \\ & 1 & \end{bmatrix}}_{B^h}\overline{v}^h(x) + \underbrace{\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 0 & 0 & \varepsilon \\ & 0 & \end{bmatrix}}_{A^h} v^h(x) = 0$$

Symbols: $\theta = (\theta_1, \theta_2)$

$$B^h e^{i\theta\underline{x}/h} = \underbrace{\frac{1}{h^2}\left(\varepsilon e^{-i\theta_1} + e^{-i\theta_2} - 2 - 2\varepsilon\right)}_{\widetilde{B}^h(\theta)} \cdot e^{i\theta\underline{x}/h}$$

$$A^h e^{i\theta\underline{x}/h} = \underbrace{\frac{1}{h^2}\left(\varepsilon e^{i\theta_1} + e^{i\theta_2}\right)}_{\widetilde{A}^h(\theta)} \cdot e^{i\theta\underline{x}/h}$$

$\Longrightarrow$ **Amplification factors:**

$$\mu(\theta) = -\frac{\widetilde{A}^h(\theta)}{\widetilde{B}^h(\theta)} = -\frac{\varepsilon e^{i\theta_1} + e^{i\theta_2}}{\varepsilon e^{-i\theta_1} + e^{-i\theta_2} - 2 - 2\varepsilon}$$

$\Longrightarrow$ **Smoothing rate:** (standard coarsening)

$$\mu = \max\{|\mu(\theta)| : \frac{\pi}{2} \leq |\theta| \leq \pi\}$$

– For $\varepsilon \ll 1$ the maximum is attained at

$$\theta_1 = \pi, \theta_2 = 0$$

The amplification factor of this component is:

$$|\mu(\pi, 0)| = \left|\frac{1-\varepsilon}{1+3\varepsilon}\right|.$$

Therefore

$$\mu = 1 - O(\varepsilon) \quad \text{for} \ \ \varepsilon \longrightarrow 0$$

– **Note:** if $\varepsilon = 1$ (isotropic case, see above), then $\mu = 0.5$

– The pointwise Gauss–Seidel relaxation has good smoothing properties only if ε is close to 1, i.e. close to the isotropic case.

ε	1	0.1	0.05	0.02	0.01
μ	0.5	0.83	0.91	0.96	0.98

Smoothing rates of pointwise Gauss–Seidel for anisotropic operator

2. The linewise Gauss–Seidel relaxation:

x–line relaxation: Simultaneous relaxation of unknowns located on horizontal gridlines.

y–line relaxation: Simultaneous relaxation of unknowns located on vertical gridlines.

i.e. sweep over all (x– or y–) gridlines and solve simultaneously for all unknowns located on this line.

Here: **y–line relaxation**: (in error terms)

$$\underbrace{\frac{1}{h^2}\begin{bmatrix} & 1 & \\ \varepsilon & -2-2\varepsilon & 0 \\ & 1 & \end{bmatrix}}_{B^h}\overline{v}^h(x) + \underbrace{\frac{1}{h^2}\begin{bmatrix} & 0 & \\ 0 & 0 & \varepsilon \\ & 0 & \end{bmatrix}}_{A^h} v^h(x) = 0$$

Symbols: $\theta = (\theta_1, \theta_2)$

$$B^h e^{i\theta \underline{x}/h} = \underbrace{\frac{1}{h^2}\left(\varepsilon e^{-i\theta_1} + e^{i\theta_2} + e^{-i\theta_2} - 2 - 2\varepsilon\right)}_{\widetilde{B}^h(\theta)} \cdot e^{i\theta \underline{x}/h}$$

$$A^h e^{i\theta \underline{x}/h} = \underbrace{\frac{1}{h^2}\left(\varepsilon e^{i\theta_1}\right)}_{\widetilde{A}^h(\theta)} \cdot e^{i\theta \underline{x}/h}$$

$\Longrightarrow$ **Amplification factors:**

$$\mu(\theta) = -\frac{\widetilde{A}^h(\theta)}{\widetilde{B}^h(\theta)} = -\frac{\varepsilon e^{i\theta_1}}{\varepsilon e^{-i\theta_1} + e^{-i\theta_2} + e^{i\theta_2} - 2 - 2\varepsilon}$$

$\Longrightarrow$ **Smoothing rate:** (standard coarsening)

$$\mu = \max\{|\mu(\theta)| : \frac{\pi}{2} \leq |\theta| \leq \pi\}$$

– For $\varepsilon \leq 1$ the maximum is attained at

$$\theta_1 = \frac{\pi}{2}, \theta_2 = 0$$

The amplification factor of this component is:

$$|\mu(\frac{\pi}{2}, 0)| = \frac{1}{\sqrt{5}} = 0.447....$$

– for $\varepsilon > 1$ the smoothing is again bad. In this case, **x–line relaxations** will help.

– **The general rule:** The strongly coupled unknowns (strong weights in difference stencil) have to be relaxed simultaneously.

ε	1	0.1	0.05	0.02	0.01
μ	0.447	0.447	0.447	0.447	0.447
ε		10	20	50	100
μ		0.83	0.91	0.96	0.98

Smoothing rates of y-line Gauss–Seidel for anisotropic operator

Multigrid performance:

- **What do the smoothing rates have to do with the performance of the multigrid iteration ?**

- Let μ be the smoothing rate of the relaxation scheme used within a multigrid cycle.
- Assume that **by the coarse grid correction cycle all low frequency error components are "exactly" approximated** and that there is **no interaction between low and high frequencies**. This is the "ideal" case !

Then: The (asymptotic) error reduction by one step of a 2–level method is determined by the reduction of the high frequency error components on the fine grid. It can, therefore, (roughly) be estimated by

$$\boxed{\rho := \mu^{\nu_1+\nu_2}}$$

where ν_1 and ν_2 are the numbers of the pre– and post–smoothing steps (provided that $\nu_1 + \nu_2$ is small).

Note: This is not a rigorous mathematical estimate of the 2–level convergence !

Extension of local mode analysis to systems of equations

Let

$$LU = F$$

be a (linear) $q \times q$ system of PDE's,

$$U = (U_1, ..., U_q)^t\,, \quad F = (F_1, ..., F_q)^t$$

or in matrix notation:

$$\underbrace{\begin{pmatrix} l_{1,1} & , & \cdots & \cdots & , & l_{1,q} \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ l_{q,1} & , & \cdots & \cdots & , & l_{q,q} \end{pmatrix}}_{L} \cdot \begin{pmatrix} U_1 \\ \vdots \\ \vdots \\ U_q \end{pmatrix} = \begin{pmatrix} F_1 \\ \vdots \\ \vdots \\ F_q \end{pmatrix}$$

with: $\quad l_{k,l} \;$: scalar differential operators.

Example: Stokes equations, a 3×3 system

$$LU = \begin{bmatrix} -\;\Delta & 0 & \partial_x \\ 0 & -\;\Delta & \partial_y \\ \partial_x & \partial_y & 0 \end{bmatrix} \cdot \begin{bmatrix} U \\ V \\ P \end{bmatrix} = \cdots$$

Systems of difference operators with constant coefficients (2D)

analogously to the **scalar case** consider:

$$\underbrace{\begin{pmatrix} l_{1,1}^h & , & \cdots & \cdots & , & l_{1,q}^h \\ \vdots & & & & & \vdots \\ \vdots & & & & & \vdots \\ l_{q,1}^h & , & \cdots & \cdots & , & l_{q,q}^h \end{pmatrix}}_{L^h} \cdot \underbrace{\begin{pmatrix} U_1^h \\ \vdots \\ \vdots \\ U_q^h \end{pmatrix}}_{U^h} = \underbrace{\begin{pmatrix} F_1^h \\ \vdots \\ \vdots \\ F_q^h \end{pmatrix}}_{F^h}$$

where: $l_{k,l}^h$ are **scalar** difference operators with constant coefficients on a grid

$$G^h = \{(mh_x, nh_y) : m, n \in \mathbb{Z}\}, \quad h = (h_x, h_y)$$

i.e.

$$l_{k,l}^h u^h(\underline{x}) := \sum_{\mu_1 \in I} \sum_{\mu_2 \in I} a_{\mu_1 \mu_2}^{k,l} u^h(x + \mu_1 h_x, y + \mu_2 h_y)$$

with $a_{\mu_1 \mu_2}^{k,l} \in R$,

$I = \{-\nu, \ldots, 0, \ldots, \nu\}$

$\underline{x} = (x, y) \in G^h$ gridpoint

Smoothing analysis for systems of equations

– $q \times q$ system of difference equations

$$L^h U^h = F^h$$

– **splitting** of the difference operator (see scalar case)

$$L^h = A^h + B^h$$

where A^h, B^h are again $q \times q$ systems.

– **Relaxation:** $u^h \longrightarrow \overline{u}^h$

$$\boxed{A^h u^h(\underline{x}) + B^h \overline{u}^h(\underline{x}) = F^h(\underline{x})}$$

or in error terms:
$$\begin{aligned} v^h &= U^h - u^h \\ \overline{v}^h &= U^h - \overline{u}^h \end{aligned}$$

$$\boxed{A^h v^h(\underline{x}) + B^h \overline{v}^h(\underline{x}) = 0}$$

– Let
$$\begin{aligned} v^h &= \mathcal{A} e^{i\theta \underline{x}/h}, \quad \mathcal{A} \in C^q \\ \overline{v}^h &= \overline{\mathcal{A}} e^{i\theta \underline{x}/h}, \quad \overline{\mathcal{A}} \in C^q \end{aligned}$$

$\Longrightarrow$
$$\boxed{\widetilde{A}^h(\theta)\mathcal{A} + \widetilde{B}^h(\theta)\overline{\mathcal{A}} = 0}$$

with Fourier symbols $\widetilde{A}^h(\theta), \widetilde{B}^h(\theta)$ of A^h and B^h.

$\Longrightarrow$

$$\boxed{\overline{\mathcal{A}} = \underbrace{-\left(\widetilde{B}^h(\theta)\right)^{-1} \widetilde{A}^h(\theta)}_{\in C^{q \times q}} \cdot \mathcal{A}}$$

This leads to the following definition of the **smoothing rate**:

$$\boxed{\mu := \max\left\{\rho\left(\widetilde{B}^h(\theta)^{-1}\widetilde{A}^h(\theta)\right) : \theta = \text{high frequencies}\right\}}$$

or equivalently

$$\boxed{\mu = \max\left\{|\lambda(\theta)| \ \text{ with } \ \det\left(\lambda(\theta)\widetilde{B}^h(\theta) + \widetilde{A}^h(\theta)\right) = 0\right\}}$$

where max is again taken over all high frequencies.

- ρ: spectral radius;
- μ is the worst (asymptotic) convergence factor of all high frequency error components;
- in scalar case, this definition is consistent to the one given there.

In practice: Work out the Fourier symbols $\widetilde{B}^h(\theta)$ and $\widetilde{A}^h(\theta)$ of your particular relaxation and find the maximum of the spectral radii in Fourier space numerically:

e.g. using the Fortran program LFA002D by R. Wienands (smoothing rates of relaxation schemes based on operator splitting, for systems and scalar equations).

Example: The biharmonic problem

$$\Delta\Delta U = \cdots \qquad\qquad \text{scalar 4th order problem}$$

or in system form:

$$\begin{array}{lccc} \Delta W & & = & \cdots \\ \Delta U & -W & = & 0 \end{array} \Longleftrightarrow \begin{pmatrix} \Delta & 0 \\ -1 & \Delta \end{pmatrix} \begin{pmatrix} W \\ U \end{pmatrix} = \cdots$$

Discretization: on a square cartesian mesh, meshsize h

$$\begin{pmatrix} \Delta^h & 0 \\ -1 & \Delta^h \end{pmatrix} \begin{pmatrix} W^h \\ U^h \end{pmatrix} = \cdots$$

with

$$\Delta^h = \frac{1}{h^2} \begin{bmatrix} & & 1 & \\ 1 & - & 4 & 1 \\ & & 1 & \end{bmatrix}$$

The predictions coming from smoothing and 2-level analysis are quite exact for the biharmonic system :

		asymptotic reduction factors W–cycle				
ν_1, ν_2	$\mu^{\nu_1+\nu_2}$	N=32	N=64	N=128	N=256	$\rho^\star$
1,0	0.5	0.373	0.388	0.403	0.402	0.400
1,1	0.25	0.206	0.214	0.216	0.217	0.193
2,1	0.125	0.084	0.092	0.108	0.117	0.119
2,2	0.0625	0.072	0.071	0.075	0.083	0.084
		4 grids	5 grids	6 grids	7 grids	

$\rho^\star$ is the prediction of multigrid performance obtained by a 2–level local mode analysis.

The reduction factors of W–cycles were obtained for the biharmonic problem with Dirichlet boundary conditions

$$U = \text{given}, \quad U_n = \text{given}.$$

Note: If the biharmonic problem is treated as scalar 4th order problem, discretized by central differences (i.e. 13–point stencil, see above), the smoothing rate of the pointwise Gauss–Seidel relaxation is much worse:

$$\mu = 0.8$$

- Furthermore, this Gauss–Seidel relaxation for the 13–point stencil is computationally more expensive !

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