Dividends

- The owner of a stock theoretically owns a piece of the company. This ownership can only be turned into cash, if he owns so many of the stocks that he can take over the company and keep all profits for himself, which is unrealistically (for most of us).
- To the average investor the value in holding the stock comes from the dividends and any growth in the stock's value. Dividends are the lump payments, paid out every quarter or every six months, to the holder of the stock.
- The amount of dividend varies from year to year depending on the profitability of the company. Companies like to try to keep the level of dividends about the same.
- The amount of dividend is decided by the board of directors of the company and is usually set a month or so before the dividend is actually paid.
- When the stock is bought it either comes with its entitlement to the next dividend (cum) or not (ex). There is a date at around the time of dividend payment when the stock goes from cum to ex. The original holder of a stock gets the dividend but the person who buys it obviously does not.

Options on dividend-paying equities

- A stock that is cum dividend is better than one that is ex dividend. Thus at the time that the dividend is paid there will be a drop in the value of the stock. (The jump in asset price is more complex in practice.)
- The price of an option on an dividend-paying asset is affected by these payments. Therefore we must modify the Black-Scholes analysis.
- Different structures are possible for the dividend payment
  - payments may be deterministic or stochastic
  - payments may be made continuously or at discrete times
- Here, we only consider deterministic dividends, whose amount and timing are known.
- Let's consider that in a time dt the underlying asset pays out a dividend $DSdt$ with D constant. The payment is independent of time except through the S-dependence. It represents a continuous and constant dividend yield.
- This structure is a good model for index options.

- Arbitrage considerations show that the asset price must fall by the amount of dividend payment. This is modeled as: $dS = \sigma S dw + (\mu - D) S dt$.
- The dividend payment also has its effect on the hedged portfolio: Since we receive $DSdt$ for every asset held and we hold $-\Delta$ of the underlying, the portfolio changes by an amount $-DS\Delta dt$. Therefore, we add to our $d\Pi$ from before this amount:

$$d\Pi = dV - \Delta dS - DS\Delta dt.$$

  We find after similar reasoning as for European options that dividend is included in the following formulation:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r - D) S \frac{\partial V}{\partial S} - rV = 0$$

- This model is also applicable to options on foreign currencies, though only for short dated options. Since holding an amount of foreign currency yields interest at the foreign rate r_f, in this case $D = r_f$
- A nonzero dividend yield also has an effect on the boundary and final conditions.

Options on dividend-paying equities

- At the time that a dividend is paid there will be a drop in the value of the stock.
- The price of an option on an dividend-paying asset is affected by these payments.
- Different structures are possible for the dividend payment (deterministic or stochastic with payments continuously or at discrete times)
- We can also consider discrete deterministic dividends, whose amount and timing are known.
- Arbitrage arguments require:

$$V(S, t_d^-) = V(S - D, t_d^+)$$

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Discretization

Change direction by $\tau = T - t$ $(\partial/\partial\tau = -\partial/\partial t)$. Look for a solution of the PDE

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r - D)S\frac{\partial V}{\partial S} + rV = 0$$

with boundary and initial conditions. An interval $[0, S_{max}]$ is divided into N parts $[x_i, x_{i+1}]$
The points S_i and the mesh width

$$\delta S_i = h = \frac{S_N - S_0}{N} = \frac{1}{N}$$

determine the 1D S-grid, on which we approximate solution V_i $(i = 1, N)$

Equivalence

Transformation to Diffusion Equation

- It can sometimes be useful to transform the basic Black-Scholes equation by a change of variables
- Writing $V(S,t) = e^{\alpha x + \beta \tau} y(x, \tau)$, with

$$S = e^x, t = T - 2\tau/\sigma^2, \alpha = -\frac{1}{2}(\frac{2r}{\sigma^2} - 1), \beta = -\frac{1}{4}(\frac{2r}{\sigma^2} + 1)^2$$

- Then $y(x, \tau)$ satisfies the basic diffusion equation

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

- With terms $S^j \partial^j V / \partial S^j$, we deal with an Eulerian differential equation: The convection-diffusion-reaction type equation can be transformed into a heat equation.

Stability of time discretization

- If the order of consistency of implicit and explicit time discretizations is identical, one typically favors an explicit discretization, because it costs less arithmetic operations per time step compared to an implicit discretization. Consistency is a statement for $k \to 0$, whereas in practice one computes with a finite $k > 0$. This give rise to another view at time discretizations. A quality criterion of a method with $k > 0$ is the stability of a discretization.
- The diffusion equation reads

$$\begin{aligned} \frac{\partial y}{\partial t} - \frac{\partial^2 y}{\partial x^2} = 0 \quad &\text{in} \qquad (0,1) \times (0,\infty) \\ y(x,t) = 0 \quad &\text{for} \qquad x \in \{0,1\}, \quad t > 0 \\ y(x,0) = y_0(x) \quad &\text{for} \qquad x \in (0,1) \end{aligned}$$

An Example

- We take $N \in \mathbb{N}^*$ grid points and approximate $\dfrac{\partial^2 y}{\partial x^2}$ at the grid:

$$x_i := i\Delta x,\ 1 \leq i \leq N,\ \Delta x := \frac{1}{N+1}$$

- Approximate by second order difference quotients

$$\frac{\partial^2 y(x_i,t)}{\partial x^2} = \frac{1}{(\Delta x)^2}[y(x_{i-1},t) - 2y(x_i,t) + y(x_{i+1},t)] + O((\Delta x)^2)\ .$$

- With the notation

$$y(t) := (y(x_1,t),\ldots,y(x_N,t))^T,\ y_0 := (y_0(x_1,t),\ldots,y_0(x_N,t))^T$$

$$A := \frac{1}{(\Delta x)^2}\begin{pmatrix} -2 & 1 & & & & \\ 1 & -2 & 1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & -2 & 1 \\ & & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{N\times N}$$

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An Example

- The PDE changes into a discrete initial value problem

$$\frac{dy}{dt} = Ay, \quad y(0) = y_0 \quad (*) \ .$$

- Define the vectors $z^{(k)} \in \mathbb{R}^N$ and the numbers $\lambda_k \in \mathbb{R}$, $1 \leq k \leq N$, by

$$\begin{aligned} z_i^{(k)} &:= \sin k\pi x_i \quad 1 \leq i \leq N, \ 1 \leq k \leq N, \\ \lambda_k &:= \frac{2}{(\Delta x)^2} \left(1 - \cos k\pi \Delta x\right), \ 1 \leq k \leq N. \end{aligned}$$

With the help of the additions theorems one easily calculates that the vectors $z^{(k)}$ represent the Eigenvectors of A w.r.t. the eigenvalues $-\lambda_k$.

An Example

- Therefore the exact solution of $(*)$ reads

$$y(t) = \sum_{k=1}^{N} \alpha_k e^{-\lambda_k t} z^{(k)}$$

with

$$\sum_{k=1}^{N} \alpha_k z^{(k)} = y_0 \quad .$$

- Apply the explicit Euler method with constant step size Δt to $(*)$, one obtains the approximation

$$\widetilde{y}(i\Delta t; \Delta t) = \sum_{k=1}^{N} \alpha_k (1 - \lambda_k \Delta t)^i z^{(k)} \quad .$$

- This equation should, for fixed Δt and $i \to \infty$ show the same asymptotic behavior as the exact solution $y(t)$ for $t \to \infty$. Therefore,

$$|1 - \lambda_k \Delta t| < 1 \quad \forall\, 1 \leq k \leq N \quad .$$

- A simple calculation reveals

$$
\begin{aligned}
0 < \lambda_1 &\leq \lambda_2 \leq \ldots \leq \lambda_N, \\
\lambda_1 &= \frac{4}{(\Delta x)^2}\sin^2\frac{\pi \Delta x}{2} = \pi^2 + O((\Delta x)^2), \\
\lambda_N &= \frac{2}{(\Delta x)^2}(1+\cos \pi \Delta x) = O((\Delta x)^{-2}).
\end{aligned}
$$

- So,

$$\Delta t < 2\lambda_N^{-1} = O((\Delta x)^2).$$

- This is a serious restriction on the time step, which is determined by the least interesting solution component.

- Would we apply the implicit Euler method with constant step size Δt to $(*)$, we obtain the approximation

$$\widetilde{y}(i\Delta t;\Delta t) = \sum_{k=1}^{N} \alpha_k (1+\lambda_k \Delta t)^{-i} z^{(k)} \ .$$

- Due to $\lambda_k > 0,\ 1 \leq k \leq N$, this equation show for every time step $\Delta t > 0$ the same asymptotic behavior as the exact solution. Even for large step sizes we find a qualitatively correct solution.

- The IVP $(*)$ is an example of a stiff problem: The different components of the solution decay in a very different way, and the components that decay fastest (and are therefore least interesting), restrict the step size of the discretization.

- The example above shows, that implicit schemes despite extra costs for the solution for the solution of (possibly non-)linear systems of equations advantageous as compared to explicit methods.

Time Discretization

$$\frac{\partial V(S,\sigma,t)}{\partial t} + A_h V_h(S,\sigma,t) = 0.$$

- Second order ($O(\Delta t^2)$) accuracy: Crank-Nicolson, Backward difference formulae (BDF), combinations of implicit and explicit schemes (IMEX)
- Crank-Nicolson: $\theta = 1/2$

$$\frac{V_h^{(m+1)} - V_h^{(m)}}{\Delta t} + \theta A_h V_h^{(m+1)} + (1-\theta) A_h V_h^{(m)} = 0$$

- BDF 2:

$$\frac{3/2 V_h^{(m+1)} - 2V_h^{(m)} + 1/2 V_h^{(m-1)}}{\Delta t} + A_h V_h^{(m+1)} = 0$$

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