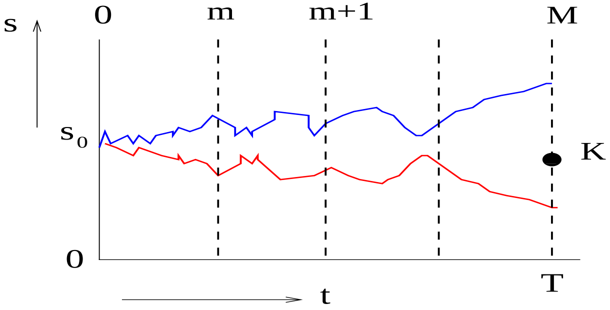
Early Exercise Option Valuation



- With $V(t_M, S(t_M)) = E(t_M, S(t_M))$ we find the option price via backward induction:

$$
\begin{cases}
V(t_M, S(t_M)) &= E(t_M, S(t_M)) \\
C(t_m, S(t_m)) &= e^{-r\Delta t}\mathbb{E}_{t_m}\left[V(t_{m+1}, S(t_{m+1}))\right] \\
V(t_m, S(t_m)) &= \max\{C(t_m, S(t_m)), E(t_m, S(t_m))\}, \\
V(t_0, S(t_0)) &= C(t_0, S(t_0)),
\end{cases}
\quad m = M-1, \ldots, 1,
$$

Discounted Expected Payoff

- Write, in the case of deterministic interest rates, as an integral:

$$C(t_m, S(t_m)) = e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|S(t_m)) dy$$

- O'Sullivan(2005): Generalization to exponential Lévy processes, as the density can be recovered via Fourier inversion.
- With the midpoint rule, the density can be approximated and resolved by the FFT. Overall complexity of $O(MN^2)$ for M-times exercisable Bermudan options.

The CONV method (Carr-Madan extended)

- The main premise of the CONV method is that $f(y|x)$ depends on x and y via
$$f(y|x) = f(y-x).$$
- Assumption is clearly satisfied in exp. Lévy models, where x and y then represent log-asset prices. The assumption means that log-returns are independent.

$$\begin{aligned} C(t_m, x) &= e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, y) f(y|x) dy \\ &= e^{-r\Delta t} \int_{-\infty}^{\infty} V(t_{m+1}, x+z) f(z) dz. \end{aligned}$$

- The key insight is the notion that, apart from the discounting, the equation is a cross-correlation of V with the density function f.

Early Exercise Option Valuation

- Premultiplying by $\exp(\alpha x)$ and taking its Fourier transform, gives:

$$
\begin{aligned}
e^{r\Delta t}\mathcal{F}\{e^{\alpha x}C(t_m,x)\} &= e^{r\Delta t}\int_{-\infty}^{\infty} e^{iux}e^{\alpha x}C(t_m,x)dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iux}e^{\alpha x}V(t_{m+1},x+z)f(z)dzdx \\
&= \int_{-\infty}^{\infty} e^{i(u-i\alpha)y}V(t_{m+1},y)dy\int_{-\infty}^{\infty} e^{-i(u-i\alpha)z}f(z)d \\
&= \widetilde{V}(t_{m+1},u-i\alpha)\phi\left(-(u-i\alpha)\right).
\end{aligned}
$$

- A computation for resolving the (conditional) density function is avoided,
  only the characteristic function ϕ is involved.
- The option price is recovered by the inverse Fourier transform and undamping.

Some Details

- The extended characteristic function

$$\phi\,(x+yi)=\int_{-\infty}^{\infty}e^{i(x+yi)z}f(z)dz,$$

  is well-defined when $\phi(yi)<\infty$, as $|\phi(x+yi)|\le|\phi(yi)|$.

$\Rightarrow$ This puts a restriction on the damping coefficient α, because $\phi(\alpha i)$ must be finite.

- The damping factor is necessary when considering e.g. a Bermudan put, as then $V(t_{m+1},x)$ tends to a constant when $x\to-\infty$, and as such is not L^1-integrable.
- The difference with the Carr-Madan approach is that we take a transform with respect to the log-spot price instead of the log-strike price.

$\Rightarrow$ The idea for GBM is already present in a presentation by Eric Reiner (2000)

Algorithm

The algorithm may now be clear, with $E(t_0, x) = 0$:

- $V(t_M, x) = E(t_M, x)$ for all x
- **For** $m = M - 1$ **to 0**
- Dampen $V(t_{m+1}, y)$ and take its Fourier transform
- Multiply with $\phi(-u + i\alpha)$
- Apply Fourier inversion and undamp
- $V(t_m, x) = \max\{E(t_m, x), C(t_m, x)\}$
- **Next** m

Expressions for hedge parameters

- The CONV formulae for two hedge parameters, Δ and Γ, defined as,

$$\Delta = \frac{\partial V}{\partial S} = \frac{1}{S}\frac{\partial V}{\partial x}, \; \Gamma = \frac{\partial^2 V}{\partial S^2} = \frac{1}{S^2}\left(-\frac{\partial V}{\partial x} + \frac{\partial^2 V}{\partial x^2}\right). \quad (1)$$

- Define, $\mathcal{F}\{e^{\alpha x}V(t_0, x)\} = e^{-r\Delta t}A(u)$, where $A(u) = \mathcal{F}\{e^{\alpha y}V(t_1, y)\} \cdot \phi(-u + i\alpha)$.
- CONV formula for Δ and Γ,

$$\begin{aligned} \Delta &= \frac{e^{-\alpha x}e^{-r\Delta t}}{S}\Big[\mathcal{F}^{-1}\{-iuA(u)\} - \alpha\mathcal{F}^{-1}\{A(u)\}\Big], \\ \Gamma &= \frac{e^{-\alpha x}e^{-r\Delta t}}{S^2}\Big[\mathcal{F}^{-1}\{(-iu)^2A(u)\} - (1+2\alpha)\mathcal{F}^{-1}\{-iuA(u)\} \\ &+ \alpha(\alpha+1)\mathcal{F}^{-1}\{A(u)\}\Big]. \end{aligned}$$

CONV Method, FFT

- Step 1 - The payoff transform

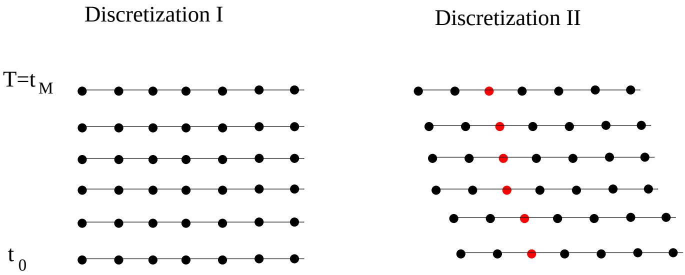
$$
\begin{aligned}
\mathcal{F}\{e^{\alpha y} V(t_{m+1}, y)\}(u) &= \int_{-\infty}^{\infty} e^{iuy} e^{\alpha y} V(t_{m+1}, y) dy \\
&\approx \Delta y \sum_{n=0}^{N-1} w_n e^{iu_j y_n} e^{\alpha y_n} V(t_{m+1}, y_n)
\end{aligned}
$$

- Can be evaluated using the FFT, use the Trapezoidal rule, for example.
- Need uniform grids for u, x (log-asset price at t_m) and y (log-asset price at t_{m+1}).
- Further, the Nyquist relation must be satisfied: $\Delta u \cdot \Delta x = 2\pi / N$.

Error analysis of the CONV method

- Rederive discretized CONV formula by a Fourier series expansion of continuation value.
- This reveals that
  - Only moment restriction on α is necessary (L^1 integrability is replaced by L^1-summability);
  - If ϕ decays faster than a polynomial, the discretized CONV formula converges as $O(1/N^2)$ for continuous payoff functions;
  - If ϕ decays as x^β, the order is $O(1/N^{\min\{1+\beta,2\}})$ for continuous payoff functions.

Dealing with discontinuities for Bermudan Options



- Consider two discretizations:
  - Discretization I: $x = y$ throughout, and $\ln S(0)$ lies on the grid;
  - Discretization II: At each time, t_m, we place d_m on the x-grid.

1. Estimate d_m in $C(t_m, d_m) = E(t_m, d_m)$;
2. Place d_m on the x-grid and recalculate $C(t_m)$;
3. Re-evaluate exercise decision and continue.

Bermudan option , Discretization II

- Pricing 10-times exercisable Bermudan put under GBM and VG
- $S_0 = 100, K = 110, T = 1, r = 0.1, q = 0;$
- For GBM: $\sigma = 0.25$, reference= 11.1352431;
- For VG: $\sigma = 0.12, \theta = -0.14, \nu = 0.2$, reference= 9.040646114;

$(N = 2^n)$	GBM			VG		
n	time(msec)	abs. error	conv.	time(msec)	abs. error	conv
7	0.23	-2.7-02	–	0.28	-9.6e-02	–
8	0.46	-7.4-03	3.7	0.55	-1.1e-02	9.0
9	0.90	-2.0e-03	3.7	1.09	-2.3e-03	4.7
10	2.00	-5.2e-04	3.8	2.15	-6.1e-04	3.8
11	3.85	-1.3e-04	4.0	4.38	-1.6e-04	3.8
12	7.84	-3.3e-05	4.0	9.29	-4.1e-05	3.9

Approximation of American option

- The value of an American option can be approximated
  - either by a Bermudan with many exercise dates,
  - or, by Richardson extrapolation on a series of Bermudan options with an increasingly number of exercise dates
- To this end assume that the Bermudan price $V(\Delta t)$, with Δt the time step between two consecutive exercise moments, can be written as:

$$V(\Delta t) = V(0) + \sum_{i=1}^{\infty} a_i (\Delta t)^{\gamma_i}$$

American option under GBM

- $\lim_{M \to \infty} P(M) = \text{American option value}$
  - Approximate the American option value by $P(M)$ with a big M.
  - Reconstruct a faster converging series $P'(M)$ by
- $S_0 = 100, K = 110, T = 1, \sigma = 0.25, r = 0.1, q = 0;$
- Reference value: $V_{ref}(0, S(0) = 12.169417$ (Black-Scholes)
- Richardson extrapolation with 128, 64 and 32 exercise opportunities

$(N = 2^n)$	P(N/2)			Richardson		
n	time(msec)	error	conv.	time(msec)	error	conv.
7	0.97	-5.9e-02	–	3.3	-3.1e-02	–
8	3.7	-2.2e-03	2.6	6.6	-7.8e-03	3.9
9	14.8	-9.3e-03	2.4	14.0	-2.1e-03	3.8
10	60.0	-4.16e-03	2.2	28.4	-5.2e-04	4.0
11	251.7	-2.0e-03	2.1	66.4	-1.2e-04	4.3
12	1108.1	-9.4e-04	2.1	151.9	-2.1e-05	5.8

Yet Another Method: Fourier-Cosine Expansion

- The COS method:
  - Exponential convergence;
  - Greeks are obtained at no additional cost.
  - For discretely-monitored barrier and Bermudan options as well;
- The basic idea:
  - Replace the density by its Fourier-cosine series expansion;
  - Series coefficients have simple relation with characteristic function.

Series Coefficients of the Density and the Ch.F.

- Fourier-Cosine expansion of density function on interval $[a, b]$:

$$f(x) = {\sum_{n=0}^{\infty}}' F_n \cos\left(n\pi \frac{x-a}{b-a}\right),$$

with $x \in [a, b] \subset \mathbb{R}$ and the coefficients defined as

$$F_n := \frac{2}{b-a} \int_a^b f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx.$$

- F_n has direct relation to ch.f., $\phi(u) := \int_{\mathbb{R}} f(x) e^{iux} dx$ ($\int_{\mathbb{R}\backslash[a,b]} f(x) \approx 0$),

$$\begin{aligned} F_n \approx A_n \quad &:= \quad \frac{2}{b-a} \int_{\mathbb{R}} f(x) \cos\left(n\pi \frac{x-a}{b-a}\right) dx \\ &= \quad \frac{2}{b-a} \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}\right) \exp\left(-i \frac{ka\pi}{b-a}\right)\right\}. \end{aligned}$$

Recovering Densities

- Replace F_n by A_n, and truncate the summation:

$$f(x) \approx \frac{2}{b-a} {\sum_{n=0}^{N-1}}' \mathrm{Re}\left\{ \phi\left(\frac{n\pi}{b-a}; x\right) \exp\left(in\pi \frac{-a}{b-a}\right)\right\} \cos\left(n\pi \frac{x-a}{b-a}\right)$$

- Example: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, $[a, b] = [-10, 10]$ and $x = \{-5, -4, \cdots, 4, 5\}$.

N	4	8	16	32	64
error	0.2538	0.1075	0.0072	4.04e-07	3.33e-16
cpu time (sec.)	0.0025	0.0028	0.0025	0.0031	0.0032

Exponential error convergence in N.

Pricing European Options

- Start from the risk-neutral valuation formula:

$$V(x,t_0) = e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[V(y,T)|x\right] = e^{-r\Delta t}\int_{\mathbb{R}} V(y,T)f(y|x)dy.$$

- Truncate the integration range:

$$V(x,t_0) = e^{-r\Delta t}\int_{[a,b]} V(y,T)f(y|x)dy + \varepsilon.$$

- Replace the density by the COS approximation, and interchange summation and integration:

$$\hat{V}(x,t_0) = e^{-r\Delta t}{\sum_{n=0}^{N-1}}{}' \mathrm{Re}\left\{\phi\left(\frac{n\pi}{b-a};x\right)e^{-in\pi\frac{a}{b-a}}\right\}\mathcal{V}_n,$$

where the series coefficients of the payoff, $\mathcal{V}_n$, are analytic.

Pricing European Options

- Log-asset prices: $x := \ln(S_0/K) \quad \text{and} \quad y := \ln(S_T/K)$,
- The payoff for European options reads

$$V(y, T) \equiv [\alpha \cdot K(e^y - 1)]^+.$$

- For a call option, we obtain

$$\begin{aligned} \mathcal{V}_k^{call} &= \frac{2}{b-a} \int_0^b K(e^y - 1) \cos\left(k\pi \frac{y-a}{b-a}\right) dy \\ &= \frac{2}{b-a} K\left(\chi_k(0,b) - \psi_k(0,b)\right), \end{aligned}$$

- For a vanilla put, we find

$$\mathcal{V}_k^{put} = \frac{2}{b-a} K\left(-\chi_k(a,0) + \psi_k(a,0)\right).$$

Characteristic Functions Heston Model

- The characteristic function of the log-asset price for Heston's model:

$$\begin{aligned}\varphi_{hes}(u;\sigma_0) &= \exp\left(iur\Delta t + \frac{\sigma_0}{\gamma^2}\left(\frac{1-e^{-D\Delta t}}{1-Ge^{-D\Delta t}}\right)(\kappa - i\rho\gamma u - D)\right)\cdot \\ &\quad \exp\left(\frac{\kappa\bar{\sigma}}{\gamma^2}\left(\Delta t(\kappa - i\rho\gamma u - D) - 2\log(\frac{1-Ge^{-D\Delta t}}{1-G})\right)\right),\end{aligned}$$

with $D = \sqrt{(\kappa - i\rho\gamma u)^2 + (u^2 + iu)\gamma^2}$ and $G = \frac{\kappa - i\rho\gamma u - D}{\kappa - i\rho\gamma u + D}$.

- For Lévy and Heston models, the ChF can be represented by

$$\begin{aligned}\phi(u;\mathbf{x}) &= \varphi_{levy}(u)\cdot e^{iu\mathbf{x}} \quad \text{with} \quad \varphi_{levy}(u) := \phi(u;0), \\ \phi(u;\mathbf{x},\sigma_0) &= \varphi_{hes}(u;\sigma_0)\cdot e^{iu\mathbf{x}},\end{aligned}$$

Characteristic Functions Lévy Processes

- For the CGMY/KoBol model:

$$\begin{aligned}\varphi_{levy}(u) &= \exp\left(iu(r-q)\Delta t - \frac{1}{2}u^2\sigma^2\Delta t\right)\cdot \\ &\quad \exp\left(\Delta t C\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]\right),\end{aligned}$$

  where $\Gamma(\cdot)$ represents the gamma function.
- The parameters should satisfy $C \geq 0, G \geq 0, M \geq 0$ and $Y < 2$.
- The characteristic function of the log-asset price for NIG:

$$\varphi_{NIG}(u) = \exp\left(iu\mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2})\right)$$

  with $\alpha, \delta > 0, \beta \in (-\alpha, \alpha - 1)$

Heston Model

- We can present the $\mathcal{V}_k$ as $\mathbf{V}_k = \mathcal{U}_k \mathbf{K}$, where

$$\mathcal{U}_k = \begin{cases} \frac{2}{b-a}\left(\chi_k(0,b) - \psi_k(0,b)\right) & \text{for a call} \\ \frac{2}{b-a}\left(-\chi_k(a,0) + \psi_k(a,0)\right) & \text{for a put.} \end{cases}$$

- The pricing formula simplifies for Heston and Lévy processes:

$$v(\mathbf{x}, t_0) \approx \mathbf{K} e^{-r\Delta t} \cdot \mathrm{Re}\left\{ {\sum_{n=0}^{N-1}}' \varphi\left(\frac{n\pi}{b-a}\right) \mathcal{U}_n \cdot e^{in\pi \frac{\mathbf{x}-a}{b-a}} \right\},$$

where $\varphi(u) := \phi(u; 0)$

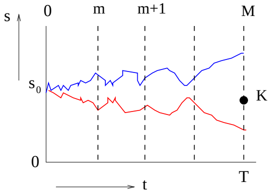
Numerical Results

Pricing for 21 strikes $K = 50, 55, 60, \cdots, 150$ under Heston's model.
Other parameters: $S_0 = 100, r = 0, q = 0, T = 1, \kappa = 1.5768, \gamma = 0.5751, \bar{\sigma} = 0.0398, \sigma_0 = 0.0175, \rho = -0.5711$.

COS	N	96	128	160
	(msec.)	2.039	2.641	3.220
	max. abs. err.	4.52e-04	2.61e-05	$4.40e-06$
Carr-Madan	N	2048	4096	8192
	(msec.)	20.36	37.69	76.02
	max. abs. error	2.61e-01	$2.15e-03$	2.08e-07

Error analysis for the COS method is provided in the COS paper.

Pricing Bermudan Options



- The pricing formulae

$$\begin{cases} C(x, t_m) & = & e^{-r\Delta t} \int_{\mathbb{R}} V(y, t_{m+1}) f(y|x) dy \\ V(x, t_m) & = & \max\left(E(x, t_m), C(x, t_m)\right) \end{cases}$$

and $V(x, t_0) = e^{-r\Delta t} \int_{\mathbb{R}} V(y, t_1) f(y|x) dy$.

  - Use Newton's method to locate the early exercise point x_m^*, which is the root of $E(x, t_m) - C(x, t_m) = 0$.
  - Recover $\mathcal{V}_n(t_1)$ recursively from $\mathcal{V}_n(t_M)$, $\mathcal{V}_n(t_{M-1}), \cdots, \mathcal{V}_n(t_2)$.
  - Use the COS formula for $V(x, t_0)$.

$\mathcal{V}_k$-Coefficients

- Once we have x_m^*, we split the integral, which defines $\mathcal{V}_k(t_m)$:

$$\mathcal{V}_k(t_m) = \begin{cases} \mathcal{C}_k(a, x_m^*, t_m) + \mathcal{G}_k(x_m^*, b), & \text{for a call,} \\ \mathcal{G}_k(a, x_m^*) + \mathcal{C}_k(x_m^*, b, t_m), & \text{for a put,} \end{cases}$$

for $m = M-1, M-2, \cdots, 1$. whereby

$$\mathcal{G}_k(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} E(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

and

$$\mathcal{C}_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_{x_1}^{x_2} \hat{C}(x, t_m) \cos\left(k\pi \frac{x-a}{b-a}\right) dx.$$

Theorem

The $\mathcal{G}_k(x_1, x_2)$ are known analytically and the $\mathcal{C}_k(x_1, x_2, t_m)$ can be computed in $O(N \log_2(N))$ operations with the Fast Fourier Transform.

Bermudan Details

- Formula for the coefficients $\mathcal{C}_k(x_1, x_2, t_m)$:

$$\mathcal{C}_k(x_1, x_2, t_m) = e^{-r\Delta t} \mathrm{Re} \left\{ \sum_{j=0}^{\prime N-1} \varphi_{levy} \left(\frac{j\pi}{b-a} \right) \mathcal{V}_j(t_{m+1}) \cdot \mathcal{M}_{k,j}(x_1, x_2) \right\},$$

where the coefficients $\mathcal{M}_{k,j}(x_1, x_2)$ are given by

$$\mathcal{M}_{k,j}(x_1, x_2) := \frac{2}{b-a} \int_{x_1}^{x_2} e^{ij\pi \frac{x-a}{b-a}} \cos\left(k\pi \frac{x-a}{b-a} \right) dx,$$

- With fundamental calculus, we can rewrite $\mathcal{M}_{k,j}$ as

$$\mathcal{M}_{k,j}(x_1, x_2) = -\frac{i}{\pi} \left(\mathcal{M}^c_{k,j}(x_1, x_2) + \mathcal{M}^s_{k,j}(x_1, x_2) \right),$$

Hankel and Toeplitz

- Matrices $\mathcal{M}_c = \{\mathcal{M}^c_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ and $\mathcal{M}_s = \{\mathcal{M}^s_{k,j}(x_1, x_2)\}_{k,j=0}^{N-1}$ have special structure for which the FFT can be employed: $\mathcal{M}_c$ is a Hankel matrix,

$$\mathcal{M}_c = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{N-1} \\ m_1 & m_2 & \cdots & \cdots & m_N \\ \vdots & & & & \vdots \\ m_{N-2} & m_{N-1} & \cdots & & m_{2N-3} \\ m_{N-1} & \cdots & & m_{2N-3} & m_{2N-2} \end{bmatrix}_{N \times N}$$

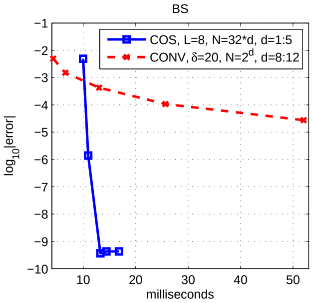
and $\mathcal{M}_s$ is a Toeplitz matrix,

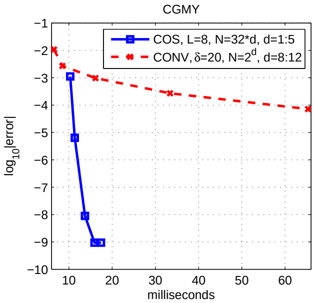
$$\mathcal{M}_s = \begin{bmatrix} m_0 & m_1 & \cdots & m_{N-2} & m_{N-1} \\ m_{-1} & m_0 & m_1 & \cdots & m_{N-2} \\ \vdots & & \ddots & & \vdots \\ m_{2-N} & \cdots & m_{-1} & m_0 & m_1 \\ m_{1-N} & m_{2-N} & \cdots & m_{-1} & m_0 \end{bmatrix}_{N \times N}$$

Bermudan puts with 10 early-exercise dates

Table: Test parameters for pricing Bermudan options

Test No.	Model	S_0	K	T	r	σ	Other Parameters
2	BS	100	110	1	0.1	0.2	–
3	CGMY	100	80	1	0.1	0	$C = 1, G = 5, M = 5, Y = 1.5$





Pricing Discrete Barrier Options

- The price of an M-times monitored up-and-out option satisfies

$$\begin{cases} C(x, t_{m-1}) & = & e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} V(x, t_m) f(y|x) dy \\ V(x, t_{m-1}) & = & \begin{cases} e^{-r(T - t_{m-1})} Rb, & x \geq h \\ C(x, t_{m-1}), & x < h \end{cases} \end{cases}$$

  where $h = \ln(H/K)$, and
  $V(x, t_0) = e^{-r(t_m - t_{m-1})} \int_{\mathbb{R}} V(x, t_1) f(y|x) dy.$
- The technique:
  - Recover $\mathcal{V}_n(t_1)$ recursively, from $\mathcal{V}_n(t_M)$, $\mathcal{V}_n(t_{M-1}), \cdots, \mathcal{V}_n(t_2)$ in $O((M-1)N \log_2(N))$ operations.
  - Split the integration range at the barrier level (no Newton required)
  - Insert $\mathcal{V}_n(t_1)$ in the COS formula to get $V(x, t_0)$, in $O(N)$ operations.

Monthly-monitored Barrier Options

Table: Test parameters for pricing barrier options

Test No.	Model	S_0	K	T	r	q	Other Parameters
1	NIG	100	100	1	0.05	0.02	$\alpha = 15, \beta = -5, \delta = 0.5$

Option Type	Ref. Val.	N	time (milli-sec.)	error
DOP	2.139931117	2^7	3.7	1.28e-3
		2^8	5.4	4.65e-5
		2^9	8.4	1.39e-7
		2^{10}	14.7	1.38e-12
DOC	8.983106036	2^7	3.7	1.09e-3
		2^8	5.3	3.99e-5
		2^9	8.3	9.47e-8
		2^{10}	14.8	5.61e-13

Credit Default Swaps

- Credit default swaps (CDSs), the basic building block of the credit risk market, offer investors the opportunity to either buy or sell default protection on a reference entity.
- The protection buyer pays a premium periodically for the possibility to get compensation if there is a credit event on the reference entity until maturity or the default time, which ever is first.
- If there is a credit event the protection seller covers the losses by returning the par value. The premium payments are based on the CDS spread.

CDS and COS

- CDS spreads are based on a series of default/survival probabilities, that can be efficiently recovered using the COS method. It is also very flexible w.r.t. the underlying process as long as it is Lévy.
- The flexibility and the efficiency of the method are demonstrated via a calibration study of the iTraxx Series 7 and Series 8 quotes.

Lévy Default Model

- Definition of default: For a given recovery rate, R, default occurs the first time the firm's value is below the "reference value" RV_0.
- As a result, the survival probability in the time period $(0, t]$ is nothing but the price of a
digital down-and-out barrier option without discounting.

$$\begin{aligned} \mathrm{P}_{surv}(t) &= \mathrm{P}_{\mathbb{Q}}\left(X_s > \ln R, \text{for all } 0 \leq s \leq t\right) \\ &= \mathrm{P}_{\mathbb{Q}}\left(\min_{0 \leq s \leq t} X_s > \ln R\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\left(\min_{0 \leq s \leq t} X_s > \ln R\right)\right] \end{aligned}$$

Survival Probability

- Assume there are only a finite number of observing dates.

$$\mathrm{P}_{surv}(\tau) = \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}\Big(X_{\tau_1} \in [\ln R, \infty)\Big) \cdot \mathbf{1}\Big(X_{\tau_2} \in [\ln R, \infty)\Big) \cdots \mathbf{1}\Big(X_{\tau_M} \in [\ln R,$$

  where $\tau_k = k\Delta\tau$ and $\Delta\tau := \tau/M$.

- The survival probability then has the following recursive expression:

$$\begin{cases} \mathrm{P}_{surv}(\tau) & := & p(x = 0, \tau_0) \\ p(x, \tau_m) & := & \int_{\ln R}^{\infty} f_{X_{\tau_{m+1}}|X_{\tau_m}}(y|x) p(y, \tau_{m+1})\, dy, \quad m = M-1, \cdots, 2, \\ p(x, \tau_M) & := & 1(x > \ln R) \text{ and equals 0 otherwise} \end{cases}$$

  $f_{X_{\tau_{m+1}}|X_{\tau_m}}(\cdot|\cdot)$ denotes the conditional probability density of $X_{\tau_{m+1}}$ given X_{τ_m}.

The Fair Spread of a Credit Default Swap

- The *fair spread*, C, of a CDS at the initialization date is the spread that equalizes the present value of the premium leg and the present value of the protection leg, i.e.

$$C = \frac{(1-R)\left(\int_0^T \exp(-r(s)s)\mathrm{d}\mathrm{P}_{def}(s)\right)}{\int_0^T \exp(-r(s)s)\mathrm{P}_{surv}(s)\mathrm{d}s},$$

- It is actually based on a series of survival probabilities on different time intervals:

$$C = \frac{(1-R)\sum_{j=0}^{J}\frac{1}{2}\left[\exp(-r_j t_j) + \exp(-r_{j+1} t_{j+1})\right]\left[\mathrm{P}_{surv}(t_j) - \mathrm{P}_{surv}(t_{j+1})\right]}{\sum_{j=0}^{J}\frac{1}{2}\left[\exp(-r_j t_j)\mathrm{P}_{surv}(t_j) + \exp(-r_{j+1} t_{j+1})\mathrm{P}_{surv}(t_{j+1})\right]\Delta t}$$

The COS Formula for Survival Probabilities

- Replace the conditional density by the COS (semi-analytical) expression, the survival probability then satisfies

$$\begin{cases} \mathrm{P}_{surv}(\tau) & = & p(x=0,\tau_0). \\ p(x,\tau_0) & = & {\sum{}'}_{n=0}^{N-1} \phi_n(x) \cdot P_n(\tau_1), \end{cases}$$

- The only thing one needs is $\{P_n(\tau_1)\}_{n=0}^{N-1}$, which can be recovered from $\{P_n(\tau_M)\}_{n=0}^{N-1}$ via backwards induction.

Backwards Induction

- Starting from the definition of $P_n(\tau_m)$, we apply the COS reconstruction of $p(y, \tau_m)$ to get

$$\mathbf{P}(\tau_m) = \mathrm{Re}\{\Omega\, \Lambda\}\ \mathbf{P}(\tau_{m+1}),$$

- Applying this recursively backwards in time, we get

$$\mathbf{P}(\tau_1) = \left(\mathrm{Re}\{\Omega\, \Lambda\}\right)^{M-1} \mathbf{P}(\tau_M)$$

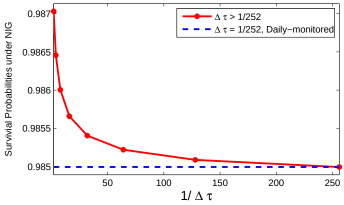
- For this recursive matrix-vector-product, there exists a fast algorithm, e.g.

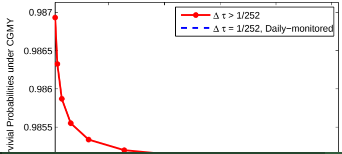
$$\mathbf{P}(\tau_1) = \mathrm{Re}\{\Omega\ [\Lambda \quad \mathrm{Re}\{\Omega\ [\Lambda \quad \mathrm{Re}\{\Omega\ [\Lambda\, \mathbf{P}(t_3)]\}\ ]\}\ ]\}\,.$$

- The FFT algorithm can be applied because $\Omega = H + T$, where H is a Hankel matrix and T is a Toeplitz matrix.

Convergence of Survival Probabilities

- Ideally, the survival probabilities should be monitored daily, i.e. $\Delta\tau = 1/252$. That is, $M = 252T$, which is a bit too much for $T = 5, 7, 10$ years.
- For Black-Scholes' model, there exist rigorous proof of the convergence of discrete barrier options to otherwise identical continuous options [Kou,2003].
- We observe similar convergence under NIG, CGMY:





Error Convergence

- The error convergence of the COS method is usually exponential in N

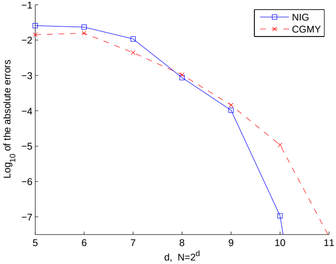


Figure: Convergence of $\mathrm{P}_{surv}(\Delta\tau = 1/48)$ w.r.t. N for NIG and CGMY

Calibration Setting

- The data sets: weekly quotes from iTraxx Series 7 (S7) and 8 (S8). After cleaning the data we were left with 119 firms from Series 7 and 123 firms from Series 8. Out of these firms 106 are common to both Series.
- The interest rates: EURIBOR swap rates.
- We have chosen to calibrate the models to CDSs spreads with maturities 1, 3, 5, 7, and 10 years.

The Objective Function

- To avoid the ill-posedness of the inverse problem we defined here, the objective function is set to

$$F_{obj} = \text{rmse} + \gamma \cdot ||\mathbf{X}_2 - \mathbf{X}_1||_2,$$

where

$$\text{rmse} = \sqrt{\sum_{\text{CDS}} \frac{(\text{market CDS spread} - \text{model CDS spread})^2}{\text{number of CDSs on each day}}}\,,$$

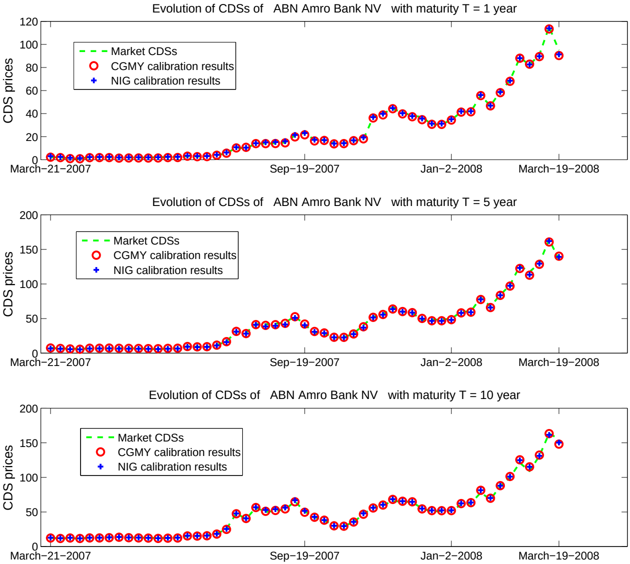
$|| \cdot ||_2$ denotes the L_2–norm operator, and $\mathbf{X}_2$ and $\mathbf{X}_1$ denote the parameter vectors of two neighbor data sets.

Good Fit to Market Data

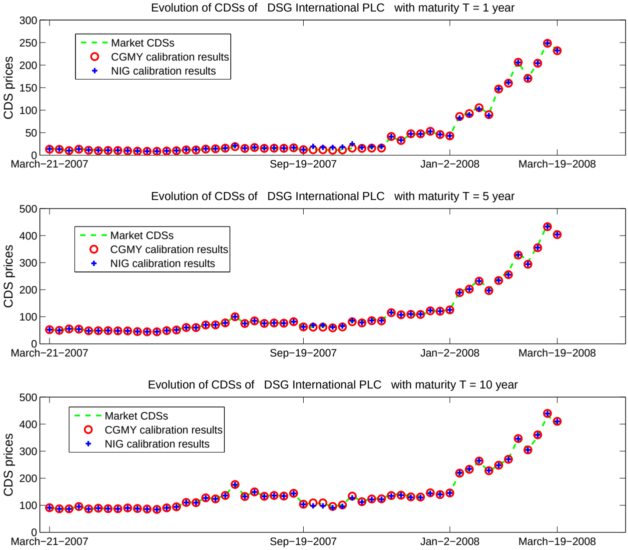
Table: Summary of calibration results of all 106 firms in both S7 and S8 of iTraxx quotes

RMSEs	NIG in S7	CGMY in S7	NIG in S8	CGMY in S8
Average (bp.)	0.89	0.79	1.65	1.54
Min. (bp.)	0.22	0.29	0.27	0.46
Max. (bp.)	2.29	1.97	4.27	3.52

A Typical Example



An Extreme Case



NIG Parameters for "ABN AMRO Bank"

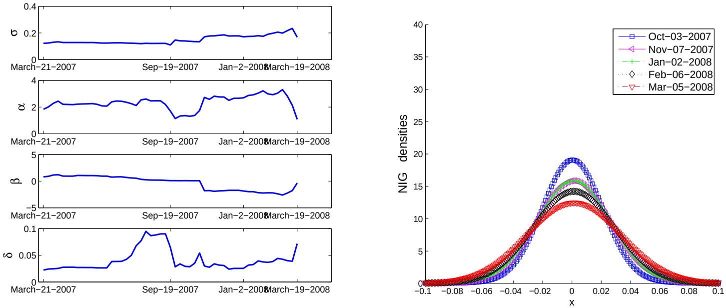


Figure: Evolution of the NIG parameters and densities of "ABN AMRO Bank"

NIG Parameters for “DSG International PLC”

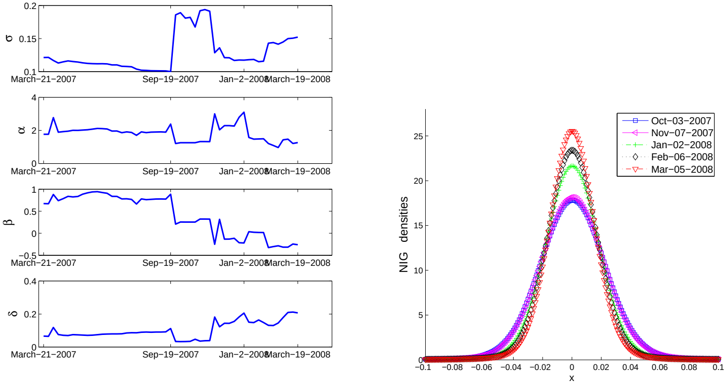


Figure: Evolution of the NIG parameters and densities of “DSG International PLC”

NIG vs. CGMY

Both Lévy processes gave good fits, but

- The NIG model returns more consistent measures from time to time and from one company to another.
- From a numerical point of view, the NIG model is also more preferable.
  - Small N (e.g. $N = 2^{10}$) can be applied.
  - The NIG model is much less sensitive to the initial guess of the optimum-searching procedure.
  - Fast convergence to the optimal parameters are observed (usually within 200 function evaluations). However, averagely 500 to 600 evaluations for the CGMY model are needed.

Truncation Range

$$[a, b] := \left[(c_1 + x_0) - L\sqrt{c_2 + \sqrt{c_4}}, \quad (c_1 + x_0) + L\sqrt{c_2 + \sqrt{c_4}} \right],$$

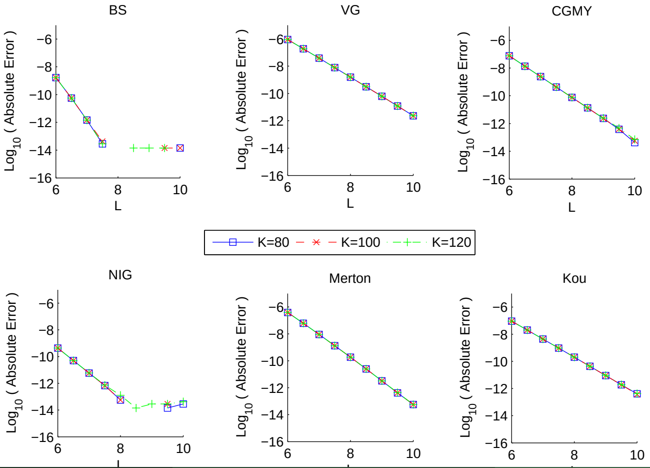


Table: Cumulants of $\ln(S_t/K)$ for various models.

BS	$c_1 = (\mu - \frac{1}{2}\sigma^2)t, \quad c_2 = \sigma^2 t, \quad c_4 = 0$	
NIG	$c_1 = (\mu - \frac{1}{2}\sigma^2 + w)t + \delta t\beta/\sqrt{\alpha^2 - \beta^2}$ $c_2 = \delta t\alpha^2(\alpha^2 - \beta^2)^{-3/2}$ $c_4 = 3\delta t\alpha^2(\alpha^2 + 4\beta^2)(\alpha^2 - \beta^2)^{-7/2}$ $w = -\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2})$	
Kou	$c_1 = t\left(\mu + \frac{\lambda p}{\eta_1} + \frac{\lambda(1-p)}{\eta_2}\right)$ $c_4 = 24t\lambda\left(\frac{p}{\eta_1^4} + \frac{1-p}{\eta_2^4}\right)$	$c_2 = t\left(\sigma^2 + 2\frac{\lambda p}{\eta_1^2} + 2\frac{\lambda(1-p)}{\eta_2^2}\right)$ $w = \lambda\left(\frac{p}{\eta_1+1} - \frac{1-p}{\eta_2-1}\right)$
Merton	$c_1 = t(\mu + \lambda\bar{\mu})$ $c_4 = t\lambda\left(\bar{\mu}^4 + 6\bar{\sigma}^2\bar{\mu}^2 + 3\bar{\sigma}^4\lambda\right)$	$c_2 = t\left(\sigma^2 + \lambda\bar{\mu}^2 + \bar{\sigma}^2\lambda\right)$
VG	$c_1 = (\mu + \theta)t$ $c_4 = 3(\sigma^4\nu + 2\theta^4\nu^3 + 4\sigma^2\theta^2\nu^2)t$	$c_2 = (\sigma^2 + \nu\theta^2)t$ $w = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2)$
CGMY	$c_1 = \mu t + Ct\Gamma(1-Y)\left(M^{Y-1} - G^{Y-1}\right)$ $c_2 = \sigma^2 t + Ct\Gamma(2-Y)\left(M^{Y-2} + G^{Y-2}\right)$ $c_4 = Ct\Gamma(4-Y)\left(M^{Y-4} + G^{Y-4}\right)$ $w = -C\Gamma(-Y)[(M-1)^Y - M^Y + (G+1)^Y - G^Y]$	

where w is the drift correction term that satisfies $\exp(-wt) = \varphi(-i, t)$.

American Options and Extrapolation

Let $v(M)$ denote the value of a Bermudan option with M early exercise dates, then we can rewrite the 3-times repeated Richardson extrapolation scheme as

$$v_{AM}(d) = \frac{1}{12}\left(64v(2^{d+3}) - 56v(2^{d+2}) + 14v(2^{d+1}) - v(2^{d})\right), \qquad (2)$$

where $v_{AM}(d)$ denotes the approximated value of the American option.

Further Reading: Fourier Pricing

[1] R. Lord, C. Kahl, Optimal Fourier inversion in semi-analytical option pricing. *J. Comp. Finance*, 10(4): 1-30, 2007.

[2] R. Lord, F. Fang, F. Bervoets and C.W. Oosterlee, A fast and accurate FFT-based method for pricing early-exercise options under Lévy processes. *SIAM J. Sci. Comput.* **30,** 1678-1705 (2008).

[3] C. O'Sullivan, *Path dependent option pricing under Lévy processes* EFA 2005 Moscow Meetings Paper, Available at SSRN: http://ssrn.com/abstract=673424|, Febr. 2005

[4] E. Reiner, Convolution methods for exotic options *Conference on Computational Intelligence for Financial Engineering,* 30 March, 1998, New York.

[5] F. Fang and C.W. Oosterlee, A Novel Pricing Method for European Options Based on Fourier-Cosine Series Expansions. *SIAM J. Sci. Comput.* 31: 826-848, 2008.

[6] F. Fang and C.W. Oosterlee, *Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions.* Techn. Report. Delft 2008.

Other Reading:

[1] A.D. Andricopoulos, M. Widdicks, P.W. Duck and D.P. Newton, Universal option valuation using quadrature methods, *J. Fin. Economics*, 67: 447-471, 2003.

[2] S.I. Boyarchenko and S.Z. Levendorskiĭ, *Non-Gaussian Merton-Black-Scholes theory*, Vol. 9 Advanced Series on Statist. Science & Appl. Probability., World Scient. Publ. Co. Inc., River Edge, NJ, 2002.

[3] C-C Chang, S-L Chung and R.C. Stapleton, Richardson extrapolation technique for pricing American-style options. *J. Futures Markets*, 27(8): 791-817, 2007.

[4] L. Feng L. and V. Linetsky, Pricing discretely monitored barrier options and defaultable bonds in Lévy process models: a fast Hilbert transform approach, To appear in *Mathematical Finance*, 2008.