Start

- The asset price follows the lognormal random walk:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

- Interest rate r and volatility σ are known functions of t.
- Transaction costs for hedging are not included in the model.
- No dividend is paid during the life of the option.
- There are no arbitrage possibilities.

$\Rightarrow$ Black-Scholes partial differential equation:
(for a European option)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

- Nobel prize in 1997 for Merton and Scholes (Black died in 1995).
- The Black-Scholes equation is a parabolic partial differential equation

Meaning of the terms

- The Black-Scholes equation can be interpreted as a convection-diffusion-reaction equation
- The second derivative is typically for diffusion-type problems, with the coefficient in front we would think of diffusion in an inhomogeneous medium.
- The first order term: $rS\partial V/\partial S$ can be thought of as a convection term
- If this would be a physical system's representation (diffusion of smoke particles in the atmosphere, then the convection term would be due to a breeze, blowing the smoke in a preferred direction.
- $-rV$ is a reaction term. It is used a a model for the decay of a radioactive body, with the half-life being related to r.

Hedging

- Writers of options use them as an insurance to reduce risk against unexpected movements in the market.
- Suppose a portfolio with S (shares) and P (puts). If the price of S falls, the value of the portfolio depends on the ratio of S and P.
- A ratio exists, which results in no movement in the value of the portfolio. This ratio is instantaneously risk-free.
- A reduction of risk, for example by combining a number of S and P in a portfolio is called hedging.
- (If a market maker is able to sell an option for more than it is worth and then hedge all the risk for the time until the options expiry, he can obtain a risk-free profit !)

Hedging

- Any reduction in randomness is generally termed hedging.
- The perfect elimination of risk, by exploiting correlation between two instruments (in this case an option and the underlying) is generally called delta hedging.
- Delta hedging is a dynamic hedging strategy: From one time step to the next Δ changes, since it is, like V a function of ever changing variables S and t. This means that the perfect hedge must be continually rebalanced.
- The resulting equation for option prices contains the obvious variables and parameters such as the underlying, time, volatility, but there is no mention of the drift rate μ !
- This means that if two people agree on the volatility of an asset they will agree on the value of its derivatives, even if they have differing estimates of the drift.

Delta Hedged Portfolios

- If we were guaranteed to get a return greater than r from the delta-hedged portfolio then what we could do is borrow from the bank, paying interest at the rate r, invest in the risk-free option/stock portfolio and make a profit.
- If, on the other hand the return were less than the risk-free interest rate we should go short on the option, delta hedge it, and invest the cash in the bank.
- Either way, we make a riskless profit in excess of the risk-free rate of interest.
- At this point we say that, all things being equal, the action of investors buying and selling to exploit the arbitrage opportunity will cause the market to move in the direction that eliminates the arbitrage.

Important Quantities

Sensitivity Analysis

- The important quantities to be calculated are the so-called hedge parameters.
- Delta: $\Delta(S,t)$ the rate of change of the value of the option (or portfolio) with respect to S. The largest random component of a portfolio is eliminated. It indicates the number of shares, that should be kept with each option issued in order to cope with a loss in the case of exercise.
$$\Delta = \frac{\partial V}{\partial S}$$
- Gamma: indicates the change in Delta
$$\Gamma = \frac{\partial^2 V}{\partial S^2}$$
- If Gamma is low, it is only necessary to sometimes change the portfolio. If it is high, the portfolio under consideration results only for a very short period of time in a risk-less scenario.
- There are several other important hedging parameters.

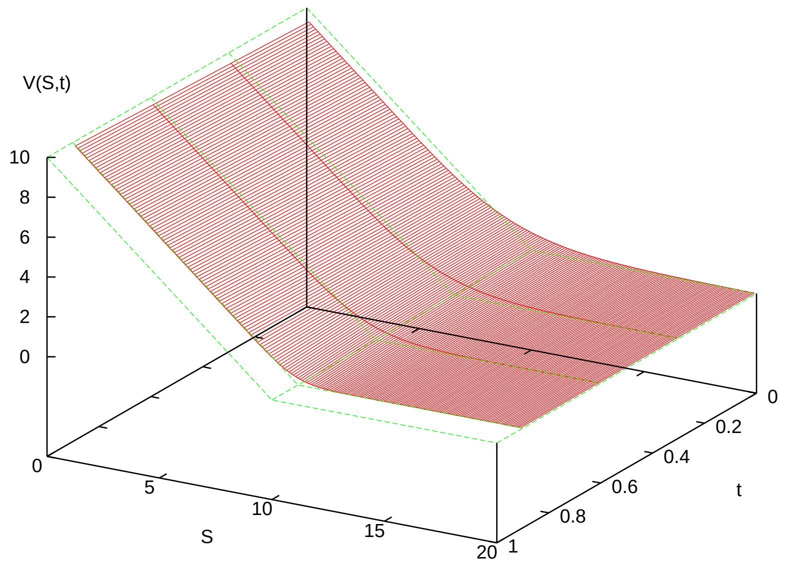
Boundary conditions

- The final conditions for a put and a call option have already been presented. For a call: $C(S,T) = \max(S-K,0)$.
- Boundary conditions for a call at $S=0$ and $S \to \infty$
  At $S=0$, we have $dS=0$, and the pay-off is 0. So, $C(0,t)=0$
- At $S \to \infty$, one uses $C(S,t) \sim S - Ke^{-r(T-t)}$
- With these conditions, the European call option can be solved exactly !
- Put option: Final condition: $P(S,T) = \max(K-S,0)$
- Boundary conditions for a put at $S=0$: $P(0,t) = Ke^{-r(T-t)}$.
- Boundary conditions for a put at $S \to \infty$: $P(S,t) \to 0$.

Numerical Solution

A result:

- In a vast majority of cases we must solve the Black-Scholes equation numerically.
- However, parabolic equations are easy to solve numerically.
- European Put, $K = 10$, $r = 0.06$, $\sigma = 0.3$, $T = 1$ year



- Finite Differences (central discretization), Crank-Nicolson implicit time discretization

Equivalence

Transformation to Diffusion Equation

- It can sometimes be useful to transform the basic Black-Scholes equation by a change of variables
- Writing $V(S,t) = e^{\alpha x + \beta \tau} y(x,\tau)$, with

$$S = e^x, t = T - 2\tau/\sigma^2, \alpha = -\frac{1}{2}(\frac{2r}{\sigma^2} - 1), \beta = -\frac{1}{4}(\frac{2r}{\sigma^2} + 1)^2$$

- Then $y(x,\tau)$ satisfies the basic diffusion equation

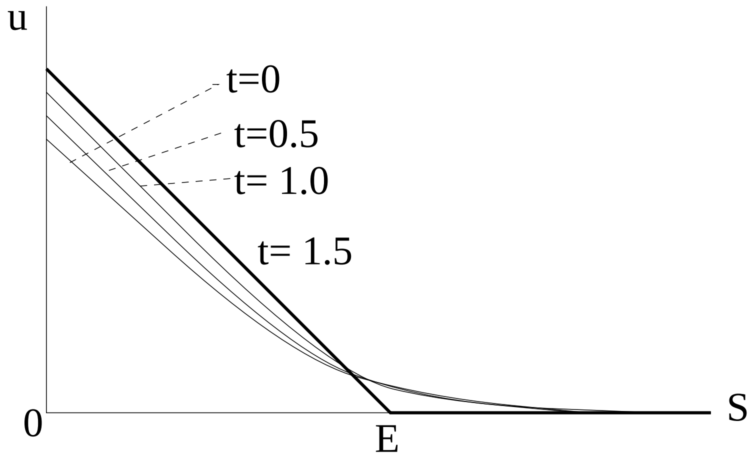
$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial x^2}$$

- With terms $S^j \partial^j V / \partial S^j$, we deal with an Eulerian differential equation: The convection-diffusion-reaction type equation can be transformed into a heat equation.

TU Delft

American Options

- American options are contracts that may be exercised early, prior to expiry. Most traded stock and future options are American style, but most index options are European.
- The right to exercise at any time is clearly valuable. The value of an American option cannot be less than the equivalent European option.
- The important question for the holder is; when should he exercise best ? This is what makes American options more interesting than the European ones.



• European put option is in a certain s range less than the pay-off function.

• For an American option exercise is permitted at any time during the life of an option. In this S range, $P(s,t) < \max(K-S,0)$.

• Buying the option for P and the asset for S, exercise the put immediately, i.e., sell the asset for K, would lead to a risk-free profit of $K-P-S$

$\Rightarrow$ When early exercise is permitted, a constraint $V(S,t) \geq \max(S-K,0)$ must be imposed. (For an American call option, a similar constraint can be formulated.)

$\Rightarrow$ A special S value exists, S^*, to one side of S^* one should hold the option, to the other side, one should exercise the option.

- We first analyze a contract without expiry; it can be exercised at any time. So the solution is independent of time, $V(S)$. It depends only on the level of the underlying.
- The option value can never go below the early-exercise payoff: $V \geq \max(K - S, 0)$.
- Since the option is independent of t, it must satisfy

$$\frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0.$$

  The general solution of this ODE is: $V(S) = AS + BS^{-2r/\sigma^2}$ with A and B constants.
- For the American put coefficient A must be zero; as $S \to \infty$ the value of the option must tend to zero. What about B ?
- While the asset is 'high' we won't exercise the option. But if it falls too low we immediately exercise the option, receiving $K - S$.
- Suppose that we decide that $S = S^*$ is the value at which we exercise, i.e. as soon as S reaches this value from above we exercise. How do we choose S^* ?

- When $S = S^*$ the option value must be the same as the exercise payoff: $V(S^*) = K - S^*$. It cannot be less, that would result in an arbitrage opportunity, and it cannot be more or we would not exercise.
- The continuity of the option value with the payoff gives us one equation:

$$V(S^*) = B(S^*)^{-2r/\sigma^2} = K - S^*.$$

Since B and S^* are unknown, we need one more equation.

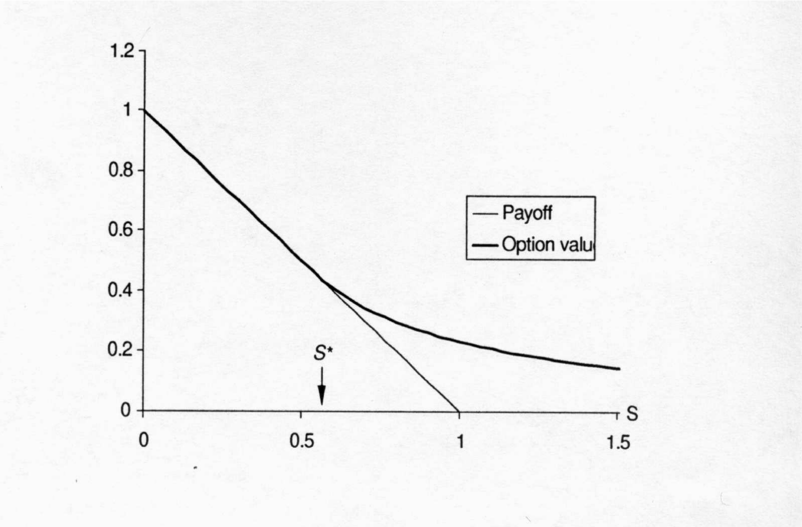
- Eliminating B, looking at the value of the option as a function of S^* gives for $S > S^*$:

$$V(S) = (K - S^*)\left(\frac{S}{S^*}\right)^{-2r/\sigma^2}.$$

- Choose S^* to maximize the option's value at any time before exercise. This value is found by differentiation with respect to S^*

$$\frac{\partial}{\partial S^*}(K - S^*)\left(\frac{S}{S^*}\right)^{-2r/\sigma^2} = \frac{1}{S^*}\left(\frac{S}{S^*}\right)^{-2r/\sigma^2}\left(-S^* + \frac{2r}{\sigma^2}(K - S^*)\right) = 0$$

- One finds that $S^* = \frac{K}{1+\sigma^2/2r}$. This choice maximizes $V(S)$ for all $S \geq S^*$. The solution with this choice for S^* is:



- The slope of the option value and the slope of the payoff function are the same.
- The American option value is maximized by an exercise strategy that makes the option value and the slope continuous.
- Position S^* is called the optimal exercise point.

“Optimal exercise point ?”

- The idea behind valuing options with early exercise is to decide when the option should be exercised.
- To correctly price American options we must place ourselves in the shoes of the option writer and assume that he is hedging his position by trading in the underlying asset (Delta hedging). The position in the underlying asset is maintained Delta neutral so as to be insensitive, to leading order, to movement of the asset
- By maintaining such a hedge, the write does not care about the direction in which the underlying moves: he eliminates all asset price risk.
- However, he does remain exposed to the exercise strategy of the option holder: If the writer makes an assumption about when the holder will exercise his option and this assumption is incorrect, this will have an impact on the writer’s profit.
- Since the writer cannot possibly know what the holder’s strategy will be, how can he reduce his exposure to this strategy ?

Holder

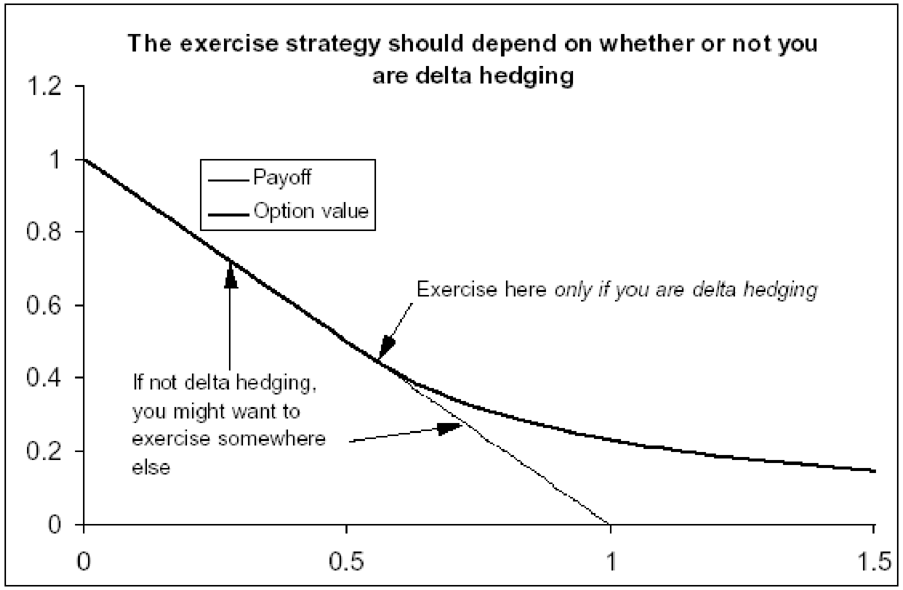
- The answer is simple: The writer assumes that the holder exercises at the worst possible time for the writer.
- He assumes that the option is exercised at the moment that gives the writer the least profit.

⇒ This is often referred to as "the optimal stopping time"

- Out of all strategies one must find the one that gives the option the least value to the writer (or equivalently the highest value to the holder).
- The holder rarely Delta hedges (he bought the option maybe as a speculative investment), and is typically not insensitive the to direction of the underlying asset.
- Should the holder therefore act in the optimal way and exercise at S^* ?

Answer

- The answer is no. The writer and the holder of the option have different priorities: which is optimal to one is not necessarily optimal to the other. The holder may simply have a gut feeling about the stock and decides to exercise, or he has adopted a different strategy.
- It is highly unlikely that his exercise time will correspond to that calculated by the writer of the option



Writer

- How does the writer feel about exercise ?
- The writer receives a sum of money in exchange for the option. That sum was calculated assuming that the holder exercises at a certain optimal time.
- This optimal exercise strategy gives the option its highest theoretical value. The writer receives this maximal amount even though the holder may exercise at any time.
- It is clear that the writer can never lose. The worst that can happen to him is that the option is exercised at this theoretical optimal time. But this has already been priced into the premium that he received.

- In the region where $V = K - S,\ (S < K)$, we find by substitution:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = -rK < 0$$

$\Rightarrow$ Inequality.

- Theory for American-style contracts with arbitrary payoff and maturity: Inequality can be derived with standard Black-Scholes analysis with minor modifications.
- The contract and option value will be a function of S and t.
- If V is the value of a long position in an American option, we can earn no more than the risk-free rate on our portfolio. This gives rise to the inequality for the whole S-region:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \le 0$$

- The problem we need to solve for an American put option contract reads:

$$
\begin{aligned}
\frac{\partial V}{\partial t} &+ \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0 \\
V(S,t) &\geq \max(K - S, 0) \\
V(S,T) &= \max(K - S, 0) \\
\frac{\partial V}{\partial S} & \quad \text{is continuous}
\end{aligned}
$$

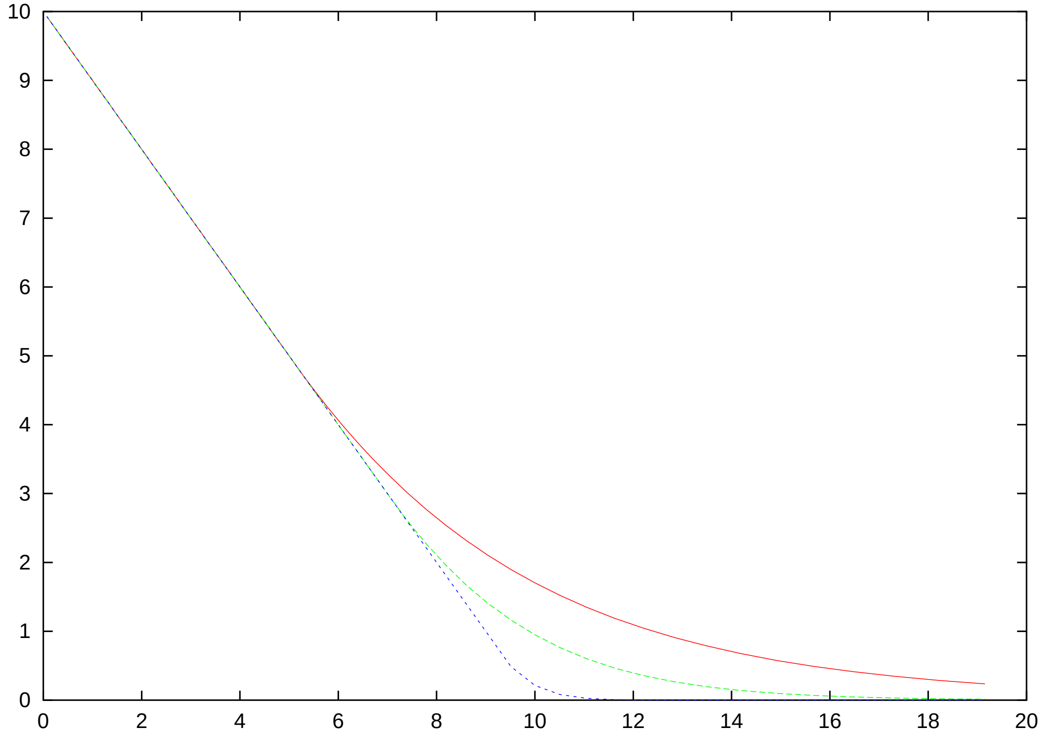
- The option value must be greater than, or equal to the payoff, the Black-Scholes equation is replaced by an inequality, the option value must be a continuous function of S, the option delta (its slope) must be continuous.
- The European call solution, in absence of dividends, satisfies the inequality ! It also satisfies the constraint $V(S,t) \geq \max(S - K, 0)$.

$\Rightarrow$ The value of an American call option is the same as the value of a European call option when the underlying pays no dividends. The American option should not be exercised prior to expiry.

A result

for an American option

- American Put option, $K = 10, r = 0.06, T = 1$ year.
- Blue: $\sigma = 0.1$, Green: $\sigma = 0.3$, Red: $\sigma = 0.5$



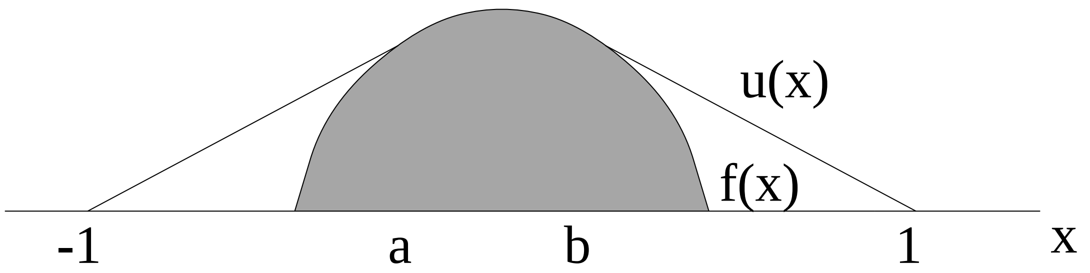
(shown $t = 0$)

Intermezzo: Obstacle problems

- Given: an "obstacle" $f(x)$, with $f(x) > 0$ for $a < x < b, f \in C^2$, $f'' < 0$ and $f(-1) < 0, f(1) < 0$.
- Over the obstacle, one spans a function u of minimal length. This string u lies on the obstacle between a and b. This obstacle problem is a simple free boundary problem:

$$\begin{aligned} -1 < x < a: &\quad u'' = 0 \quad (u > f) \\ a < x < b: &\quad u = f \quad (u'' = f'' < 0) \\ b < x < 1: &\quad u'' = 0 \quad (u > f) \end{aligned}$$

- The string must be above or on the obstacle, must have negative or zero curvature, must be continuous, the slope must be continuous



Linear complementarity

- $u > f$, then $u'' = 0$
  $u = f$, then $u'' < 0$.
- Reformulation of the obstacle problem:
  Find $u(x)$, so that

$$u''(u-f) = 0, \quad -u'' \geq 0, \quad u - f \geq 0,$$
$$u(-1) = u(1) = 0, \quad u \in C^1[-1,1]$$

- This formulation is beneficial for iterative numerical solution processes.
- A similar complementarity follows for the American options:
  For a put: $V > \max(K-S,0) \quad (S > S^*(t))$,
  then Black-Scholes equation,
  $V = K - S \quad (S \leq S^*(t))$,
  then Black-Scholes inequality