The Binomial model

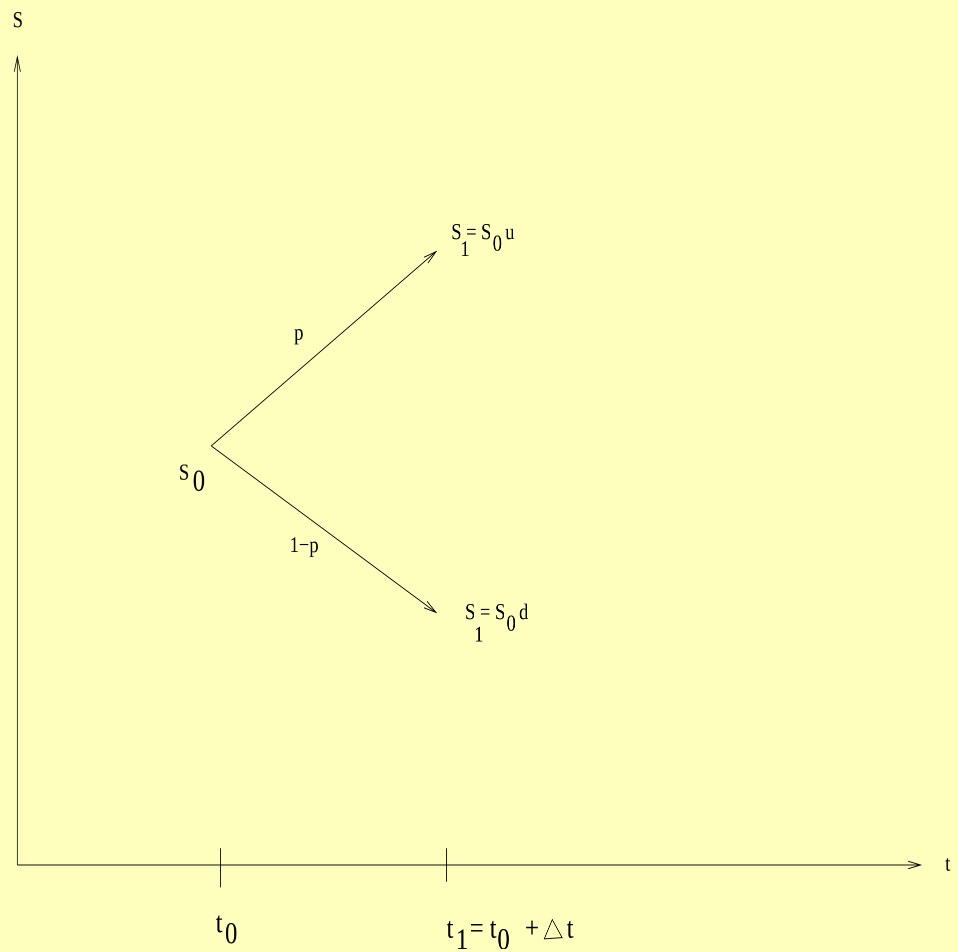
We have a sequence of prices $S_0, S_1, \ldots$, separated by a given time interval Δt, with the following characteristics:

1. Only two possible outcomes are allowed. If now the price is S_i, after some Δt (say one day) it will be either $S_{i+1} = S_i u$ or $S_{i+1} = S_i d$, u for up, d for down, $0 < d < u$.
2. Which one of them is going to happen is decided by a probability p:

$$\begin{aligned} P(S_{i+1} = S_i u) &= p \\ P(S_{i+1} = S_i d) &= 1 - p \end{aligned}$$

3. The asset price S_i is "expected to grow" according to the interest rate

$$E(S_{i+1}) = S_i e^{r \Delta t}$$


S
S = S u
1 0
p
S
0
1−p
S = S d
1 0
t
t
0
t = t + △t
1 0

Since the expectation is w.r.t the 'risk-neutral' probability P (defined by p), then from (1)-(3)

$$pu + (1-p)d = e^{r\Delta t} \quad (1)$$

On the other hand, from the continuous model

$$E(S_{i+1}^2) = S_i^2 e^{2r+\sigma^2 \Delta t} \quad (2)$$

i.e.,

$$Var(S_{i+1}) = S_i^2 e^{2r\Delta t}(e^{\sigma^2 \Delta t} - 1) \quad (3)$$

and from the discrete:

$$Var(S_{i+1}) = p(S_i u)^2 + (1-p)(S_i d)^2 - S_i^2(pu + (1-p)d)^2. \quad (4)$$

Equating (3) and (4):

$$e^{2r\Delta t+\sigma^2\Delta t} = pu^2 + (1-p)d^2. \tag{5}$$

Finally imposing the symmetry condition

$$ud = 1 \tag{6}$$

one has 3 equations (1),(5) and (6) with 3 unknowns u, d and p.
This gives the equation:

$$u^2 - 2\beta u + 1 = 0, \quad \beta := \frac{1}{2}(e^{-rt} + e^{(r+\sigma^2)\Delta t})$$

the solution is

$$u = \beta + \sqrt{\beta^2 - 1}$$
$$d = \beta - \sqrt{\beta^2 - 1}$$
$$p = \frac{e^{r\Delta t} - d}{u - d}.$$

a good approximation to u is the number $e^{\sigma\Delta t}$:

$$u = e^{\sigma\Delta t} + O((\Delta t)^{3/2})$$

(Ex. 1.6)

The algorithm

Fix a time horizon T. Divide the interval $[0, T]$ into M subintervals, and let

$$t_0 = 0, t_1 = \Delta t, \text{ etc, with } \Delta t = T/M.$$

The tree is now constructed (forward) in the following way:

For i = 1 until M

$$S_{ji} := S_0 u^j d^{i-j}, \quad j = 0, 1, \ldots, i$$

EndFor

setting $S_0 = S_{00}$.
The European option price is computed (backwards) as:

$$V_i = e^{-r\Delta t} E[V_{i+1}]$$

The European algorithm

This gives in the (ij) notation:

$$V_{j,i} = e^{-r\Delta t}(pV_{j+1,i+1} + (1-p)V_{j,i+1}) \quad (7)$$

We have the following pseudo-code for a European call (put is similar):

- Input: r, σ, S_0, T, K and M
- Define the quantities: $\Delta t, u, d, p$ and $S_{00} := S_0$
- Find the asset values $S_{j,M}$ at the end of the period, i.e., $S_{j,M} = S_{00}u^j d^{M-j}$ for $j = 0, 1, \ldots, M$
- Compute the payoff $V_{j,M} = (S_{j,M} - K)^+$, for $j = 0, 1, \ldots, M$
- Iterate (7) backwards for $i = M-1, \ldots, 0$, and for all $j = 0, 1, \ldots, i$
- return the price V_{00}

The American algorithm

Define the iteration:

$$V_{j,i} = \max\{(S_{j,i} - K)^+, e^{-r\Delta t}(pV_{j+1,i+1} + (1-p)V_{j,i+1})\} \qquad (8)$$

We have the following pseudo-code for an American call:

- Input: r, σ, S_0, T, K and M
- Define the quantities: $\Delta t, u, d, p$ and $S_{00} := S_0$
- Find the tree $S_{j,i}$ for ALL times, i.e., $S_{j,i} = S_{00}u^j d^{i-j}$ for $i = 1, \ldots, M$ and $j = 0, 1, \ldots, i$
- Compute the payoff $V_{j,M} = (S_{j,M} - K)^+$, for $j = 0, 1, \ldots, M$
- Iterate (8) backwards for $i = M-1, \ldots, 0$, and for all $j = 0, 1, \ldots, i$
- return the price V_{00}

Remarks

- The European and the American algorithms give the same value for a call, provided no dividends are paid (which is the case now). The European and the American put values are always different.
- The European algorithm generates prices V_{00}^{M} that converge towards the price of the continuous model for $M \to \infty$ (Exercise 1.8)
- The American algorithm also converges to the price of the continuous model (more difficult to prove)
- The algorithms may be extended to the case of discrete dividend, the tree might not be 'recombining'.
- It is possible to extend the tree such that 3 outcomes are allowed, thus 3 probabilities $p1, p2, p3$ should be found. The trinomial method is more accurate.

Risk-Neutral valuation and replication

Now we derive the European price from a different principle. The principle is called replication: we construct a portfolio that 'replicates' the option price at maturity. We construct a so-called 'Hedging' portfolio.

Assume a one-period model, i.e., only one time step. Let us drop the assumption $E(S_1) = e^{rt}S_0$ and that we now the probabilities of up and down movements.

Say that I bought one option, from a bank, and I paid V_0.

What can I do to eliminate the risk? Make a portfolio such that the return is the same as the return from a bank account.

The portfolio consists of the asset S and the option V, I long Δ units of the asset, and I am short V.

Today the balance is:

$$\Pi_0 = S_0\Delta - V_0 \tag{9}$$

Risk-Neutral valuation and replication, cont

Two possibilities arise: S_0 becomes $S_1 = S_0 u$ or $S_1 = S_0 d$. Therefore, two possibilities arise for Π:

$$\Pi^u = S_0 u \Delta - V^u, \quad \text{or} \quad \Pi^d = S_0 d \Delta - V^d$$

The risk is eliminated if this quantities are equal $\Pi_T = \Pi^u = \Pi^d$, that gives the strategy:

$$\Delta = \frac{V^u - V^d}{S_0(u-d)} \tag{10}$$

On the other hand, the 'no-arbitrage' assumption yields

$$\Pi_T = \Pi_0 e^{rT} \tag{11}$$

(I cannot make more or less money from my portfolio than by investing on a bank account)

Risk-Neutral valuation and replication, cont

We have

$$S_0\Delta - V_0 = \Pi_0 = e^{-rT}\Pi_T = e^{-rT}(S_0 u\Delta - V^u),$$

and after substituting (10) here:

$$V_0 = e^{-rT}\{qV^u + (1-q)V^d\}, \quad \text{where} \quad q := \frac{e^{rT} - d}{u - d}.$$

If q is to be interpreted as a probability, then $0 < q < 1$, which is equivalent to $d < e^{rT} < u$. Violating this bound leads to arbitrage.
The value of the option is obtained by 'discounting' and 'averaging' with respect to the prob. measure Q defined by q:

$$V_0 = e^{-rT}E_Q[V_T]$$

Q is sometimes called risk-neutral measure, or equivalent martingale measure.

Risk-Neutral valuation and replication, cont

One also finds

$$E_Q(S_T) = S_0 e^{rT}$$

which is the so-called martingale property for discounted prices. Summarizing:

- We have obtained the same expression for p and q, so replication and risk-neutral valuation give the same price.
- The real-world is NOT risk-neutral. Moreover, pricing with the real-world probability DOES NOT give the right answer. Pricing with the risk-neutral probability is a tool that gives the right answer, in the sense that it can be perfectly hedged.
- The principles here are also valid for multi-periods and the continuous model.
- In the limit Δ becomes $\Delta = \frac{\partial V(t,S)}{\partial S}$, which appears in continuous models