Previously

We have already seen the market:

$$\begin{cases} dB_t &= \quad rB_t dt, \\ dS_t &= \quad \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}. \end{cases}$$

Whereas under $\mathbb{Q}$ measure $\mu = r$, i.e.:

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}}.$$

In an alternative process we aim to generalize the assumptions about constant parameters r and σ.

We can choose:

1. Constant: r, σ.
2. Deterministic- Piecewise constant: r_i, σ_i, on $[T_{i-1}, T_i]$.
3. Stochastic- time dependent: $r_t = f(t, W_t^r)$, $\sigma_t = g(t, W_t^{\sigma})$.

Stochastic Volatility: Model of Heston

Let us start with a stochastic volatility:

For the state vector $X_t = [S_t, \sigma_t]^T$ let us fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_n = \{\mathcal{F}_t : t \geq 0\}$ which satisfies the usual conditions, and X_t is assumed to be Markov relative to $(\mathcal{F}_t)$. The model that we are consider next is the so-called Heston Stochastic Volatility model:

$$\begin{cases} dS_t &= rS_t dt + \sqrt{\sigma_t} S_t dW_t^S & \text{Heston Equity process} \\ d\sigma_t &= -\kappa(\sigma_t - \overline{\sigma})dt + \gamma\sqrt{\sigma_t} dW_t^\sigma & \text{CIR process} \\ dB_t &= rB_t dt & \text{bank account} \end{cases}$$

And:

$$dW_t^S dW_t^\sigma = \rho dt$$

Stochastic Volatility: Model of Heston

Parameters interpretation.

- r is the rate of the return,
- $\overline{\sigma}$ is the long vol, or long run average price volatility ($\lim_{t\to\infty} \mathbb{E}\sigma_t = \overline{\sigma}$)
- κ is the rate at which σ_t reverts to $\overline{\sigma}$,
- γ is the vol- vol, or volatility of the volatility; as the name suggests, this determines the variance of σ_t.

Let us set: $T = 2$; $v_0 = 0.1$; $r = 0.05$; $S_0 = 1$; $\kappa = 0.2$; $\overline{\sigma} = 0.3$; $\gamma = 0.1$; $\rho = -0.8$;

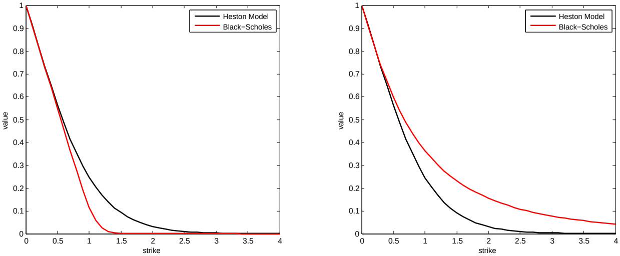
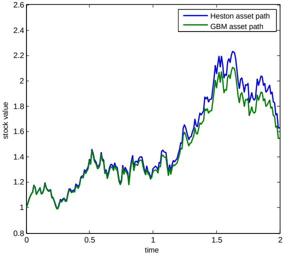
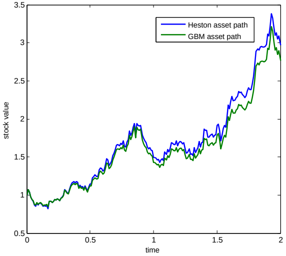


Figure: LEFT: $\sigma^{BS} = v_0$, RIGHT: $\sigma^{BS} = 60\%$

Sample paths of a geometric Brownian motion and the spot process in the Heston's model obtained with the same set of random numbers. Despite the fact that the volatility in the GBM is constant, whereas in Heston's model it is driven by a mean reverting process the sample paths are indistinguishable by mere eye.

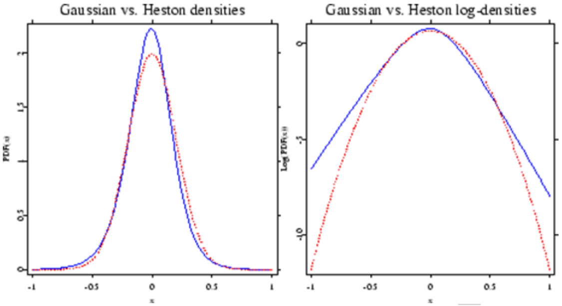




Stochastic Volatility: Model of Heston

A closer inspection of Heston's model does, however, reveal some important differences with respect to GBM.

1. the probability density functions of (log-)returns have heavier tails - exponential compared to Gaussian
2. they are similar to hyperbolic distributions (Weron; 2004), i.e. in the log-linear scale they resemble hyperbolas (rather than parabolas)



Stochastic Volatility: Model of Heston

```
clear all; clc; close all;
T=2; v_0=0.1; r=0.05; S_0=1; kappa=0.2; sigmabar=0.3; gamma=0.1; rho=-0.8; sigma2=0.3;
N=10000; %number of paths
M=200;   %number of steps
dt=T/M;
Vol=zeros(N,M+1);
S=zeros(N,M+1);
S2=zeros(N,M+1);
Vol(:,1)=v_0;
S(:,1)=S_0;
S2(:,1)=S_0;
for i=1:M
    Sigma=     [1 ,         rho      %r
                rho,          1,]*dt;      %s
    W=random('normal',0,1,[N,2]);
    C=chol(Sigma,'lower');
    W=(C*W')';
     Vol(:,i+1)=Vol(:,i)-kappa*(Vol(:,i)-sigmabar)*dt+   gamma*sqrt(Vol(:,i)).*W(:,1);
     S(:,i+1)=S(:,i)+r*S(:,i)*dt+sqrt(Vol(:,i)).*S(:,i).*W(:,2);
     S2(:,i+1)=S2(:,i)+r*S2(:,i)*dt+sigma2.*S2(:,i).*W(:,2);
end
VALUE=[];
VALUE2=[];
Strikes=0:0.1:4;
for k=Strikes
    VALUE(end+1)=mean(exp(-r*T).*max(S(:,end)-k,0));    %% CALL PRICE
    VALUE2(end+1)=blsprice(S_0, k, r , T, 0.6);
end
```

Financial mathematics aspects

- Knowledge: What product are we dealing with?
    - Contract specification (contract function),
    - Early-Exercise product, or not,
    - Product's lifetime,

  $\Rightarrow$ Determines the model for underlying asset (stochastic interest rate, or not...)

- Financial sub-problem: Product pricing or parameter calibration,
  $\Rightarrow$ All this determines the choice of numerical method.

Semi-Exact Solutions for option pricing

- It is generally difficult to find an analytic solution for multi-dimensional correlated stochastic differential equations;
- Monte-Carlo methods are straightforward but:
    - Depends on the sampling seed;
    - Involves sampling error;
    - Requires powerful computing machines;

Alternative methods need to be used!

- Although for complicated models, the distribution is unknown analytically, the corresponding characteristic function can be often derived analytically/semi-analytically;
- Alternatives to Monte-Carlo methods for pricing derivatives are Fourier based algorithms, which are based on determining characteristic function.

A pricing approach

$$V(S(t_0), t_0) = e^{-r(T-t_0)} \mathbb{E}^Q \{V(S(T), T) | S(t_0)\}$$

Quadrature:

$$V(S(t_0), t_0) = e^{-r(T-t_0)} \int_{\mathbb{R}} V(S(T), T) f(S(T)|S(t_0)) dS$$

- Trans. PDF, $f(S(T)|S(t_0))$, typically not available, but the characteristic function, ϕ, often is.

Motivation Fourier Methods

- Derive pricing methods that
  - are computationally fast
  - are not restricted to Gaussian-based models
  - should work as long as we have a characteristic function,

$$\phi(u) = \int_{-\infty}^{\infty} e^{iux} f(x) dx;$$

    (available for Lévy processes and also for Heston's model).
  - In probability theory a characteristic function of a continuous random variable X, equals the Fourier transform of the density of X.

Fourier Transformation

- The continuous Fourier transform is one of the most important transforms in the signal analysis.
- It transforms one function into another, which is called the frequency domain representation of the original function (where the original function is often a function in the time-domain).
- In this specific case, both domains are continuous and unbounded.
- There are several common conventions for defining the Fourier transform of a complex-valued Lebesgue integrable functions.
- In communications and signal processing,

Fourier Transformation

Suppose we have given a function $f : \mathbb{R} \to \mathbb{R}$ which is in L^1, i.e.,

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty,$$

and if $f(x)$ is continuous, then the Fourier transform of $f(x)$ is defined as:

$$\phi(u) = \mathbb{E}\left(e^{iuX}\right) = \int_{-\infty}^{+\infty} e^{iux} f(x) dx = \int_{-\infty}^{+\infty} e^{iux} dF(x),$$

where $x \in \mathbb{R}$.

Class of AJD processes

Suppose we have given a following system of SDEs:

$$d\mathbf{X}_t = \mu(\mathbf{X}_t)dt + \sigma(\mathbf{X}_t)d\mathbf{W}_t + d\mathbf{Z}_t,$$

Moreover, for processes in the affine jump diffusion (AJD) class it is assumed that drift, volatility, jump intensities and interest rate components are of the affine form, i.e.

$$\begin{aligned}
\mu(\mathbf{X}_t) &= a_0 + a_1\mathbf{X}_t \text{ for } (a_0, a_1) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \\
\lambda(\mathbf{X}_t) &= b_0 + b_1^T\mathbf{X}_t, \text{ for } (b_0, b_1) \in \mathbb{R} \times \mathbb{R}^n, \\
\sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^T &= (c_0)_{ij} + (c_1)_{ij}^T\mathbf{X}_t, (c_0, c_1) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \\
r(\mathbf{X}_t) &= r_0 + r_1^T\mathbf{X}_t, \text{ for } (r_0, r_1) \in \mathbb{R} \times \mathbb{R}^n.
\end{aligned}$$

Characteristic function for AJD

Duffie, Pan and Singleton (2000) have shown that for affine jump diffusion processes the discounted characteristic function defined as:

$$\phi(\mathbf{X_t}, \mathbf{t}, \mathbf{T}, \mathbf{u}) \equiv \mathbb{E}^{\mathbb{Q}} \left(e^{-\int_t^T r(\mathbf{X_s})ds} e^{i\mathbf{u}\mathbf{X_T}} | \mathcal{F}_t \right) \text{ for } \mathbf{u} \in \mathbb{C}^{\mathbf{n}},$$

with boundary condition:

$$\phi(\mathbf{X_T}, \mathbf{T}, \mathbf{T}, \mathbf{u}) = e^{i\mathbf{u}^T \mathbf{X_T}},$$

has a solution of a following form:

$$\phi(\mathbf{X_t}, \mathbf{t}, \mathbf{T}, \mathbf{u}) = e^{A(\mathbf{u}, t, T) + \mathbf{B}(\mathbf{u}, t, T)^T \mathbf{X_t}},$$

How to find the coefficients $A(\mathbf{u}, t, T)$ and $\mathbf{B}(\mathbf{u}, \mathbf{t}, \mathbf{T})^{\mathbf{T}}$?

Characteristic function for AJD

The coefficients $A(\mathbf{u}, t, T)$ and $\mathbf{B}(\mathbf{u}, \mathbf{t}, \mathbf{T})^{\mathbf{T}}$ have to satisfy the following system of Riccati-type ODEs[1]:

$$
\begin{aligned}
\frac{d}{d\tau} A(\mathbf{u}, \tau) &= -r_0 + \mathbf{B}^T a_0 + \frac{1}{2} \mathbf{B}^T c_0 \mathbf{B} \\
\frac{d}{d\tau} \mathbf{B}(\mathbf{u}, \tau) &= -r_1 + a_1^T \mathbf{B} + \frac{1}{2} \mathbf{B}^T c_1 \mathbf{B}.
\end{aligned}
$$

[1] Note that we do not consider jumps any more.

An example: Black-Scholes

For a given stock-process

$$dS_t = rS_t dt + \sigma S_t dW_t^{\mathbb{Q}},$$

with the money savings account B_t :

$$dB_t = rB_t dt,$$

the pricing PDE is given by:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial S^2} - rV = 0. \tag{1}$$

We know that the stock process S_t is not affine, therefore we define a transform:

$$x_t = \log S_t.$$

Black Scholes Model

For GBM we have the following SDE:

$$dS_t = rS_t dt + \sigma S_t dW_t^Q,$$

The process is not affine because of

$$\sigma(S_t)\sigma(S_t) = \sigma^2 S_t^2,$$

To consider the process into the affine class we define:

$$x_t = \log S_t,$$

which gives following SDE

$$d \log S_t = \left(r - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t^{\mathbb{Q}}$$

Black Scholes Model

The model is in the AJD class of processes, moreover we have:

$$\mu(x_t) = \underbrace{r - \frac{1}{2}\sigma^2}_{a_0} + \underbrace{0}_{a_1} x_t,$$

$$\sigma(x_t)\sigma(x_t) = \underbrace{\sigma^2}_{c0} + \underbrace{0}_{c_1} x_t,$$

and

$$r(x_t) = \underbrace{0}_{r_0} + \underbrace{0}_{r_1} x_t$$

Black Scholes Model

In order to find the characteristic function:

$$\phi(\tau) = e^{A(\tau)+B_x(\tau)x_0}$$

we set up the system of ODEs

$$\begin{cases} \dfrac{dB_x(\tau)}{d\tau} &= -r_1 + a_1 B_x(\tau) + \frac{1}{2} B_x(\tau) c_1 B_x(\tau) \\ \dfrac{dA(\tau)}{d\tau} &= -r_0 + a_0 B_x(\tau) + \frac{1}{2} B_x(\tau) c_0 B_x(\tau) \end{cases}$$

which reads:

$$\begin{cases} \dfrac{dB_x(\tau)}{d\tau} &= 0 \\ \dfrac{dA(\tau)}{d\tau} &= \left(r - \frac{1}{2}\sigma^2\right) B_x(\tau) + \frac{1}{2}\sigma^2 B_x(\tau) B_x(\tau) \end{cases}$$

Black Scholes Model

By taking the boundary conditions:

$$B_x(0, u) = iu,$$

and

$$A(0, u) = 0,$$

we finally obtain:

$$\begin{cases} B_x(\tau, u) = iu, \\ A(\tau, u) = \left[iu \left(r - \frac{1}{2}\sigma^2 \right) - \frac{1}{2}u^2\sigma^2 \right] \tau. \end{cases}$$

The characteristic function for GBM is now given by:

$$\phi(\tau) = e^{iu \log S_0 + iu\left(r - \frac{1}{2}\sigma^2\right)\tau - \frac{1}{2}u^2\sigma^2\tau}$$

An example: Black-Scholes Case

With this substitution we have:

$$V_u(x_t, t) = V(S_t, t).$$

So:

$$\begin{cases} \dfrac{\partial V}{\partial t} = & \dfrac{\partial V_u}{\partial_t}, \\ \dfrac{\partial V}{\partial S} = & \dfrac{\partial V_u}{\partial x}\dfrac{\partial x}{\partial S} = \quad \dfrac{1}{S}\dfrac{\partial V_u}{\partial x}, \\ \dfrac{\partial^2 V}{\partial S^2} = & -\dfrac{1}{S^2}\dfrac{\partial V_u}{\partial x} + \dfrac{1}{S^2}\dfrac{\partial^2 V_u}{\partial x^2} \end{cases}$$

The pricing PDE now reads:

$$\frac{\partial V_u}{\partial t} + rS\frac{1}{S}\frac{\partial V_u}{\partial x} + \frac{1}{2}\sigma^2 S^2\left(-\frac{1}{S^2}\frac{\partial V_u}{\partial x} + \frac{1}{S^2}\frac{\partial^2 V_u}{\partial x^2}\right) - rV_u = 0,$$

which simply becomes:

$$\frac{\partial V_u}{\partial t} + r\frac{\partial V_u}{\partial x} + \frac{1}{2}\sigma^2\left(-\frac{\partial V_u}{\partial x} + \frac{\partial^2 V_u}{\partial x^2}\right) - rV_u = 0.$$

An example: Black-Scholes Case

By setting

$$\tau = T - t,$$

we have:

$$-\frac{\partial V_u}{\partial \tau} + r\frac{\partial V_u}{\partial x} + \frac{1}{2}\sigma^2\left(-\frac{\partial V_u}{\partial x} + \frac{\partial^2 V_u}{\partial x^2}\right) - rV_u = 0.$$

By the results of Duffie-Pan-Singelton, we know that the discounted characteristic function has the following form:

$$\phi(u,\tau) = e^{A(u,\tau)+B(u,\tau)x},$$

with boundary condition: $\phi(u,0) = e^{iux}$. By partial differentiation we have:

$$\begin{cases} \dfrac{\partial \phi}{\partial \tau} = & \phi\left(\dfrac{\partial A}{\partial \tau} + x\dfrac{\partial B}{\partial \tau}\right), \\ \dfrac{\partial \phi}{\partial x} = & \phi B, \\ \dfrac{\partial^2 \phi}{\partial x^2} = & \phi B^2. \end{cases} \tag{2}$$

An example: Black-Scholes Case

Now, by substituting these quantities in the pricing PDE we have:

$$-\phi\left(\frac{\partial A}{\partial \tau}+\frac{\partial B}{\partial \tau}\right)+\left(r-\frac{1}{2}\sigma^2\right)\phi B+\frac{1}{2}\sigma^2\phi B^2-r\phi=0,$$

or

$$-\left(\frac{\partial A}{\partial \tau}+\frac{\partial B}{\partial \tau}\right)+\left(r-\frac{1}{2}\sigma^2\right)B+\frac{1}{2}\sigma^2 B^2-r=0.$$

From above we obtain the set of ODEs in the following way:

$$\begin{cases} \dfrac{\partial B}{\partial \tau}= & 0, \\ \dfrac{\partial A}{\partial \tau}= & \left(r-\dfrac{1}{2}\sigma^2\right)B+\dfrac{1}{2}\sigma^2 B^2-r. \end{cases} \tag{3}$$

An example: Black-Scholes Case

By using the boundary conditions we find

$$\begin{cases} B(u,\tau) = & iu, \\ A(u,\tau) = & \left(r - \frac{1}{2}\sigma^2\right) iu\tau - \frac{1}{2}\sigma^2 u^2 \tau - r\tau. \end{cases} \quad (4)$$

So the discounted characteristic function is given by

$$\phi(u,\tau) = e^{\left(r - \frac{1}{2}\sigma^2\right) iu\tau - \frac{1}{2}\sigma^2 u^2 \tau - r\tau + iux}.$$

Heston Model

From definition of Heston we have:

$$\begin{cases} dS_t &= r_t S_t dt + \sqrt{\sigma_t} S_t dW_t^1 \\ d\sigma_t &= -\kappa \left(\sigma_t - \overline{\sigma}\right) dt + \gamma \sqrt{\sigma_t} dW_t^2 \end{cases}$$

Is it affine?

$$\sigma(\mathbf{X_t})\sigma(\mathbf{X_t})^T = \begin{bmatrix} \sigma_t S_t^2 & S_t \sigma_t \gamma \rho_{x,\sigma} \\ S_t \sigma_t \gamma \rho_{x,\sigma} & \gamma^2 \sigma_t \end{bmatrix}$$

IT IS NOT AFFINE!

Heston Model

Let us define the log transform:

$$x_t = \log S_t,$$

$$\begin{cases} dx_t & = & \left(r_t - \frac{1}{2}\sigma_t\right) dt + \sqrt{\sigma_t} dW_t^1, \\ d\sigma_t & = & -\kappa\left(\sigma_t - \overline{\sigma}\right) dt + \gamma\sqrt{\sigma_t} dW_t^2. \end{cases}$$

Is it affine?? Let us have a look at the instantaneous covariance matrix:

$$\sigma(\mathbf{X_t})\sigma(\mathbf{X_t})^T = \left[\begin{array}{ll} \sigma_t & \sigma_t\gamma\rho_{x,\sigma} \\ \sigma_t\gamma\rho_{x,\sigma} & \gamma^2\sigma_t \end{array}\right]$$

Fourier Transformation

Suppose we have given a function $f : \mathbb{R} \to \mathbb{R}$ which is in L^1, i.e.,

$$\int_{-\infty}^{+\infty} |f(x)|\, dx < \infty,$$

and if $f(x)$ is continuous, then the Fourier transform of $f(x)$ is defined as:

$$\phi(u) = \mathbb{E}\left(e^{iuX}\right) = \int_{-\infty}^{+\infty} e^{iux} f(x) dx = \int_{-\infty}^{+\infty} e^{iux} dF(x),$$

where $x \in \mathbb{R}$.

Fourier Transformation

Assuming that $\phi(u)$ is in L^1, the original function can be recovered from its Fourier transform by inversion:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\phi(u)du.$$

Now, suppose that we discretize the domain for x, and u into N grid points, then we consider the vectors $\mathbf{f}, \phi \in \mathbb{C}^N$:

$$\mathbf{f} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{pmatrix}, \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \\ \phi_N \end{pmatrix}. \quad (5)$$

Fourier Transformation

If we let

$$\omega_N = e^{-\frac{2\pi i}{N}},$$

the discretized -Fourier Transform- matrix $M \in \mathbb{C}^{\mathbb{N}\times\mathbb{N}}$ is then defined as:

$$M = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N^1 & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{N(N-1)} & \dots & \omega_N^{(N-1)(N-1)} \end{pmatrix}, \qquad (6)$$

that is,

$$M_{nk} = \omega_N^{(n-1)(k-1)}.$$

Fourier Transformation

Now, the discrete Fourier transform $\mathbf{f}$ of ϕ is given by the matrix multiplication:

$$\mathbf{f} = M\phi,$$

or equivalently:

$$f_k = \sum_{n=1}^{N} \phi_n e^{-\frac{2\pi i}{N}(n-1)(k-1)} = \sum_{n=1}^{N} \phi_n \omega_N^{(n-1)(k-1)}.$$

Fourier Transformation

Lemma (Inversion Lemma)

Let $\phi(u)$ be a characteristic function and $f(x)$ be a probability density function of some continuous variable X. Then we have:

$$f(x) = \frac{1}{\pi} \Re \left(\int_0^\infty e^{-iux} \phi(u) du \right)$$

Proof

From Fourier inverse we have:

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iux}\phi(u)du = \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-iux}\phi(u)du\right. \\
&+ \left.\int_{0}^{+\infty} e^{-iux}\phi(u)du\right).
\end{aligned}
$$

where the first integral on the RHS can be written as:

$$
\begin{aligned}
\int_{-\infty}^{0} e^{-iux}\phi(u)du &= \int_{0}^{\infty} e^{ivx}\phi(-v)dv \\
&= \int_{0}^{\infty} \overline{e^{-iux}\phi(u)}du \\
&= \overline{\int_{0}^{+\infty} e^{-iux}\phi(u)du}
\end{aligned}
$$

Proof

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\left(\int_{-\infty}^{0} e^{-iux}\phi(u)du + \int_{0}^{+\infty} e^{-iux}\phi(u)du\right) \\
&= \frac{1}{2\pi}\left(\int_{0}^{+\infty} e^{-iux}\phi(u)du + \overline{\int_{0}^{+\infty} e^{-iux}\phi(u)du}\right), \\
&= \frac{1}{\pi}\Re\left(\int_{0}^{\infty} e^{-iux}\phi(u)du\right).
\end{aligned}
$$

The lemma above shows that we need to find the integral

$$
\int_{0}^{\infty} e^{-iux}\phi(u)du = \int_{0}^{\infty} \gamma(u)du.
$$

Now, we define a trapezoidal integration over domain $[0, \tau]$, for which we have:

$$\begin{aligned}\int_0^{\tau} \gamma(u)du &\approx \frac{\Delta_u}{2}\left[\gamma(u_1) + 2\sum_{n=2}^{N-1}\gamma(u_n) + \gamma(u_N)\right] \\ &= \Delta_u\left[\sum_{n=2}^{N-1}\gamma(u_n) + \frac{1}{2}\left(\gamma(u_1) + \gamma(u_N)\right)\right].\end{aligned}$$

If we set

$$\tau = N\Delta_u,$$

$$u_n = (n-1)\Delta_u$$

$$x_k = -b + \Delta_x(k-1),$$

where: $k = 1, \ldots, N$ to be the grid in the x-domain.

Fourier Transform Derivations

The constant b is a tuning parameter which can be freely chosen, but here we take:

$$b = \frac{N\Delta_x}{2}.$$

So now, we have:

$$\begin{aligned}\int_0^\tau \gamma(u)du &\approx \Delta_u \Bigg[\sum_{n=1}^{N} e^{-i[(n-1)\Delta_u][-b+\Delta_x(k-1)]}\phi(u) \\ & \quad -\frac{1}{2}\left[e^{-ixu_1}\phi(u_1)+e^{ixu_N}\phi(u_N)\right]\Bigg]\end{aligned}$$

$$\begin{aligned}\int_0^\tau \gamma(u)du &\approx \Delta_u \Bigg[\sum_{n=1}^{N} e^{-i\Delta_x\Delta_u(n-1)(k-1)}e^{i(n-1)b\Delta_u}\phi(u) \\ & - \quad \frac{1}{2}\left[e^{-ixu_1}\phi(u_1)+e^{ixu_N}\phi(u_N)\right]\Bigg]\end{aligned}$$

Fourier Transform Derivations

If we set

$$\Delta_x \Delta_u = \frac{2\pi}{N},$$

we obtain

$$\begin{aligned}\int_0^\tau \gamma(u)du &\approx \Delta_u \left[\sum_{n=1}^{N} e^{-i\frac{2\pi}{N}(n-1)(k-1)} e^{i(n-1)b\Delta_u} \phi(u) \right. \\ &- \left. \frac{1}{2}\left[e^{-ixu_1}\phi(u_1) + e^{ixu_N}\phi(u_N) \right] \right].\end{aligned}$$

FFT Implementation

So finally we obtain:

$$
\begin{aligned}
f(x) &= \frac{1}{\pi}\Re\left(\int_0^{\infty} e^{-iux}\phi(u)du\right) \\
&= \frac{1}{\pi}\Re\left\{\Delta_u\left[\sum_{n=1}^{N} e^{-i\frac{2\pi}{N}(n-1)(k-1)} e^{i(n-1)b\Delta_u}\phi(u) - \frac{1}{2}(\gamma_1+\gamma_2)\right]\right\}.
\end{aligned}
$$

where

$$\gamma_1 = e^{-ixu_1}\phi(u_1),$$

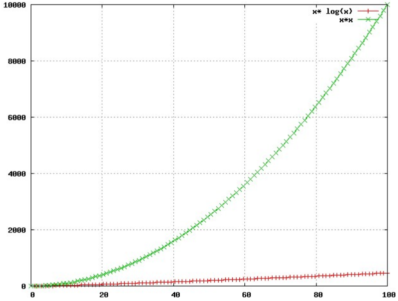
and

$$\gamma_2 = e^{ixu_N}\phi(u_N).$$

Why this kind of representation?

FFT Implementation

- This is a matrix multiplication, which requires about N^2 (complex) multiplications and N^2 (complex) additions. The number of arithmetic operations is of order N^2, i.e., $O(N^2)$.
- In 1965 Cooley and Tukey showed that it is possible to have the DFT evaluated in $O(N \log_2 N)$ operations.
- The algorithm was called the Fast Fourier Transform, FFT. Standard routines are available in many computer languages.



FFT Implementation

Let us have a look at the FFT algorithm in Matlab!

```
%FFT Discrete Fourier transform.
%   FFT(X) is the discrete Fourier transform (DFT) of vector X.  For
%   matrices, the FFT operation is applied to each column. For N-D
%   arrays, the FFT operation operates on the first non-singleton
%   dimension.
%
%   FFT(X,N) is the N-point FFT, padded with zeros if X has less
%   than N points and truncated if it has more.
%
%   FFT(X,[],DIM) or FFT(X,N,DIM) applies the FFT operation across the
%   dimension DIM.
%
%   For length N input vector x, the DFT is a length N vector X,
%   with elements
%                    N
%      X(k) =       sum  x(n)*exp(-j*2*pi*(k-1)*(n-1)/N), 1 <= k <= N.
%                   n=1
%   The inverse DFT (computed by IFFT) is given by
%                    N
%      x(n) = (1/N) sum  X(k)*exp( j*2*pi*(k-1)*(n-1)/N), 1 <= n <= N.
%                   k=1
```

Let us make an experiment!

FFT Implementation

We take the characteristic function of the normal distribution:

$$\phi(t) = \exp\left(\mu i t - \frac{1}{2}\sigma^2 t^2\right)$$

and we take $\mu = 1$, $\sigma = 1$. Now, we compare the original pdf and the FFT approximation, with Simpson's rule,

N	2^4	2^6	2^8	2^{10}	2^{12}
time [ms]	0.98	1.1	1.4	2.9	10.0
SSE	5.6	4.3	7.8E-4	5.7E-7	5.7E-7
$F(\infty)$	6.6	4.3	9.9922	1.000	1.000

Pricing

How to get price of a Call Option if the CHF of the asset is known?

- Gil-Palaez Inverse theorem,

$\Rightarrow$ Carr-Madan Pricing,

$\longrightarrow$ CONV method for early-exercise Bermudan options

$\Rightarrow$ COS Method

Carr-Madan Pricing Technique

Let us assume that the discounted characteristic function is found. To price plain vanilla options, we define: S_T denote the price at maturity of the underlying asset of a European call with strike K, moreover $S \equiv \log(S_T)$ with associated risk neutral density given by $f_T(s)$ under measure $\mathbb{Q}$. Then the Fourier transform of $f_T(s)$, or equivalently the characteristic function of S, can be written as

$$\phi_T(u) = \int_{-\infty}^{+\infty} e^{ius} f_T(s) ds$$

If we take $k \equiv \log K$, risk neutral valuation then yields:

$$\Pi(t, T, K) \quad = \quad \int_{-\infty}^{+\infty} e^{-\int_t^T r_s ds} \left(e^S - e^k\right)^+ f_T(s) ds.$$

Carr-Madan Pricing Technique

Since

$$\lim_{K \to 0} \Pi(t, T, K) = \lim_{k \to -\infty} \Pi(t, T, e^k) = S_0,$$

$\Pi(t, T, e^k)$ is not in L^1, as $\Pi(t, T, e^k)$ does not tend to zero for $k \to -\infty$.
Let us therefore consider the modified call price

$$\pi(t, T, k) \equiv e^{\alpha k} \Pi(t, T, e^k)$$

for $\alpha > 0$ assuming existence of Fourier transform of $\pi(t, T, k)$ we have:

$$\psi_T(v) \equiv \pi\widehat{(t, T, k)} = \int_{-\infty}^{+\infty} e^{ivk} \pi(t, T, k) dk.$$

Inverting gives:

$$\pi(t, T, k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-ivk} \psi_T(v) dv.$$

Carr-Madan Pricing Technique

We see that the last expression is equivalent with

$$\begin{aligned} \Pi(t,T,K) &= \frac{e^{-\alpha \log K}}{2\pi} \int_{-\infty}^{+\infty} e^{-iv \log K} \psi_T(v) dv \\ &= \frac{e^{-\alpha \log K}}{\pi} \Re \left(\int_0^{+\infty} e^{-iv \log K} \psi_T(v) dv \right), \end{aligned}$$

where

$$\psi_T(v) = \frac{1}{\alpha + \alpha^2 - v^2 + iv(2\alpha + 1)} \mathbb{E}^{\mathbb{Q}} \left(e^{-\int_t^T r_s ds} e^{S_T(1+\alpha+iv)} \right).$$

Carr-Madan Pricing Technique

To simplify computations we follow Duffie, Pan and Singleton and derive a discounted characteristic function of equity under the risk neutral measure:

$$\phi(u, S_T, t, T) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} e^{iuS_T}\right)$$

so:

$$\phi\left((v - i(1+\alpha)), S_T, t, T\right) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds} e^{(1+\alpha+iv)S_T}\right).$$

Carr-Madan Pricing Technique

So finally the call price is:

$$\Pi(t, T, K) = \frac{e^{-\alpha \log K}}{\pi} \Re \left(\int_0^{+\infty} e^{-iv \log K} \psi_T(v) dv \right),$$

where:

$$\psi_T(v) = \frac{\phi\left((v - i(1+\alpha)), S_T, t, T\right)}{\alpha + \alpha^2 - v^2 + iv(2\alpha + 1)}.$$

We know that this can be approximated by the trapezoidal or the Simpson rule

Carr-Madan Pricing Technique

The approximation is given by:

$$\begin{aligned}\Pi(t, T, k_u) &\approx \frac{e^{-\alpha k_u}}{\pi}\Re\left(\Delta v\left(\sum_{n=1}^{N}\omega_N^{(n-1)(k-1)}e^{iv_n b}\psi_T(v_n).\right).\right) \\ &= -\frac{1}{2}\left(g(v_1)+g(v_N)\right)\Bigg)\Bigg),\end{aligned}$$

with the condition:

$$\Delta v \Delta k = \frac{2\pi}{N}$$

and where:

$$g(v) \equiv e^{-ivk}\psi_T(v), k_u = -b + \Delta k(u-1).$$

Example: Black-Scholes model

Example The characteristic function for the Black-Scholes asset price is given by:

$$\phi(u) = \exp\left(i(\log(S_0) + (r - \frac{1}{2}\sigma^2)T)u - \frac{1}{2}\sigma^2 T u^2 \right),$$

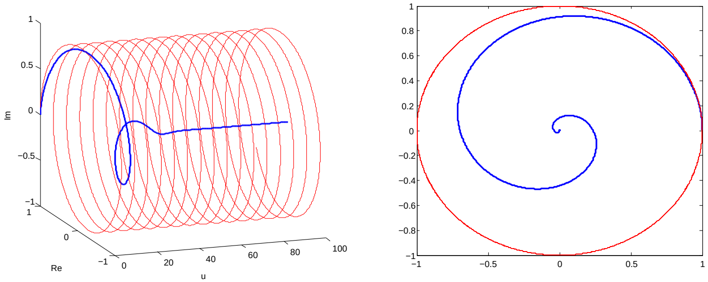


Figure: Characteristic Function for lognormal distribution

Example- Black, Scholes model

We set: $\sigma = 0.3$, $T = 1$, $r = 0.06$ and $S_0 = 1$. We have generated 10000 paths with step 1000, Time needed for calculation:

Monte Carlo: $5[s]$,

Car Madan- FFT: $0.1[s]$,

Exact Solution: $0.06[s]$.

Table: Comparison of the results.

Strike K	0.01	0.3	0.5	1	1.5	2
Error MC	-7E-4	-7E-4	-7E-4	-0.03	-7E-4	-2E-3
Error FFT	9E-6	8E-6	2E-6	1E-5	-4E-5	3E-5

How the results ar influenced by the Maturity T?

Example: Black-Scholes model

```
clear all; clc; close all;
T=1;  sigma=0.3; r=0.08; S_0=1;
N=10000; %number of paths
M=100;  %number of steps
dt=T/M;Strikes=0.01:0.1:10;
%Monte Carlo
tic
    noise=random('normal',0,sqrt(dt),[N,M]);
    S=zeros(N,M);
    S(:,1)=S_0*ones(N,1);
        for i=1:1:M-1
            S(:,i+1)=S(:,i)+r*S(:,i)*dt+S(:,i)*sigma.*noise(:,i);
        end
        Call_MC=zeros(length(Strikes),1);
        for k=1:length(Strikes)
            Call_MC(k)=exp(-r*T)*mean(max(S(:,end)-Strikes(k),0));
        end
toc
%FFT- CAR Madan
tic
    CallValue= exp(-r*T)*BS(r,S_0,T,sigma,Strikes); %%FFT MODEL
toc
plot(Strikes,Call_MC,'k','LineWidth',1.5)
hold on;
plot(Strikes,CallValue,'r','LineWidth',1.5)%, Strikes,VALUE2,'r')
%Exact Solution
tic
    [ExactCall]=blsprice(S_0, Strikes, r, T, sigma, 0);
toc
plot(Strikes,ExactCall,'--b','LineWidth',1.5)
legend('MC','FFT','Exact')
xlabel('Strikes [K]')
ylabel('Call Value')
```

Example: Black-Scholes model

```
function [CallValue]= BS(r,S_0,tau,sigma,strike)
cF=20; N=4096; i=complex(0,1); alphaF=0.75;
etaF = cF/N;                % discretization grid on the frequency axis = delta omega
bF =pi/etaF;                % the log strike range from -b to b where b = N*lambda/2=pi/eta
U1 = [0:N-1]'*etaF;
U2= (U1 - (alphaF+1)*i);    % shifting the frequency because of the carr-madan derivation
lambdaF = 2*pi/(N*etaF);
u=U2;

Bx=i*u;

phi=i*u*r*tau+Bx*log(S_0)-i*0.5*sigma^2*tau*u-0.5*sigma^2*tau*u.^2;
value=exp(phi); %CHF

psi=value./(alphaF^2 + alphaF - U1.^2 + i*(2*alphaF +1)*U1);
SimpsonW = (3 + (-1).^[1:N] - [1, zeros(1,N-1)]);
SimpsonW(N)=0;
SimpsonW(N-1)=1;
FftFunc = exp(i*bF*U1).*psi.*SimpsonW';
payoff = real(etaF*fft(FftFunc)/3);
K=exp(-bF:lambdaF:bF-lambdaF);
intepolated_payoff= spline(K, payoff);%spline interpolation
cT=ppval(intepolated_payoff,strike);
CallValue = exp(-log(strike)*alphaF).*cT/pi;
```

- Generalization to stochastic interest rates
- We have already derived the discounted characteristic function for the Black-Scholes model and can make a next step, defining a simple hybrid model.
- The hybrid consists of two parts: An equity part, modeled by Black-Scholes Geometric Brownian Motion and a second part: The stochastic interest rate part will be done via a Hull-White process.
- For the state vector $X_t = [S_t, r_t]^T$ let us fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathcal{F}_n = \{\mathcal{F}_t : t \geq 0\}$ which satisfies the usual conditions, and X_t is assumed to be Markov relative to $(\mathcal{F}_t)$.

Black-Scholes-Hull-White Model

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma S_t dW_t^S \\ dr_t &= \lambda \left(\theta_t - r_t\right) dt + \eta dW_t^r \end{aligned}$$

The interest rate part can be decomposed into two parts: stochastic and deterministic, i.e.: $r_t = \widetilde{r}_t + \psi_t$ where

$$\begin{aligned} d\widetilde{r}_t &= -\lambda \widetilde{r}_t dt + \eta dW_t^r \\ \widetilde{r}_0 &= 0 \end{aligned}$$

and

$$\psi(t) = e^{-\lambda t} r_0 + \lambda \int_0^t e^{-\lambda(t-s)} \theta_s ds.$$

Black-Scholes-Hull-White Model

Let us define $x_t = log(S_t)$, then by the Ito formula we have

$$dx_t = \left(r_t - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t^S$$

so the system of SDE's becomes:

$$\begin{aligned} dx_t &= \left(\widetilde{r}_t + \psi_t - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \\ d\widetilde{r}_t &= -\lambda \widetilde{r}_t dt + \eta dW_t^r \end{aligned}$$

In order to simplify the calculations we introduce a new variable $x_t = \widetilde{x}_t + \Phi_t$ where $\Phi_t = \int_0^t \psi_s ds$ with

$$d\widetilde{x}_t = \left(\widetilde{r}_t - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t$$

Black-Scholes-Hull-White Model

Finally we obtain simplify the system of SDEs:

$$\begin{aligned} d\widetilde{x}_t &= \left(\widetilde{r}_t - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \\ d\widetilde{r}_t &= -\lambda \widetilde{r}_t dt + \eta dW_t^r \end{aligned}$$

Following Duffie, Pan and Singleton we have the following form for the discounted characteristic function

$$\phi(u, X(t), t, T) = e^{-\int_t^T \psi_s ds + iu^T [\Phi_T, \psi_T]^T} e^{A(u,\tau) + B_x(u,\tau)\widetilde{x}_t + B_r(u,\tau)\widetilde{r}_t}$$

where $X^* = [\widetilde{x}_t, \widetilde{r}_t]^T$, where $\tau = T - t$.

Black-Scholes-Hull-White Model

If we look at the chf at time T we got obvious boundary condition (price at time T is already known so no randomness is involved):

$$\phi(u, X^*(T), T, T) = \mathbb{E}_T^{\mathbb{Q}}\left(e^{iu^T X^*(T)}\right) = e^{iu^T X^*(T)} = e^{iu\widetilde{x}_T}$$

as a vector u we have taken $u = [1, 0]^T$- we are only interested in one dimensional characteristic function for equity. The boundary conditions that we have to consider are following

- $\tau = 0, (t = T) \Rightarrow B_x(u, 0) = iu, A(u, 0) = 0, B_r(u, 0) = 0$

We need to obtain the solution of:

$$\begin{aligned}\frac{dA}{d\tau} &= -r_0 + B^T a_0 + \frac{1}{2} B^T c_0 B \\ \frac{dB}{d\tau} &= -r_1 + a_1^T B + \frac{1}{2} B^T c_1 B\end{aligned}$$

Black-Scholes-Hull-White Model

After some calculations we get:

$$
\begin{aligned}
\frac{dB_x}{d\tau} &= 0 \Rightarrow B_x = iu \\
\frac{dB_r}{d\tau} &= -1 + B_x - \lambda B_r \Rightarrow \frac{dB_r}{d\tau} = -1 + iu - \lambda B_r \\
\frac{dA}{d\tau} &= -\frac{1}{2}\sigma^2 iu + B_x^2\sigma^2 + 2B_r B_x \sigma\eta\rho_{x,r} + B_r^2\eta^2
\end{aligned}
$$

Simple calculations give following result:

$$
\begin{aligned}
B_x &= iu \\
B_r &= (iu - 1)\lambda^{-1}(1 - e^{-\lambda\tau}) \\
A &= -\frac{1}{2}\sigma^2 iu\tau - u^2\sigma^2\tau + 2iu\sigma\eta\rho_{x,r}(1 + iu)\lambda^{-1}\left(\tau + \frac{e^{-\tau\lambda} - 1}{\lambda}\right) \\
&\quad - \frac{(1 + iu)^2\left(3 + e^{-2\lambda\tau} - 4e^{-\lambda\tau} - 2\lambda\tau\right)}{2\lambda^3}
\end{aligned}
$$

If one is assuming that θ_t is just a constant, then we have:

$$\begin{aligned}\frac{dB_x}{d\tau} &= 0 \Rightarrow B_x = iu \\ \frac{dB_r}{d\tau} &= -1 + B_x - \lambda B_r \Rightarrow \frac{dB_r}{d\tau} = -1 + iu - \lambda B_r \\ \frac{dA}{d\tau} &= \alpha - \beta B_r + \theta B_r^2\end{aligned}$$

where: $\alpha = -\frac{1}{2}\sigma^2 iu - \frac{1}{2}u^2\sigma^2$,$\beta = -\lambda\theta - iu\rho\eta\sigma$, $\gamma = \frac{1}{2}\eta^2$ resulting:

$$B_x = iu \quad (7)$$

$$B_r = (iu-1)\lambda^{-1}(1 - e^{-\lambda\tau}) \quad (8)$$

$$A = \frac{\beta - D}{2\gamma}\left(\frac{1 - e^{-\tau D}}{1 - e^{-\tau D\left(\frac{a}{b}\right)}}\right) \quad (9)$$

where: $a = \frac{\beta + D}{2\gamma}$, $b = \frac{\beta - D}{2\gamma}$, $D = \sqrt{\beta^2 - 4\alpha\gamma}$.

Black-Scholes-Hull-White Model

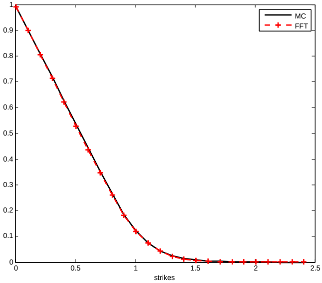


Figure: Call prices for a strip of strikes: Results for $\lambda = 1$, $T = 0.5$, $\theta = 0.1$, $\rho = -0.6$, $\eta = 0.1$, $\sigma = 0.3$, $r_0 = 0.2$, $S_0 = 1$ for 1000 paths with 100 steps.

Black-Scholes-Hull-White Model

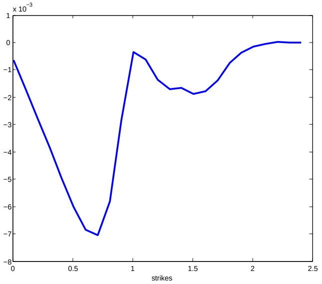


Figure: Difference between approaches (FFT-MC): Results for $\lambda = 1$, $T = 0.5$, $\theta = 0.1$, $\rho = -0.6$, $\eta = 0.1$, $\sigma = 0.3$, $r_0 = 0.2$, $S_0 = 1$ for 1000 paths with 100 steps.

Black-Scholes-Hull-White Model

```
clear all; clc; close all;
lambda=1; T=0.5; theta=0.1; rho=-0.6; eta=0.1; sigma=0.3; r_0=0.2; S_0=1;

N=1000; %number of paths
M=100;  %number of steps
dt=T/M;
IntegralOverR=[];
for i=1:N
    V=zeros(2,M);
    V(1,1)=r_0;
    V(2,1)=S_0;
    for j=1:M
        C=[1,        rho      %x
           rho,      1, ]*dt; %r
        Noise=mvnrnd([0,0]',C);
    V(1,j+1)=V(1,j)+lambda*(theta-V(1,j))*dt+eta*Noise(1);
    V(2,j+1)=V(2,j)+V(1,j)*V(2,j)*dt+sigma*V(2,j)*Noise(2);
    end
    IntegralOverR=[IntegralOverR;sum(V(1,:))*dt];
    Frate(i)=V(1,end);              %Final Interest Rate
    Fasset(i)=V(2,end);             %Final Asset Price
end
mean(exp(-IntegralOverR)'.*Fasset)
VALUE=[];
VALUE2=[];
Strikes=0.01:0.1:2.5;
for k=Strikes
    VALUE(end+1)=mean(exp(-IntegralOverR)'.*max(Fasset-k,0));   %% CALL PRICE
    %VALUE2(end+1)=blsprice(S_0, k,r_0 , T, sigma);
end
plot(Strikes,VALUE,'k')%, Strikes,VALUE2,'r')
[CallValue]= BSHW3(r_0,S_0,T,lambda,theta,rho,eta,sigma,Strikes); %%FFT MODEL
hold on;
plot(Strikes,CallValue,'r')%, Strikes,VALUE2,'r')
```

Black-Scholes-Hull-White Model

```
function [CallValue]= BSHW3(r_0,S_0,tau,lambdaH,theta,rho,etaH,sigma,strike)
cF=100; N=2*4096; i=complex(0,1); alphaF=0.75;
etaF = cF/N;                 % discretization grid on the frequency axis = delta omega
bF =pi/etaF;                 % the log strike range from -b to b where b = N*lambda/2=pi/eta
U1 = [0:N-1]'*etaF;
U2= (U1 - (alphaF+1)*i);     % shifting the frequency because of the carr-madan derivation
lambdaF = 2*pi/(N*etaF);
u=U2;
x_0=log(S_0);
        beta=(-lambdaH*theta-i*u*rho*etaH*sigma);
        theta=(0.5*etaH^2);
        alfa=(-0.5*sigma^2*i*u-0.5*u.^2*sigma^2);%0.5*(i-u).*u*sigma^2;
        D=sqrt(beta.^2-4*alfa*theta);
        b=(beta-D)./(2*theta);
        G=(beta-D)./(beta+D);
[value1]=Bx(u,i);
[value2]=Br(lambdaH,i,u,tau);
[value3]=A(D,G,b,tau);
value=exp(value3+value2*r_0+value1*x_0); %CHF

psi=value./(alphaF^2 + alphaF - U1.^2 + i*(2*alphaF +1)*U1);
SimpsonW = (3 + (-1).^[1:N] - [1, zeros(1,N-1)]);
SimpsonW(N)=0;
SimpsonW(N-1)=1;
FftFunc = exp(i*bF*U1).*psi.*SimpsonW';
payoff = real(etaF*fft(FftFunc)/3);
K=exp(-bF:lambdaF:bF-lambdaF);
intepolated_payoff= spline(K, payoff);%spline interpolation
cT=ppval(intepolated_payoff,strike);
CallValue = exp(-log(strike)*alphaF).*cT/pi;
```

Further Reading: Basics

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