Financial industry; Work at banks

Front office	⇔	Back office
Pricing and selling products	⇔	Validation of prices, research into alternative models

- Pricing approach:
  1. Start with some financial product
  2. Model asset prices involved (SDEs)
  3. Calibrate the model to market data (numerics, optimization)
  4. Model product price correspondingly (P(I)DE or integral)
  5. Price the product of interest (numerics, MC)
  6. Set up a hedge to remove the risk to the product (optimization)

Risk Neutral valuation

We already know:
For a given market described by the equations:

$$\begin{cases} dB(t) &= rB(t)dt \\ dS(t) &= \mu S(t)dt + \sigma S(t)dW(t)^{\mathbb{P}}, \end{cases}$$

and a contingent claim of the form

$$\chi = V(T, S(T)).$$

Then, the arbitrage free price is given, via Ito's Lemma, by $V(t, S(t))$, where function $V(t, S(t))$ satisfies the Black-Scholes equation:

$$\begin{cases} \frac{\partial V}{\partial t} + rS(t)\frac{\partial V}{\partial S} + \frac{1}{2}S^2(t)\sigma^2\frac{\partial^2 V}{\partial S^2} - rV &= 0 \\ V(T, S(T)) &= \chi. \end{cases}$$

Pricing: Feynman-Kac Theorem

Given the system of stochastic differential equations:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

and an option, V, such that

$$V(S,t) = e^{-r(T-t)}\mathbb{E}^Q\{V(S(T),T)|S(t)\}$$

with the sum of the first derivatives of the option square integrable.

Then the value, $V(S(t),t)$, is the unique solution of the final condition problem

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - rV = 0, \\ V(S,T) = \text{given} \end{cases}$$

A pricing approach

$$V(S(t_0), t_0) = e^{-r(T-t_0)} \mathbb{E}^Q \{ V(S(T), T) | S(t_0) \}$$

Quadrature:

$$V(S(t_0), t_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} V(S(T), T) f(S(T)|S(t_0)) dS$$

- Trans. PDF, $f(S(T)|S(t_0))$, typically not available, but the characteristic function, ϕ, often is.

Risk Neutral valuation

Feynman-Kac:
The Black-Scholes equation is of the form which can be solved using a stochastic representation formula via Feynman-Kac.

Is there an exact solution?

Analytical Solution of BS prices

In order to solve the Black-Scholes Equation, we apply Itô's formula to the function

$$g_t = \log S(t)$$

we then have:

$$\begin{aligned}
dg &= g_s dS + g_t dt + \frac{1}{2} g_{s,s} (dS)^2, \\
&= \frac{1}{S(t)} dS - \frac{1}{2} \frac{1}{S(t)^2} (dS)^2, \\
&= \frac{1}{S(t)} \left(rS(t)dt + \sigma S(t) dW_t^{\mathbb{Q}} \right) + \frac{1}{2} \frac{1}{S(t)^2} \sigma^2 S(t)^2 dt, \\
&= \left(r - \frac{1}{2}\sigma^2 \right) dt + \sigma dW_t^{\mathbb{Q}},
\end{aligned}$$

Analytical Solution of BS prices

Finally we have:

$$\begin{aligned}\int_t^T d\log S(u) &= \int_t^T \left(r - \frac{1}{2}\sigma^2\right) dt + \int_t^T \sigma dW_t^{\mathbb{Q}},\\ \log\frac{S(T)}{S(t)} &= \left(r-\frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W(T)^{\mathbb{Q}} - W(t)^{\mathbb{Q}}\right),\end{aligned}$$

So, we find:

$$S(T) = S_0 \exp\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t) + \sigma\left(W(T)^{\mathbb{Q}} - W(t)^{\mathbb{Q}}\right)\right).$$

Feynman-Kac theorem gives:

$$\begin{aligned}V(t,S(t)) &= e^{-r(T-t)}\int_{-\infty}^{+\infty} V(T,S(T))f(s)ds\\ &= e^{-r(T-t)}\int_{-\infty}^{+\infty} V(T,S_0e^{Z})f(z)dz\end{aligned}$$

Analytical Solution of BS prices

where Z is a random variable with the distribution:

$$N\left(\left(r-\frac{1}{2}\sigma^2\right)(T-t),\sigma\sqrt{T-t}\right).$$

If we now take

$$V(T,S(T))=\max(S(T)-K,0)$$

we have:

$$\mathbb{E}^{\mathbb{Q}}\left(\max(S_0e^Z,0)|\mathcal{F}_t\right)=0\cdot\mathbb{Q}\left(S_0e^Z\leq K\right)+\int_{\log\frac{K}{S_0}}^{\infty}\left(S_0e^z-K\right)f(z)dz.$$

After simple calculations we end up with the Black-Scholes pricing theorem.

Analytical Solution of BS prices

Theorem (Black-Scholes formula)

The price of a European call option with strike price K and maturity T is given by the formula:

$$
\begin{aligned}
V(t, S(t)) &= S_0 \phi(d_1(t, S_0)) - e^{-r(T-t)} K \phi(d_2(t, S_0)), \textit{ with} \\
d_1(t, S_0) &= \frac{1}{\sigma\sqrt{T-t}} \left(\log \frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2 \right) (T-t) \right), \\
d_2(t, S_0) &= d_1(t, S_t) - \sigma\sqrt{T-t},
\end{aligned}
$$

where ϕ is the cumulative distribution function for standard normal distribution i.e., N(0,1).

Deficiencies of the Black-Scholes Model

Implied Volatility

- Suppose we solve the 1D Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

  for σ, since V is known from the market.
- In practice, the implied volatility for different strikes K and maturities T on the same asset is not constant.
- This shape is commonly known as the volatility smile, or skew.

Summary of BS model

In the Black-Scholes formula we have:

- time to maturity: T (known)
- strike : K (known)
- risk free rate: r (known)
- current underlying price: S_0 (known)

What about σ ? The risk is a driving factor for options. Under normal circumstances the option's theoretical value is a monotonic increasing function of the volatility. This means there is a one-to-one relation between the option price and the volatility.

How can we test whether this is true in reality?

Historical Volatility

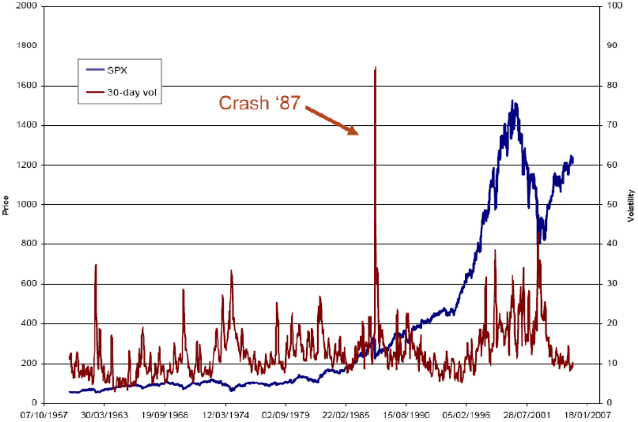


Figure: $S\&P500$ Spot level and 30-day realized volatility. Picture was adopted from work of Hans Buehler.

Black-Scholes model (1973)

Some general information: http://en.wikipedia.org/wiki/Black-Scholes
Consequences:
(Ajay Shah) "Black, Merton and Scholes: Their work and its consequences"
http://www.mayin.org/ajayshah/PDFDOCS/Shah1997_bms.pdf

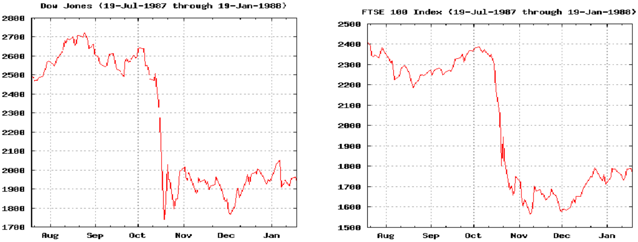


Figure: Market Crash on 19 October 1987, LEFT: Dow Jones, RIGHT: FTSE

Market Implied volatility

Example:
Suppose we have given a standard call option C on 100 shares of company Z. The strike $K = \$75$ and expires in 55 days. The risk free rate is 5%. The current stock price is \$85, and from historical data we have obtained $\sigma = 0.25$. So, the call price given by the BS model is:

$$BS(\sigma, r, T, K, S_0) = BS(25\%, 5\%, \frac{55}{365}, 75, 85) = 10.8667$$

But on the market the price of such a call option is €12.25.

What does it mean?
An Arbitrage?

Market Implied volatility

Example continuation

Based on the standard BS pricing model, the volatility implied by the market price C is 43.89% i.e.,

$$\sigma_{C^{market}} = f_\sigma(C_{market}) = 43.89\%.$$

In order to check the calculation we put back σ_{market} in the pricing model, i.e.,

$$\begin{aligned} V(S_t, t) &= BS(\sigma_{market}, r, T, K, S_0) \\ &= BS(43.89\%, 5\%, \frac{55}{365}, \$75, \$85) = \$12.25 \end{aligned}$$

How to find this implied volatility?

Market Implied volatility

Implied Volatility: *"The wrong number in the wrong formula to get the right price".* [Rebonato 1999]

We have:

$$V(S_t, t) = BS(\sigma, r, T, K, S_0),$$

where BS is monotonically increasing in σ (higher volatility corresponds to higher price). Now by assuming the existence of some inverse function

$$f_\sigma(\cdot) = BS^{-1}(\cdot)$$

so that

$$\sigma_{C^{market}} = f_\sigma(C^{market}, r, T, K, S_0)$$

we can compute the implied volatility for traded options with different strikes and maturities, and test the Black-Scholes model.

Solving the inverse pricing model function

How to find implied volatility?

The BS pricing function BS does not have a closed form solution for the inverse $f_\sigma(\cdot)$. Instead, a root finding technique is used to solve the equation:

$$BS(\sigma, r, T, K, S_0) - C^{market} = 0.$$

There are many ways to solve this equation, one of the most popular method are methods of "Newton-Raphson" and "Brent"[1]. Since the option prices can move quickly, it is often important to use the most efficient method when calculating implied volatilities.

[1] http://en.wikipedia.org/wiki/Brent%27s_method

Method of Newton-Raphson

The idea of the method is as follows: One starts with an initial guess which is reasonably close to the true root, then the function is approximated by its tangent line, and one computes the x-intercept of this tangent line.
Suppose $f : [a, b] \to \mathbb{R}$ is a differentiable function. From basic calculus we have:

$$g'(x_n) = \frac{g(x_n) - 0}{x_n - x_{n+1}} = \frac{0 - g(x_n)}{x_{n+1} - x_n} \tag{1}$$

which gives the iteration:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \tag{2}$$

with some arbitrary initial value x_0. In the case of BS we have:

$$\sigma_{i+1} = \sigma_i - \frac{BS(\sigma_i, \cdot) - C^{market}}{\frac{\partial BS(\sigma_i, \cdot)}{\partial \sigma_i}}.$$

Method of Newton

Theorem

Suppose g has a continuous second derivative, and suppose $x_{ex} \in \mathbb{R}$ satisfies $g(x_{ex}) = 0$ and $g'(x_{ex}) \neq 0$. Then there exists a $\delta > 0$ such that for $|x_0 - x_{ex}| < \delta$ the sequence given in (2) is defined for all $n > 0$, $\lim_{n\to+\infty} |x_n - x_{ex}| = 0$, and there exists a constant C such that:

$$|x_{n+1} - x_{ex}| \leq C|x_n - x_{ex}|^2.$$

Newton's method provides rapid convergence, however it requires the first partial derivative of the option's theoretical value with respect to volatility, i.e. $\frac{\partial C}{\partial \sigma}$ which in many pricing models has to be found numerically.

Method of Newton

```
clear all; clc; close all;
T=2; v_0=0.1; r=0.05; S_0=1; kappa=0.2; sigmabar=0.3; gamma=0.1; rho=-0.8; sigma2=0.3;
N=10000; %number of paths
M=200;   %number of steps
dt=T/M;
Vol=zeros(N,M+1);
S=zeros(N,M+1);
S2=zeros(N,M+1);
Vol(:,1)=v_0;
S(:,1)=S_0;
S2(:,1)=S_0;
for i=1:M
    Sigma=     [1 ,        rho      %r
                rho,         1,]*dt;      %s
    W=random('normal',0,1,[N,2]);
    C=chol(Sigma,'lower');
    W=(C*W')';
     Vol(:,i+1)=Vol(:,i)-kappa*(Vol(:,i)-sigmabar)*dt+   gamma*sqrt(Vol(:,i)).*W(:,1);
     S(:,i+1)=S(:,i)+r*S(:,i)*dt+sqrt(Vol(:,i)).*S(:,i).*W(:,2);
     S2(:,i+1)=S2(:,i)+r*S2(:,i)*dt+sigma2.*S2(:,i).*W(:,2);
end
VALUE=[];
VALUE2=[];
Strikes=0:0.1:4;
for k=Strikes
    VALUE(end+1)=mean(exp(-r*T).*max(S(:,end)-k,0));   %% CALL PRICE
    VALUE2(end+1)=blsprice(S_0, k, r , T, 0.6);
end
```

Implied volatility and Black-Scholes

Why is the implied volatility so important?

Implied volatility: Model

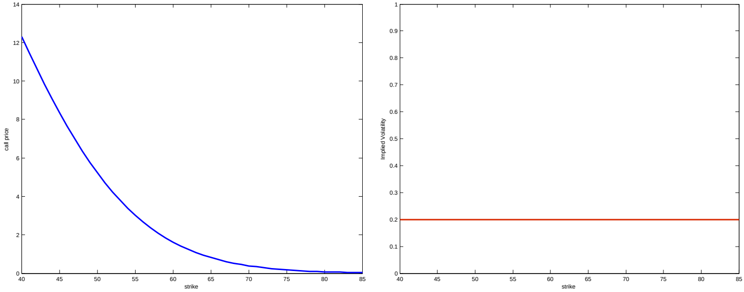
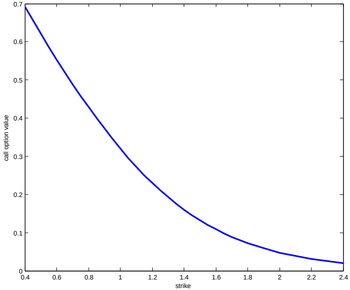


Figure: MODEL–LEFT: BS Call Prices, RIGHT: Implied Volatilities.

Why is the implied volatility so important?

Implied volatility: Market



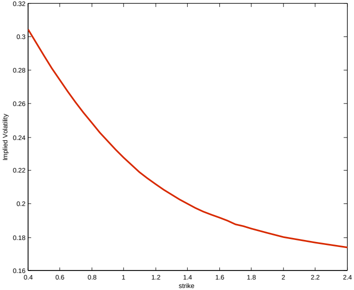
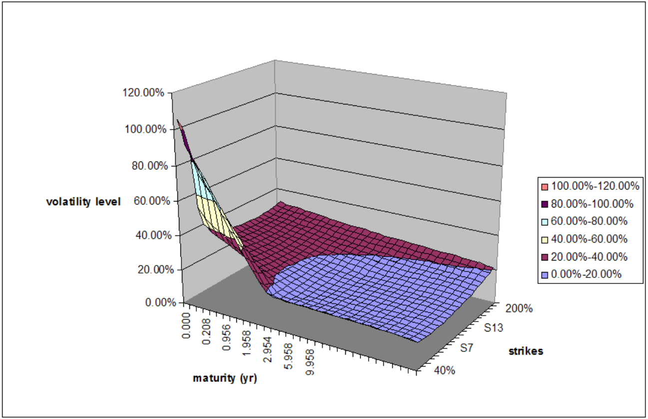


Figure: MARKET DATA– LEFT: Market Call Prices, RIGHT: Implied Volatilities.

Implied volatility and Black-Scholes

Why is the implied volatility so important?

Implied volatility Surface



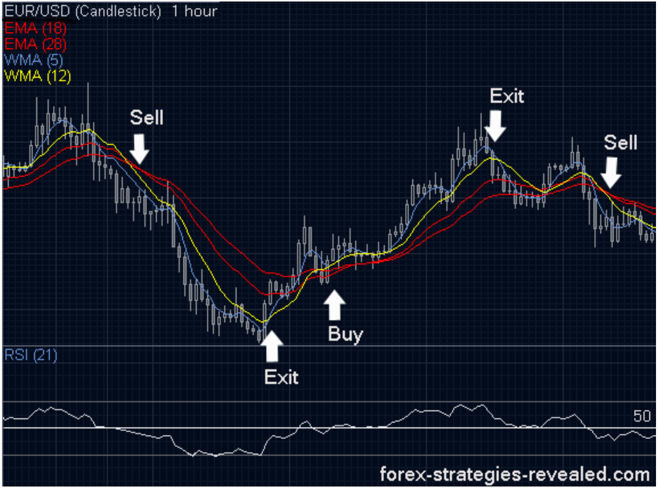
Deficiencies of the Black-Scholes Model

$\Rightarrow$ The idea of implied volatility does not fit in the Black-Scholes model

- Look for market consistent asset price models.

$\Rightarrow$ Use local volatility, Heston's model, or a a process with jumps, to better fit market data, and allow for smile effects

Market and Jumps



Market and Jumps

- Brownian motion has continuous paths S_t, however real stock data may contain jumps in the prices.
- It is common to include jumps in the stock price model by means of a Poisson process.
- Including jumps may give more realistic asset price simulation, BUT the risk of an option cannot be hedged away to zero !

Definition (Jumps and counting variable)

Let us denote the time instances for which a jump occurs by τ_j, with $\tau_1 < \tau_2 < \tau_3 \ldots$. We define the number of jumps be counted by the counting variable J_t, with

$$\tau_j = \inf\{t \geq 0, J_t = j\}.$$

Bernoulli approach

Definition (Jumps via Bernoulli approach)

Let us define a subinterval of length $\delta_t = \frac{T}{N}$ and allow for only two outcomes, a jump happened, or a jump didn't happen with the probabilities:

$$
\begin{aligned}
P(J_t - J_{t-\delta_t} = 1) &= \lambda\delta_t \\
P(J_t - J_{t-\delta_t} = 0) &= 1 - \lambda\delta_t,
\end{aligned}
$$

for some λ such that $0 < \lambda\delta_t < 1$. The parameter λ is referred to as the intensity of the jump process. Then, the occurance of k jumps in $0 \leq \tau \leq t$ has the probability:

$$
P(J_t - J_0 = k) = \binom{n}{k} (\lambda\delta_t)^k (1 - \lambda\delta_t)^{n-k}.
$$

where the trials in each subinterval are considered independent.

Bernoulli approach

Suppose now that: $n \to \infty$ then the Bernoulli process converges to

$$\frac{(\lambda t)^k}{k!} e^{-\lambda t} \sim \text{Poisson}(\lambda)$$

Definition (Poisson process)

The stochastic process $\{J_t, t \geq 0\}$ *is called a Poisson process if the following conditions hold:*

1. $J_0 = 0$,
2. $J_t - J_s$ *are integer-valued for* $0 \leq s \leq t \leq \infty$ *and*

$$P(J_t - J_s = k) = \frac{\lambda^k (t-s)^k}{k!} e^{-\lambda(t-s)} \text{ for } k = 0, 1, 2, \ldots$$

3. *The increments* $J_{t_2} - J_{t_1}$ *and* $J_{t_4} - J_{t_3}$ *are independent for all* $0 \leq t_1 < t_2 < t_3 < t_4$.

Bernoulli approach

Properties (Poisson Process)

1. *J_t is right-continuous and nondecreasing,*
2. *The times between successive jumps are independent and exponentially distributed with parameter λ. Thus,*

$$P(\tau_{j+1} - \tau_j > \delta_\tau) = e^{-\lambda \delta_\tau} \text{ for each } \delta_\tau.$$

3. *J_t is a Markov process.*
4. $\mathbb{E}(J_t) = \lambda t$, *and* $Var(J_t) = \lambda t$.

Jumps

Suppose we have given a stock variable S_t which jumps at time τ_j. We denote τ^+ the moment after one particular jump and τ^- the moment before.

- The absolute size of the jump is:

$$\delta S = S_{\tau^+} - S_{\tau^-},$$

which we model as a *proportional jump*,

- $S_{\tau^+} = qS_{\tau^-}$ with $q > 0$, so $\delta S = qS_{\tau^-} - S_{\tau^-} = (q-1)S_{\tau^-}$.
- The jump sizes equal $q-1$ times the current asset price.
- Assuming that for given set of i.i.d. $q_{\tau_1}, q_{\tau_2}, \dots$ r.v.the process

$$dS_t = (q_t - 1)S_t dJ_t,$$

is called Compound Poisson Process.

Simulating Jumps- Jump Diffusion Process

If we combine geometric Brownian motion and jump process we obtain:

$$dS_t = \mu S_t dt + \sigma S_t dW_t + (q_t - 1) S_t dJ_t.$$

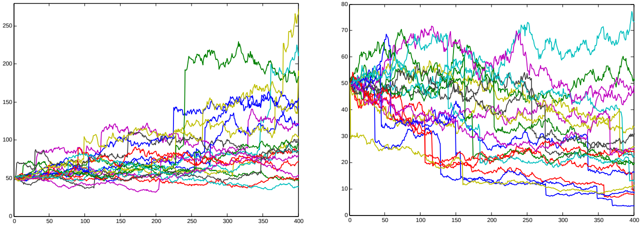


Figure: Geometric Brownian motion with jumps: $\delta = 0.01$, $\mu = 0.04$, $\sigma = 0.2$, $\lambda = 0.5$, LEFT: $q = 1.4$,RIGHT $q = 0.6$.

Simulating Jumps- Jump Diffusion Process

An analytical solution of the equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t + (q_t - 1) S_t dJ_t,$$

can be calculated on each of the jump-free subintervals $\tau_j < t < \tau_{j+1}$ where the SDE is just a GBM.
When at time τ_1 a jump of size:

$$(\delta S) = (q_{\tau_1} - 1) S_{\tau_1^-},$$

occurs, and thereafter the solution is given by:

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t \right) + (q_{\tau_1} - 1)\, S_{\tau_1^-}$$

In general we obtain:

$$S_t = S_0 \exp\left((\mu - \frac{\sigma^2}{2})t + \sigma W_t \right) + \sum_{j=1}^{J_t} S_{\tau_j^-} \left(q_{\tau_j} - 1 \right).$$

Market modeled by alternative processes

$$
\begin{aligned}
dS_t &= rS_t dt + \sigma S_t dW_t^Q \\
\Rightarrow S_t &= S_0 e^{X_t},\ X_t = (r - \frac{\sigma^2}{2})t + \sigma W_t^Q.
\end{aligned}
$$

- Compound Poisson (jump diffusion model)

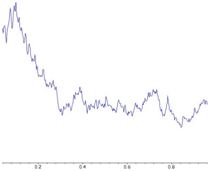
$$X_t = (\mu - \frac{\sigma^2}{2})t + \sigma W_t + \sum_{i=1}^{N_t} Y_i,$$

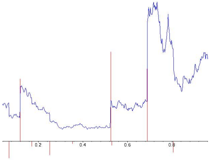
where N_t is Poisson: $P(N_t = n) = e^{-\lambda t}(\lambda t)^n / n!$, with intensity λ, Y_i i.i.d. with law F, for example, normally distributed (mean μ_J, variance σ_J^2).

Lévy Processes

- Lévy process $\{X_t\}_{t \geq 0}$: process with stationary, independent increments.
- Brownian motion and Poisson processes belong to this class
- Combinations of these give Jump-Diffusion processes
- Replace deterministic time by a random business time given by a Gamma process: the Variance Gamma process [Carr, Madan, Chang 1998]. Infinite activity jumps:
  - small jumps describe the day-to-day "noise" that causes minor fluctuations in stock prices;
  - big jumps describe large stock price movements caused by major market upsets

SDE Simulation





- Variance Gamma process with different gamma distributed times, positive drift