An Efficient Pricing Algorithm for Swing Options Based on Fourier Cosine Expansions

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Abstract

Swing options give contract holders the right to modify amounts of future delivery of certain commodities, such as electricity or gas. In this paper, we assume that these options can be exercised at any time before the end of the contract, and more than once. However, a recovery time between any two consecutive exercise dates is incorporated as a constraint to avoid continuous exercise. We introduce an efficient way of pricing these swing options, based on the Fourier cosine expansion method, which is especially suitable when the underlying is modeled by a Lévy process.

1 Introduction

A swing option usually consists of two contract parts: a future part and a swing part. The future contract guarantees that the option seller delivers certain amounts of a commodity (base load) to the option buyer at certain times, $T_0 < T_1 \leq T_2 \cdots \leq T_N \leq T$, with T the maturity time. The swing part gives the option buyer the right to order extra or deliver back amounts. Usually, the motivation behind the purchase of a swing option is to hedge away the uncertainty in the future demand of a commodity. The future part of a swing option can be priced as the discounted expected price of the underlying commodity at the delivery times, whereas the swing part, the focus of the present paper, can vary in contract complexity and is most interesting from a numerical point-of-view.

In the literature the swing option is often modeled as a Bermudan-style option with swing actions being allowed at the (fixed) delivery times of the base load, combined with some constraints. Pflug and Broussev [8] model the bid and ask prices as the least acceptable contract price and the maximal expected profit over demand patterns, respectively, and those prices are determined by stochastic programming. They present an algorithm to find the equilibrium prices from a game theoretic point-of-view.

Jaillet, Ronn and Tompaidis [10] use a trinomial forest model where a so-called usage level is discretized. Their model is a multiple layer tree which

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captures the information of the number of exercise rights remaining, the total amount exercised, and the price scenario. By a swing action one moves from one tree to another. A discrete binomial methodology is also applied by Lari-Lavassani, Simchi and Ware [14], where a transition probability matrix is used to calculate the expected profit, to be maximized over different swing actions at each time step.

Carmona and Touzi [13] view swing options as American-style contingent claims with multiple exercise opportunities and address the problem from the perspective of multiple optimal stopping problems, dealt with by means of Monte Carlo methods and Malliavin calculus. They focus on the Black-Scholes dynamics. Zeghal and Mnif [12] extend that method to Lévy processes.

Unlike the models in which swing actions are only allowed at discrete times, Dahlgren [1] proposes a continuous time model to price the commodity-based swing options. Here the option buyer can exercise the swing option any time before expiry, and more than once, with an upper bound for the maximum amount of additional commodity that can be ordered or delivered back (specified in the contract). After a swing action, the option buyer cannot exercise again unless a recovery time, $\tau_R(D, t)$, has elapsed, where D represents the amount of commodity and t is the exercise time. This recovery time can be constant, or dependent on the amount of the last swing action. Dahlgren [1] connects the price of the swing option to a system of discrete variational inequalities of Hamilton-Jacobi-Bellman-type, that are solved by means of finite elements and a projected successive over-relaxation (PSOR) algorithm [15]. A combination of dynamic programming and a finite difference approximation of the resulting partial integro-differential equation (PIDE) under Lévy jump processes has been presented in [11].

The purpose of the present paper is to develop an efficient alternative solution method for the continuous time model in [1], which is at least competitive with PIDE solvers or Monte Carlo methods in terms of efficiency, accuracy and flexibility. Our solution method for the swing option is based on dynamic programming, backward recursion and Fourier cosine expansions, as in [2, 3]. For the dynamics of the underlying prices, we employ the Ornstein-Uhlenbeck mean-reversion process, commonly used in commodity derivatives, and the CGMY Lévy jump process [7]. The present work can be seen as a generalization, in terms of the financial products, of the work in [2, 3].

The paper is organized as follows. Details of pricing swing options are presented in Section 2. In Section 3, our contribution to pricing swing options is described in detail. We consider both constant and state-dependent recovery times. Numerical results are presented in Section 4. We focus in this paper on the algorithmic description, which is somewhat technical at places. An error analysis is not included here, but it is included in [2, 3] for European and Bermudan options, which are the building blocks of the present swing option algorithm.

2 Details of the Swing Option

In our discussion, we ignore the future part of the swing option and concentrate on the swing part. Whenever we mention the term 'swing option', it indicates the swing part of the option.

2.1 Contract Details

As a start, our assumptions for the swing option are listed below.

- We adopt the concept of *recovery time*, denoted by $\tau_R(D)$, which means that if the option buyer has already exercised the swing option with an amount D at time point t, he/she has to wait $\tau_R(D)$ time before a next swing action can be conducted. Two different models of recovery time will be discussed:

  - Constant recovery time: If $D \neq 0$, $\tau_R(D) \equiv C$, where C is constant.
  - State-dependent recovery time: Here the recovery time is assumed to be an increasing function of D and independent of time t, i.e. $\tau_R(D) = f(D)$.

  Moreover, $\tau_R(D) = 0$ if and only if $D = 0$, and this statement holds for both types of recovery time.

- A swing option can be exercised at any time after a recovery time delay until the expiry date T. It implies that we deal with an American-style continuous problem.

- With the constraint of recovery time, a swing option can be exercised *more than once* before expiry.

- The amount of commodity at each swing action, D, is assumed to range from $-L, \cdots, -1, 0, 1, \cdots, L$, where a negative amount implies back delivery and a positive amount means ordering. The upper bound, L, is necessary as otherwise it may be optimal to order or deliver back an infinite amount of commodity, and thus receive an unrealistic profit.

- The price the option holder has to pay for *ordering* extra units of the commodity is given by:

$$\begin{cases} S & \text{if } S \leq K_a \\ K_a & \text{if } K_a \leq S \leq S_{max} \\ S - (S_{max} - K_a) & \text{if } S \geq S_{max}, \end{cases}$$

  Here S is the price of the underlying commodity, based on a Stochastic Differential Equation (SDE) for S_t, and the values of the strikes K_a and S_{max} are specified in the contract.

- The price the option holder will receive for *delivering back* units of the commodity is

$$\begin{cases} K_d - S_{min} + S & \text{if } S \leq S_{min} \\ K_d & \text{if } S_{min} \leq S \leq K_d \\ S & \text{if } S \geq K_d, \end{cases}$$

  where the values of the strikes K_d and S_{min} are also specified in the contract.

Based on the last two assumptions the payoff function of a swing option is of the form:

$$\begin{aligned} g(S,T,D) = \quad & D \cdot (\max(S - K_a, 0) - \max(S - S_{max}, 0) \\ & + \max(K_d - S, 0) - \max(S_{min} - S, 0)), \end{aligned} \tag{1}$$

with $S = S(T)$. This implies that there can be no profit unless the price of the underlying fluctuates below or above the thresholds K_d or K_a. The two other thresholds, S_{min} and S_{max}, are defined to protect an option writer against extreme fluctuations, see [1]. Figure 1 shows an example of the payoff for varying S and D.

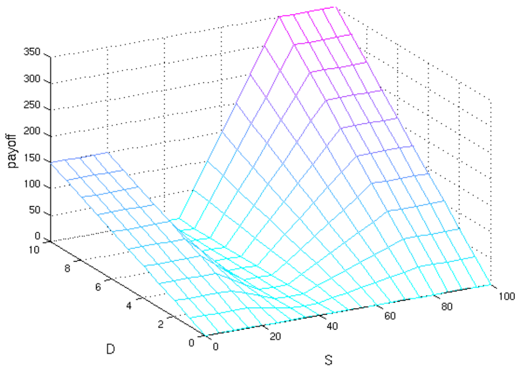


Figure 1: Example of a payoff of a swing option with $S_{min} = 20, K_d = 35, K_a = 45$, and $S_{max} = 80$, and S and D varying.

2.2 Pricing Details

Assume that the first possible time at which a swing action is allowed [1] is T_0: $0 < T_0 < T$. Let

$$n_s := min\{n | n \in \mathbb{N}_+, n \geq (T - T_0)/\tau_R(1)\}, \tag{2}$$

where $\tau_R(1)$ is the recovery time when $D = 1$. Then n_s represents the maximum number of swing actions that can be performed in the interval $[T_0, T]$.

We set $t_k^* := T - k\tau_R(1)$, so that t_k^* is the last point in time for which we can have $k + 1$ swing actions, $k = 1, \cdots, n_s - 1$. Moreover, let $I_k = (t_k^*, T]$ and $I_{n_s} = [T_0, T]$ as shown in Figure 2 [2].

On I_1, there is only one chance left for a swing action, which implies that the recovery time has no further influence for the future. Hence, if it is profitable to exercise the swing option during $(t_1^*, T]$ one should exercise the maximum

[1] If $T_0 > T$ we deal with a futures contract, and with $T_0 = T$ the price of the swing option is just the payoff, $g(S, T, 0)$, if a swing action is not profitable, and $g(S, T, L)$ otherwise.

[2] A division of the time interval into portions $I_{k+1} \backslash I_k$ was first proposed by M. Dahlgren in [1]. Our analysis is based on the appendix in [1].

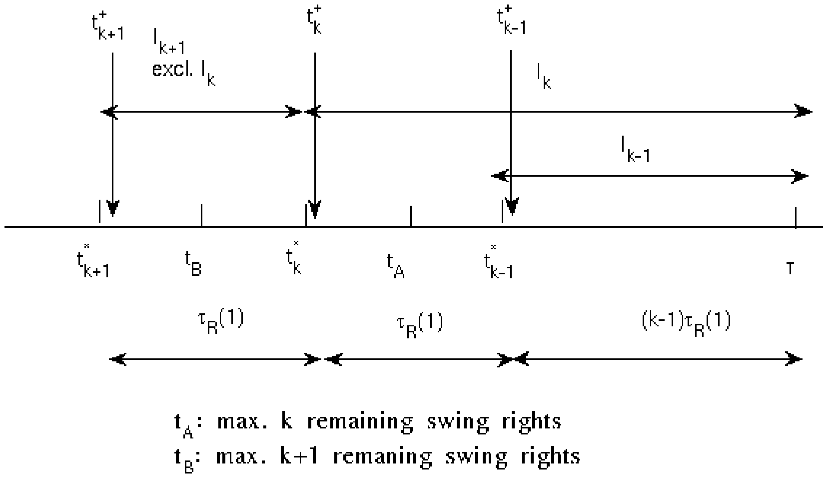


Figure 2: Notation for the division of the time axis and the maximum remaining number of swing rights.

possible amount, L. In this time interval the only issue which needs to be decided is the optimal exercise time. So, the problem is equivalent to an American-style option pricing problem, and the swing option value for any $t \in (t_1^*, T]$ is equal to the value of an American option, starting from t and expiring at T, with payoff $g(S, t', L)$, $t' \in (t, T]$.

At any time $t \in I_{k+1} \backslash I_k$, where $t \neq t_k^*$, $k = 1, \cdots, n_s - 1$, see Figure 2, the option holder basically has two choices: Either exercise the swing option at any time in $[t, t_k^*]$ or not exercise until t_k^+, the time point directly after t_k^*:

$$t_k^+ = \lim_{\delta \downarrow 0} t_k^* + \delta.$$

Note here that the length of interval $I_{k+1} \backslash I_k$ equals $\tau_R(1)$, the recovery time for $D = 1$. It is therefore not possible to exercise more than once within $I_{k+1} \backslash I_k$. In the case of exercise, the problem reduces to the decision of the optimal exercise time within $I_{k+1} \backslash I_k$. So, for each possible amount, D, the problem is equivalent to an American-style option problem, starting at $t \in I_{k+1} \backslash I_k$ and ending at t_k^*, with payoff

$$\overline{g}(S, t', D) = g(S, t', D) + \phi_D^{t'}(S, t'), \quad t' \in [t, t_k^*], t \in I_{k+1} \backslash I_k \tag{3}$$

where

$$\phi_D^{t'}(S, t') = e^{-r\tau_R(D)} \mathbb{E}_{S,t'}(v(S, t' + \tau_R(D))), \tag{4}$$

and $\mathbb{E}_{S,t'}$ represents the conditional expectation of $v(S, t' + \tau_R(D))$ given $S(t')$.

For each possible value of D, i.e., $D = -L, \cdots, L$, we compute the corresponding values of the swing option at t, assuming that D commodities are bought/sold within $I_{k+1} \backslash I_k$, by an American-style option pricing method. After taking the maximum over all values of D, we obtain the swing option value at $t \in I_{k+1} \backslash I_k$ with $t \neq t_k^*$ *if exercise takes place before* t_k^+. We denote the corresponding option value by $v_1(S, t)$.

On the other hand, if the option holder decides *not to exercise* before t_k^+ he/she has an option worth the discounted expected value:

$$v_2(S, t) = e^{-r(t_k^+ - t)} \mathbb{E}_{S,t}(v(S, t_k^+)), t \in I_{k+1} \backslash I_k \tag{5}$$

where

$$v(S, t_k^+) = \lim_{\delta \downarrow 0} v(S, t_k^* + \delta).$$

The value $v(S, t_k^+)$ with $t_k^+ \in I_k \backslash I_{k-1}$, is already obtained at the latest step in the backward recursion. After another, European-type, backward recursion procedure (5), value $v_2(S, t)$ is obtained. From the view of a profit maximizing agent, we find that

$$v(S, t) = \max \left(v_1(S, t), v_2(S, t) \right), \quad t \in I_{k+1} \backslash I_k$$

Moreover, at each t_k^*, the last time point to perform $k + 1$ swing actions, which is also in $I_{k+1} \backslash I_k$, the option value is the maximum of the payoff $\bar{g}(S, t_k^*, D)$ from (3), over all values of D, and the value of $v(S, t_k^+)$.

Finally, for $t \in [0, T_0)$, a time interval in which swing actions are not yet allowed, we have

$$v(S, t) = e^{-r(T_0 - t)} \mathbb{E}_{S,t}(v(S, T_0)),$$

which is computed by one step of a European option pricing algorithm.

This concludes the global description of the algorithm for the swing option pricing method.

Summarizing, we can distinguish two major parts in the pricing algorithm:

- For $t \in (t_1^*, T]$, we are faced with an American option pricing problem with payoff $g(S, t, D)$, given by (1), which can take five different forms in five different regions of the spot price of the underlying (see Figure 1). As mentioned, if it is profitable to exercise the swing option in this time interval, then $D_{opt} = L$. Hence the swing option price is the maximum of $g(S, t, L)$ and the continuation value.
- For the other time regions $t \in [T_0, t_1^*)$, we compute the following two quantities and compare them within each time region $I_{k+1} \backslash I_k$:
  - The value of an American option, $v_1(S, t)$, with payoff $\bar{g}(S, t, D) := g(S, t, D) + \phi_D^t(S, t)$, as in (3), and ϕ_D^t as in (4).
  - The discounted value $v_2(S, t) = \mathbb{E}_{S,t}(v(S, t_k^+))$.

  For the values $v(S, t_k^+)$ we only have to calculate the value of $v_1(S, t_k^+)$. This is due to the fact that the discounted value of $\mathbb{E}_{S,t}(v(S, t_{k-1}^+))$ equals $\phi_{D=1}^{t_k^+}(S, t_k^+)$ which is less than (or equal to) the payoff with $D = 1$ (since g is non-negative), and thus less than (or equal to) the corresponding American option value, $v_1(S, t_k^+)$.

2.3 Commodity Processes

The commodity underlying for the swing option is modeled by a stochastic differential equation for $x(t) = \ln S(t)$. State variables x and y are defined as the logarithms of the asset price, $S(t)$:

$$x := \ln(S(t_{m-1})) \quad \text{and} \quad y := \ln(S(t_m)),$$

respectively. Consequently, (1) can be rewritten (keeping the same notation, g, for the function based on $x(t)$) as

$$\begin{aligned} g(x,t,D) := \quad & D \cdot (\max(e^x - K_a, 0) - \max(e^x - S_{max}, 0) \\ & + \max(K_d - e^x, 0) - \max(S_{min} - e^x, 0)), \end{aligned} \tag{6}$$

with $x = x(t)$. Function $\overline{g}$ from Equation (3) can be generalized accordingly, also keeping the same notation, $\bar{g}$, for the function based on $x(t)$.

Two underlying processes are considered in this section, an exponential Ornstein-Uhlenbeck (OU) mean-reverting process and a CGMY Lévy jump process.

For the exponential OU process, the log-asset process $x(t) = \log(S(t))$ is assumed to be mean-reverting:

$$dx(t) = \kappa(x(t) - \bar{x})dt + \sigma dW(t), \tag{7}$$

where κ is speed of mean-reversion, $\bar{x}$ is long term mean and σ is the volatility. Moreover, under the risk-neutral measure, we should adjust $\bar{x}$ by subtracting a market price of risk parameter λ from $\bar{x}$, as in [1].

The characteristic function, $\varphi(\omega; x)$, of the conditional probability density function, $f(y|x)$, is defined as:

$$\varphi(\omega; x) = \mathbb{E}(e^{i\omega y}|x). \tag{8}$$

The well-known characteristic function for the OU process reads:

$$\varphi_{OU}(\omega; \tau) = \exp\left(x_0 B_x(\omega, \tau) + A(\omega, \tau)\right),$$

with

$$\left\{ \begin{array}{ll} B_x(\omega,\tau) & = i\omega \mathrm{e}^{-\kappa\tau}, \\ A(\omega,\tau) & = \frac{1}{4\kappa}\left(\mathrm{e}^{-2\kappa\tau} - \mathrm{e}^{-\kappa\tau}\right)\left(\omega^2\sigma^2 + \omega e^{\kappa\tau}\left(\omega\sigma^2 - 4i\kappa\overline{x}\right)\right). \end{array} \right. \tag{9}$$

The CGMY process, as defined in [7], is a Lévy jump process, a generalization of the Variance Gamma process, with as the characteristic function:

$$\varphi_{CGMY} = exp(i\omega x_0)\psi_{CGMY}(\omega, t) \tag{10}$$

with

$$\psi_{CGMY}(\omega, t) = \exp(tC\Gamma(-Y)[(M - i\omega)^Y - M^Y + (G + i\omega)^Y - G^Y]). \tag{11}$$

It is governed by four parameters. Parameter $Y < 2$ controls whether the process has finite ($Y < 1$) or infinite ($1 < Y < 2$) activity. Parameter $C > 0$ controls the kurtosis of the distribution and the non-negative parameters G, M

give control over the rate of exponential decay on the right and left side of the density, respectively.

So, we deal with general characteristic functions of the form:

$$\varphi(\omega; t) = \exp(\beta i\omega x_0) \cdot \psi(\omega, t), \tag{12}$$

in which, for the OU process β takes the value $e^{-\kappa \Delta t}$ whereas for Lévy processes $\beta = 1$. So, the first term in the expression of the characteristic function for the OU process contains the term $\beta = \exp -\kappa\Delta t$, which is not equal to one. As a result, the Fast Fourier Transform cannot be implemented in a straightforward way (as explained in Section 3 below).

3 Fourier Cosine Algorithm for Swing Options

In Section 2, we argued that the price of a swing option can be obtained by a series of Bermudan- and American-style option pricing procedures. In [2, 3] an efficient algorithm, based on the Fourier cosine series expansion (called the COS algorithm), for European and Bermudan early-exercise options was developed. The COS algorithm can be applied to processes for which the characteristic function is available. In this section, we briefly review the COS algorithm, and extend it to pricing swing options.

3.1 Fourier Cosine Expansions

Starting from the risk-neutral valuation formula

$$v(x, t_0) = e^{-r\Delta t} \int_{-\infty}^{\infty} v(y, T) f(y|x) dy,$$

where $v(x, t)$ is the option value, and x, y can be any increasing functions of the underlying, $S(t)$, at t_0 and T, respectively, and $\Delta t = T - t_0$. We truncate the integration range to $[a, b]$, so that

$$v(x, t_0) \approx e^{-r\Delta t} \int_{a}^{b} v(y, T) f(y|x) dy, \tag{13}$$

with $| \int_{\mathbb{R}} f(y|x) dy - \int_a^b f(y|x) dy| < TOL$.

We take the following integration range, from [2]:

$$[a, b] := \left[c_1 - 10\sqrt{c_2 + \sqrt{c_4}}, c_1 + 10\sqrt{c_2 + \sqrt{c_4}} \right], \tag{14}$$

where c_n denote the n^{th} cumulant of $\log S$.

The conditional density function of the underlying is approximated via the characteristic function by a truncated Fourier cosine expansion, as follows:

$$f(y|x) \approx \frac{2}{b-a} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a}; x) \exp(-i\frac{ak\pi}{b-a})) \cos(k\pi\frac{y-a}{b-a}), \tag{15}$$

where Re denotes taking the real part of the input argument.

The prime at the sum symbol in (15) indicates that the first term in the expansion is multiplied by one-half. Replacing $f(y|x)$ in (13) by its approximation in (15) and interchanging integration and summation gives us the COS algorithm to approximate the value of a European option [2]:

$$v(x,t_0) = e^{-r\Delta t}{\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a};x)e^{-ik\pi\frac{a}{b-a}})V_k, \tag{16}$$

where

$$V_k = \frac{2}{b-a}\int_a^b v(y,T)\cos{(k\pi\frac{y-a}{b-a})}dy$$

is the Fourier cosine coefficient of $v(y,T)$, which is available in closed form for several European option payoff functions.

Formula (16) can be directly applied to calculate the value of a European option, but it also forms the basis for the pricing of Bermudan options.

For a Bermudan option the COS algorithm was generalized in [3] as follows: Choose $t_m,\ m = 1,2,\cdots,\mathcal{M}$, as the "early-exercise dates". The backward recursion dynamic programming scheme for a Bermudan option with $\mathcal{M}$ exercise dates and $T = t_{\mathcal{M}}$ then reads:

For $m = \mathcal{M}, \mathcal{M}-1, \ldots, 2$,

$$\begin{cases} c(x,t_{m-1}) &= e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_m)f(y|x)dy, \\ v(x,t_{m-1}) &= \max{(\text{payoff}, c(x,t_{m-1}))}, \end{cases} \tag{17}$$

followed by

$$v(x,t_0) = e^{-r\Delta t}\int_{\mathbb{R}} v(y,t_1)f(y|x)dy. \tag{18}$$

Functions $v(x,t)$, $c(x,t)$ and "payoff" are the option value, the continuation value and the payoff at time t, respectively.

The Fourier cosine series expansion coefficients, V_k, are now time-dependent and their computation requires an efficient algorithm. The algorithm to compute V_k for swing options is discussed in detail in Sections 3.2 and 3.3.

The value of an *American option* can be obtained by the backward recursion procedure for discrete Bermudan options, explained above, in combination with a Richardson extrapolation procedure. In particular, a four-point repeated Richardson extrapolation scheme using the prices of Bermudan options for four different numbers of exercise dates, $\mathcal{M}, 2\mathcal{M}, 4\mathcal{M}, 8\mathcal{M}$,

$$\hat{v}_{AM}(\mathcal{M}) = \frac{1}{21}\left(64\hat{v}(8\mathcal{M}) - 56\hat{v}(4\mathcal{M}) + 14\hat{v}(2\mathcal{M}) - \hat{v}(\mathcal{M})\right), \tag{19}$$

has been successfully applied in [4, 3]. Here, $\hat{v}(\mathcal{M})$ denotes the Bermudan option value, $v(x,t_0)$ from (18) with $\mathcal{M}$ exercise dates; $\hat{v}_{AM}(\mathcal{M})$ is the approximation for the American option price with the extrapolation based on $\mathcal{M}$ exercise dates.

The COS algorithm exhibits an exponential convergence rate for European and Bermudan options, for asset processes whose conditional density $f(y|x) \in C^{\infty}((a,b) \subset \mathbb{R})$.

In the following subsections we generalize the COS algorithm to pricing swing options.

Remark 3.1. *Subscript k in t^*_k, as well as in t^+_k, decreases, from $n_s - 1$ to 1, if we move forward in time, with t from 0 to T, see Figure 2. In contrast, subscript m, denoting the early-exercise dates, in t_m increases and goes from 1 to $\mathcal{M}$ if we move forward in time. Further, there are $N_R = \tau_R(1)/\Delta t \equiv \tau_R(1)\mathcal{M}/T$ early-exercise dates in each time interval $I_{k+1}\backslash I_k$, i.e. between time points t^*_{k+1} and t^*_k.*

3.2 Algorithm for the Last Time Interval, $t \in I_1$

We start the detailed description of our pricing algorithm for swing options by considering the last time interval, defined as I_1, see Figure 2.

As mentioned in Subsection 2.2, in I_1, the swing option is equivalent to an American option. We can thus generalize the algorithm based on the Fourier cosine expansions for Bermudan options to the swing option payoff and combine it with a 4-point repeated Richardson extrapolation to obtain an approximation of an American option price.

3.2.1 Fourier Cosine Coefficients

At $t_{\mathcal{M}} = T$, we have for the Fourier cosine coefficients of the swing option value:

$$V_k(t_{\mathcal{M}}) \;=\; G_k(a, ln(K_d), D) + G_k(ln(K_a), b, D), \tag{20}$$

with $D = L$, and a, b as in (13). Here

$$G_k(x_1, x_2, D) = \frac{2}{b-a}\int_{x_1}^{x_2} g(x, t_{\mathcal{M}}, D)\cos(k\pi\frac{x-a}{b-a})dx \tag{21}$$

is the Fourier cosine coefficient of the swing option payoff.
In detail, we find, with $D = L$:

$$\begin{aligned} V_k(t_{\mathcal{M}}) \;=\; & \frac{2L}{b-a}\left((K_d - S_{min})\psi_k(a, ln(S_{min}))\right. \\ + \;& K_d\psi_k(ln(S_{min}), ln(K_d)) - \chi_k(ln(S_{min}), ln(K_d)) \\ + \;& \chi_k(ln(K_a), ln(S_{max})) - K_a\psi_k(ln(K_a), ln(S_{max})) \\ + \;& \left.(S_{max} - K_a)\psi_k(ln(S_{max}), b)\right), \end{aligned} \tag{22}$$

with

$$\begin{aligned} \chi_k(x_1, x_2) \;=\; & \frac{1}{1 + (\frac{k\pi}{b-a})^2}\left(\cos(k\pi\frac{x_2-a}{b-a})e^{x_2} - \cos(k\pi\frac{x_1-a}{b-a})e^{x_1}\right. \\ + \;& \left.\frac{k\pi}{b-a}\left(\sin(k\pi\frac{x_2-a}{b-a})e^{x_2} - \sin(k\pi\frac{x_1-a}{b-a})e^{x_1}\right)\right), \end{aligned} \tag{23}$$

and

$$\psi_k(x_1, x_2) = \left(\sin(k\pi\frac{x_2-a}{b-a}) - \sin(k\pi\frac{x_1-a}{b-a})\right)\frac{b-a}{k\pi}, \quad (k \neq 0), \tag{24}$$

and for $k = 0$, $\psi_k(x_1, x_2) = x_2 - x_1$.

At each time step, $t_m,\ m = \mathcal{M} - 1, \cdots, 2$, as in the case of a regular Bermudan option, the log-asset values for which the payoff equals the continuation value are determined by Newton's method. Based on these values we can determine the maximum of the two, as in (17). In the case of the swing option, there are *two* early-exercise points at each time step, as it is profitable to exercise the option when the underlying is less than K_d or larger than K_a. We denote the lower and upper early-exercise points for time t_m by x_m^d and x_m^a, respectively. To determine the two early-exercise points by Newton's method, we need the values of $c(x, t_m), g(x, t_m, D), \partial c(x, t_m)/\partial x$, and $\partial g(x, t_m, D)/\partial x$ with the help of the following formulae:

$$c(x, t_m) = e^{-r\Delta t} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a}; x)e^{-ik\pi\frac{a}{b-a}})V_k(t_{m+1}). \tag{25}$$

$$\frac{\partial c(x, t_m)}{\partial x} = e^{-r\Delta t} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a}; x) \cdot i\frac{k\pi}{b-a} \cdot e^{-ik\pi\frac{a}{b-a}})V_k(t_{m+1}), \tag{26}$$

with $\varphi(\omega; x)$ in (25) and (26) defined in (8). Function g is defined in (6) and its derivative is given by the following expression:

$$\frac{\partial g(x, t_m, D)}{\partial x} = \begin{cases} -De^x, & \text{if } ln(S_{min}) \leq x \leq ln(K_d), \\ De^x, & \text{if } ln(K_a) \leq x \leq ln(S_{max}), \\ 0, & \text{otherwise.} \end{cases} \tag{27}$$

Once x_m^d and x_m^a are determined, we split the Fourier coefficients V_k into three parts, for $m = \mathcal{M} - 1, \cdots, 1$:

$$V_k(t_m) = G_k(a, x_m^d, D) + C_k(x_m^d, x_m^a, t_m) + G_k(x_m^a, b, D),$$

with the Fourier cosine coefficient of the continuation value given by:

$$C_k(x_1, x_2, t_m) = \frac{2}{b-a} \int_{x_1}^{x_2} c(x, t_m) \cos(k\pi \frac{x-a}{b-a}) dx, \tag{28}$$

and $c(x, t_m)$ defined in (25), so that the value of $V_k(t_m)$ is obtained from $V_k(t_{m+1})$.

From basic calculus we have that, if $x_m^d < ln(S_{min})$,

$$G_k(a, x_m^d, D) = D \cdot \frac{2}{b-a}(K_d - S_{min})\psi_k(a, x_m^d), \tag{29}$$

and otherwise,

$$\begin{aligned} G_k(a, x_m^d, D) &= D \cdot \frac{2}{b-a}((K_d - S_{min})\psi_k(a, ln(S_{min})) \\ &+ K_d\psi_k(ln(S_{min}), x_m^d) - \chi_k(ln(S_{min}), x_m^d)). \end{aligned} \tag{30}$$

If $x_m^a > ln(S_{max})$, we have

$$G_k(x_m^a, b, D) = D \cdot \frac{2}{b-a}(S_{max} - K_a)\psi(x_m^a, b), \tag{31}$$

and otherwise,

$$\begin{aligned} G_k(x_m^a, b, D) &= D \cdot \frac{2}{b-a}(\chi_k(x_m^a, ln(S_{max})) - K_a\psi_k(x_m^a, ln(S_{max})) \\ &+ (S_{max} - K_a)\psi_k(ln(S_{max}), b)), \end{aligned} \tag{32}$$

where χ_k and ψ_k are defined by (23) and (24), respectively.

Next we discuss the computation of $C_k(x_m^d, x_m^a, t_m)$ in (28). To determine the value of $C_k(x_1, x_2, t_m)$, we have to compute:

$$\begin{aligned} C_k(x_1, x_2, t_m) &= -\frac{i}{\pi} \cdot e^{-r\Delta t} {\sum_{j=0}^{N-1}}' \mathrm{Re}(\phi(\frac{j\pi}{b-a}) V_j(t_{m+1}) \cdot \\ &\quad (M_{k,j}^c(x_1, x_2) + M_{k,j}^s(x_1, x_2))). \end{aligned} \tag{33}$$

We can write the equations (33) as a matrix-vector product representation, i.e.,

$$\mathbf{C}(x_1, x_2, t_m) = \frac{e^{-r\Delta t}}{\pi} \mathrm{Im}\left\{(M_c + M_s)\mathbf{u}\right\}, \tag{34}$$

where $\mathrm{Im}\{\cdot\}$ denotes taking the imaginary part of the input argument, and

$$\mathbf{u} := \{u_j\}_{j=0}^{N-1}, \quad u_j := \varphi\left(\frac{j\pi}{b-a}\right) V_j(t_{m+1}), \; u_0 = \frac{1}{2}\varphi(0)\, V_0(t_{m+1}). \tag{35}$$

Based on the general characteristic function from (12), the matrix elements of $M_{k,j}^c(x_1, x_2)$ and $M_{k,j}^s(x_1, x_2)$ read:

$$M_{k,j}^c(x_1, x_2) = \begin{cases} \dfrac{(x_2 - x_1)\pi i}{b-a}, & \text{if } k = j = 0, \\ \frac{1}{(j\beta + k)} \left[\exp\left(\dfrac{((j\beta + k)x_2 - (j+k)a)\pi i}{b-a}\right) - \exp\left(\dfrac{((j\beta + k)x_1 - (j+k)a)\pi i}{b-a}\right)\right], & \text{otherwise.} \end{cases} \tag{36}$$

and

$$M_{k,j}^s(x_1, x_2) = \begin{cases} \dfrac{(x_2 - x_1)\pi i}{b-a}, & \text{if } k = j = 0, \\ \frac{1}{(j\beta - k)} \left[\exp\left(\dfrac{((j\beta - k)x_2 - (j-k)a)\pi i}{b-a}\right) - \exp\left(\dfrac{((j\beta - k)x_1 - (j-k)a)\pi i}{b-a}\right)\right], & \text{otherwise.} \end{cases} \tag{37}$$

The matrices M_s and M_c have a Toeplitz and Hankel structure, respectively, only if $M_s(i,j) = M_s(i+1, j+1)$ and $M_c(i,j) = M_c(i-1, j-1)$. In that case, the Fast Fourier Transform can be applied directly for highly efficient matrix-vector multiplication [3], and the resulting computational complexity [3] will be $O((\mathcal{M}-1)N\log_2 N)$. We obtain however terms of the form $j\beta - k, j\beta + k$ in the matrix elements in (36) and (37), in particular for the OU process with $\beta \neq 1$, instead of terms with $j - k, j + k$ as obtained for the Lévy jump processes, with $\beta = 1$ in (12). Terms with $\beta \notin \mathbb{N} \cup \{0\}$ hamper an efficient computation of the matrix-vector products, leading to computations with $O((\mathcal{M}-1)N^2)$ complexity. For the mean-reverting OU process and the parameter values of interest here, however, we can resort to a *reformulated process*, as described in Appendix A.

[3] To be precise, we need three times of the forward Fast Fourier Transform (FFT) and twice the Inverse Fast Fourier Transform (FFT^{-1}).

Since the computation of $G_k(x_1, x_2)$ is linear in N, the overall complexity to determine the V_k-coefficients is dominated by the computation of $C(x_1, x_2, t_m)$, whose complexity is $O(N \log_2 N)$ with the FFT. As a result, the overall computational complexity for pricing a Bermudan option with $\mathcal{M}$ exercise dates is $O((\mathcal{M} - 1)N \log_2 N)$, as the work needed for the final step, from t_1 to t_0, is $O(N)$.

Although the algorithm above is only the first step towards solving the pricing problem, it can also be viewed as the complete algorithm for swing options if the option holder is only allowed to conduct a swing action once.

3.3 Algorithm for Interval $t \in I_{n_s} \backslash I_1$

Recall that n_s represents the upper bound for the number of swing rights that can be exercised, as defined in (2). In the time interval $I_{n_s} \backslash I_1$, the option holder has more than one possibility to exercise the swing option. Therefore, apart from the exercise time, the optimal number of commodities to be exercised, D, should also be determined, due to its influence on the recovery time.

Remark 3.2. *In our discussion we deal with the following three functions:*

- $c(x, t_m)$*, the continuation value, which is typically continuous and differentiable. Moreover, its derivative is usually also continuous.*
- $g(x, t_m, D)$*, the payoff, which is continuous and piecewise differentiable (see Figure 1).*
- $v(x, t_m)$*, the option value, which is piecewise continuous in time.* $v(x, t)$ *jumps at* t_k^* *where the number of swing rights is decreased by 1.*

Note that for any $k = 1, \cdots, n_s - 1$*, the equality* $v(t_k^*) = v(t_k^+)$ *may not be satisfied, since the number of possible exercise times is reduced by 1 from* t_k^* *to* t_k^+ *. However, numerically we assume that* t_k^+ *is "arbitrarily close" to* t_k^**. They are considered to lie at the same discrete time point. So, we assume* $t_k^* - t = t_k^+ - t$*, so that* $c(x, t_k^*) = c(x, t_k^+)$ *and* $v(x, t_k^*) \geq v(x, t_k^+)$*.*

Under these assumptions we have that

$$e^{-r(t_k^* - t)}\mathbb{E}_{x,t}(v(x, t_k^*)) \geq e^{-r(t_k^+ - t)}\mathbb{E}_{x,t}(v(x, t_k^+))$$

3.3.1 Model Analysis

By Q and Q_k we denote the continuous interval $\{(x, t)|x \geq 0, t \in [T_0, t_1^*]\}$ and the discrete set $\{(x, t)|x \geq 0, t \in [T_0, t_1^*], t \equiv t_k^* := T - k\tau_R(1), k = 1, \cdots, n_s - 1\}$, respectively.

The swing option value for $(x, t) \in Q \backslash Q_k$ is then given by

$$v(x, t) = \max(\max_D \tilde{v}_{AM}(\bar{g}(x, t, D)), e^{-r(t_k^+ - t)}\mathbb{E}_{x,t}(v(x, t_k^+))), \quad (x, t) \in Q \backslash Q_k \tag{38}$$

where $\tilde{v}_{AM}(\bar{g}(x, t, D))$ represents the value of an American-style option in any interval $I_{k+1} \backslash I_k$ with payoff $\bar{g}(x, tD) = g(x, t, D) + \phi_D^t(x, t)$.

The quantity $e^{-r(t_k^+ - t)}\mathbb{E}_{x,t}(v(x, t_k^+))$ represents the value of a European option, which cannot be larger than the American option. The term $e^{-r(t_k^+ - t)}\mathbb{E}_{x,t}(v(x, t_k^+))$

is therefore implicitly already included in the first term in (38), so that we find, for (38),

$$\begin{aligned} v(x,t) &= \max_D \tilde{v}_{AM}(g(x,t,D) + \phi_D^t(x,t)) \\ &= \max_D(\max(g(x,t,D) + \phi_D^t(x,t), c(x,t))) \\ &= \max(\max_D g(x,t,D) + \phi_D^t(x,t), c(x,t)), \quad (x,t) \in Q \backslash Q_k, \end{aligned} \tag{39}$$

where $c(x,t)$ is the continuation value. Therefore, the price for $(x,t) \in Q\backslash Q_k$ is reduced to the maximum of American option values over D, i.e. $v_1(x,t)$ as defined in Section 2.2.

On the other hand, for $(x,t_k^*) \in Q_k$, the value $v(x,t_k^*)$ is defined by

$$v(x,t_k^*) = \max(\max_D \bar{g}(x,t_k^*,D), v(x,t_k^+)). \tag{40}$$

After application of (39) to the right-hand side of (40), we can rewrite (40) as

$$v(x,t_k^*) = \max(\max_D \bar{g}(x,t_k^*,D), \max_D \bar{g}(x,t_k^+,D), c(x,t_k^+)), \tag{41}$$

where we assume $c(x,t_k^+) = c(x,t_k^*)$, and $\bar{g}$ is as in (3),(4).

If $t_k^* + \tau_R(D) \in Q\backslash Q_k$, with the number of exercise possibilities the same for $t_k^* + \tau_R(D)$ and $t_k^+ + \tau_R(D)$, we have $v(x,t_k^* + \tau_R(D)) = v(x,t_k^+ + \tau_R(D))$. If $t_k^* + \tau_R(D) \in Q_k$, we have $v(x,t_k^* + \tau_R(D)) \geq v(x,t_k^+ + \tau_R(D))$.

So, $v(x,t_k^* + \tau_R(D)) \geq v(x,t_k^+ + \tau_R(D))$ for any x, thus from (4) we have $\phi_D^{t_k^*}(x,t_k^*) \geq \phi_D^{t_k^+}(x,t_k^+)$. Equation (41) is now given by:

$$v(x,t_k^*) = \max(\max_D g(x,t_k^*,D) + \phi_D^{t_k^*}(x,t_k^*), c(x,t_k^*)) \tag{42}$$

As a result, from (39) and (42), we find that for all $t \in [T_0, t_1^*]$:

$$v(x,t) = \max(\max_D g(x,t,D) + \phi_D^t(x,t), c(x,t)) \tag{43}$$

Equation (43) tells us that the swing option is an American-style option with recovery time and multiple exercise opportunities. Its pricing algorithm is therefore different from a standard American option. Instead of taking the maximum of the payoff and the continuation value, we take the maximum over the resulting payoff for all possible values of D, and the continuation value from the previous time step. Another difference is that for any amount, D, the payoff also includes the term $\phi_D^t(x,t)$ from an earlier time step.

It is easy to determine the value of $g(x,t,D)$ for any x,t,D according to (6). We therefore focus on the values $\phi_D^t(x,t)$ and $c(x,t)$, which are both obtained in the recursion of Fourier cosine coefficients V_k. To calculate $c(x,t_m)$, one only needs the value of $V_k(t_{m+1})$, like in the case of a Bermudan option. However, to compute the value of $\phi_D^t(x,t)$ we need the coefficients $V_k(t+\tau_R(D))$, that depend on the function for the recovery time.

Remark 3.3. *In time interval $t \in [0,T_0]$ swing actions are not yet allowed. Therefore, we have:*

$$v(t,x) = e^{-r(T_0-t)} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a};x)e^{-ik\pi\frac{a}{b-a}})V_k(T_0),$$

where $V_k(T_0)$ is obtained by a backward recursion procedure.

3.3.2 The Early-exercise Points

In this section we consider the state-dependent recovery time, $\tau_R(D)$, which is assumed to be an increasing function of D.

The option value is obtained by means of a backward recursion on $V_k(t_m), m = \mathcal{M}-1, \cdots, 1$. At each time step, as shown in Section 3.3.1, the payoff, $\bar{g}(x, t_m, D)$, for all possible values of D and the continuation value, $c(x, t_m)$, are compared. The largest value represents the swing option value at t_m. We therefore need to identify the following regions in our pricing domain:

- $A_D, D = 1, \cdots, L$: the regions in which exercising the swing option with D commodity units will result in the highest profit $g(x, t_m, D) + \phi_D^{t_m}(x, t_m)$.
- A_c: The region in which $c(x, t)$ is the maximum. In other words, with the commodity price in A_c, it is profitable *not* to exercise the swing option.

With these regions determined, the Fourier cosine coefficients, $V_k(t_m)$, for the swing option can be determined with a splitting, as follows,

$$\begin{aligned} V_k(t_m) &= \frac{2}{b-a}\Bigg(\int_{A_c} c(x, t_{m+1}) \cos(\frac{k\pi(x-a)}{b-a})dx \\ &+ \sum_{D=1}^{L} \int_{A_D} g(x, t_m, D) \cos(\frac{k\pi(x-a)}{b-a})dx\Bigg) \end{aligned} \tag{44}$$

We now describe the procedure to locate the different regions A_c and $A_D, D = 1, \cdots, L$. As an example, let us first look at the payoff functions for *two* values $D = D_1$ and $D = D_2$ where $D_1 > D_2$, shown in Figure 3. Points $x^d(D_1, D_2)$

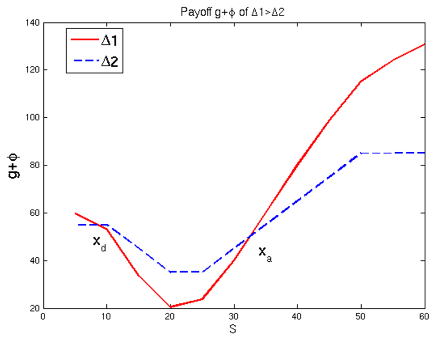


Figure 3: Payoff function $g + \phi$ for two different D.

and $x^a(D_1, D_2)$ denotes the "early-exercise points", where the strategy of exercising D_1 or D_2 units results in the same $\bar{g}$-values. Between $x^d(D_1, D_2)$ and $x^a(D_1, D_2)$, the value for D_2 is largest, in other words, it is profitable to exercise a smaller amount of commodity. Beyond $x^d(D_1, D_2)$ and $x^a(D_1, D_2)$, it is profitable to exercise the larger amount D_1.

Remark 3.4. *A rough explanation of the behavior of the two payoff functions in Figure 3 is as follows. The payoff is a sum of $g(x, t, D)$ and $\phi_D^t(x, t)$. For*

D increasing, the true payoff $g(x, t, D)$ increases, but the quantity $\phi_D^t(x, t)$ decreases because of the longer recovery time penalty. So, if $g(x, t, D)$ is the largest term in the sum, it is profitable to exercise with a larger value of D, whereas if $\phi_D^t(x, t)$ is the dominating part, it is profitable to exercise the smaller amount.

From the payoff in Figure 1 we see that payoff g equals zero when asset price S is between K_d and K_a, so that the quantity ϕ_D^t will be the main contribution. With S goes beyond K_d and K_a, payoff g increases and contributes more to the sum. Note that this explanation as well as the behavior of the two different payoff functions in Figure 3, form the basis for any two payoff functions with different D-values.

Based on the insight in Remark 3.4, let us look at a second simple example with $L = 4$ and determine A_2, i.e. the region where it is profitable to exercise with $D = 2$. The example is detailed in Figure 4, where the relation between the payoffs for any two different amounts of commodity is graphically sketched.

In the figure, a zero "0", implies taking the continuation value $c(x, t)$. $x^d(2, D_j)$, $x^a(2, D_j)$, $j = 0, 1, 3, 4$ are the two sets of points where $D = D_j$ returns the same payoff value as $D = 2$. In order to determine the region A_2, we need to find the sub-regions in which $D = 2$ gives the largest payoff compared to the other D-values.

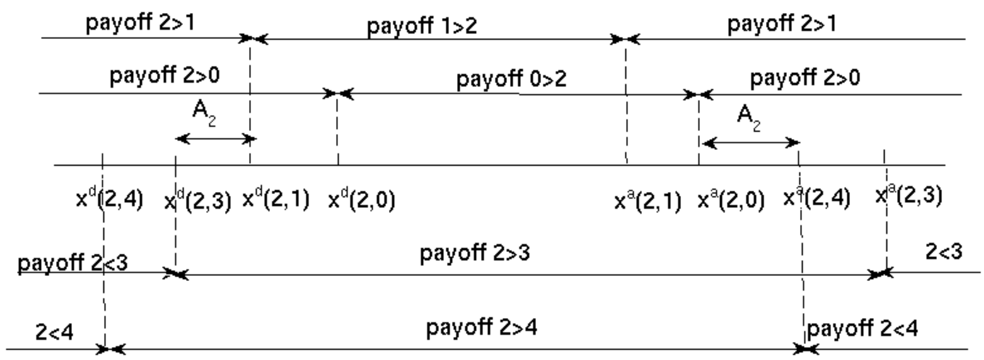


Figure 4: An example to illustrate the exercise region A_2 with $L = 4$.

The value $D = 2$ returns a larger value than $c(x, t)$, if $x < x^d(2, 0)$ or $x > x^a(2, 0)$; Similarly, $D = 2$ returns a larger value than $D = 1$, if $x < x^d(2, 1)$ or $x > x^a(2, 1)$. So, $D = 2$ returns larger values than both $c(x, t)$ and $D = 1$, if x is either smaller than both $x^d(2, 0)$ and $x^d(2, 1)$, or larger than both $x^a(2, 0)$ and $x^a(2, 1)$. To determine these regions we compute the following early-exercise points (see again Figure 4 for the values of U and W for this example):

- $U := \min(x^d(2, 0), x^d(2, 1)) \equiv x^d(2, 1)$,

- $W := \max(x^a(2,0), x^a(2,1)) \equiv x^a(2,0)$.

$D = 2$ now returns a larger value for $x < U$ or $x > W$.

We proceed in the same spirit: To make sure that $D = 2$ returns larger values than $D = 3$ and $D = 4$, x should be *larger* than both $x^d(2,3)$ and $x^d(2,4)$, or *smaller* than both $x^a(2,3)$ and $x^a(2,4)$. This is again related to the global behavior of the payoff functions with $D_1 > D_2$, as in Figure 3. Therefore we calculate

- $P := \max(x^d(2,3), x^d(2,4)) \equiv x^d(2,3)$
- $Q := \min(x^a(2,3), x^a(2,4)) \equiv x^a(2,4)$

Now $D = 2$ returns a larger value than $D = 3$ and $D = 4$ for $x > P$ or $x < Q$.

So, $D = 2$ returns the largest value, if $P < x < U$ or $W < x < Q$; Therefore, $A_2 = [P, U] \cup [W, Q]$, as shown in Figure 4.

More generally, for each $D = 1, \cdots, L$, we determine:

$$P_D = \max_{j>D} x^d(D,j),\ Q_D = \min_{j>D} x^a(D,j),\ U_D = \min_{j<D} x^d(D,j),\ W_D = \max_{j<D} x^a(D,j),$$

and set $A_D = [P_D, U_D] \cup [W_D, Q_D]$. Here P_D, Q_D represent the early-exercise interval boundaries, within which exercising D units of commodity returns a larger payoff than exercising more units. U_D, W_D are the left and right boundary, respectively, beyond which exercising D units returns a larger value than when fewer or no units are exercised. Similarly, we have

$$\begin{aligned} A_L &= [a, \min_{j<L} x^d(L,j)] \cup [\max_{j<L} x^a(L,j), b], \\ A_c &= [\max_{j>0} x^d(0,j), \min_{j>0} x^a(0,j)] \end{aligned}$$

All early-exercise points, $x^d(D,j), x^a(D,j),\ j = 0, \ldots, L$, are computed by Newton's method.

With the regions A_c and $A_D, D = 1, \cdots, L$ fixed, Equation (44) can be rewritten as:

$$\begin{aligned} V_k(t_m) &= C_k(\max_{j=1,\cdots,L} x^d(0,j), \min_{j=1,\cdots,L} x^a(0,j), t_m) + \sum_{D=1}^{L} G_k(P_D, U_D, D) \\ &+ \sum_{D=1}^{L} G_k(W_D, Q_D, D) + G_k(a, \min_{j=0,\cdots,L-1} x^d(L,j), L) \\ &+ G_k(\max_{j=0,\cdots,L-1} x^a(L,j), b, L). \end{aligned} \tag{45}$$

The computation of $C_k(x_1, x_2, t_m)$ in (45) is as in (34). The G_k differ from the expressions (29) ,. . . , (32), which will be described in detail in Subsection 3.3.3.

In the Newton procedure to find the points $x^d(D_i, D_j)$ and $x^a(D_i, D_j)$ we need to find the values of $c(x, t_m), g(x, t_m, D), \partial c/\partial x$ and $\partial g/\partial x$ as in Subsection 3.2. The values of $\phi_D^{t_m}(x, t_m)$ and $\partial \phi_D^{t_m}/\partial x$ are found by:

$$\begin{aligned} \phi_D^{t_m}(x, t_m) &= e^{-r\tau_R(D)} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a}; x, \tau_R(D)) e^{-ik\pi\frac{a}{b-a}}) V_k(t_m + \tau_R(D)), \\ \frac{\partial \phi_D^{t_m}}{\partial x} &= e^{-r\tau_R(D)} {\sum_{k=0}^{N-1}}' Re(\varphi(\frac{k\pi}{b-a}; x, \tau_R(D)) \cdot i\frac{k\pi}{b-a} e^{-ik\pi\frac{a}{b-a}}) \\ &\quad \cdot V_k(t_m + \tau_R(D)). \end{aligned}$$

Remark 3.5 (Computation of $V_k(t_m+\tau_R(D))$)**.** *To calculate $V_k(t_m+\tau_R(D))$, we determine a time step, Δt, so that $T-t$ and $\tau_R(D)$ are both time points. So, we set $\mathcal{M}=T-t/\Delta t, N_D=\tau_R(D)/\Delta t$, $D=1,\cdots,L$. For $t_m+\tau_R(D)=t_m+N_D\Delta t\le T$, the value $V_k(t_m+\tau_R(D))=V_k(t_m+N_D\Delta t)$. The values $V_k(t_m+\tau_R(D))=0$ for all k if $t_m+N_D\Delta t>T$. In that case, $\phi_D^{t_m}$ and $\partial\phi_D^{t_m}/\partial x$ are zero, as they are linear combinations of $V_k(t_m+\tau_R(D))$. In this setting, $V_k(t_m)$ and $V_k(t_m+\tau_R(D)), D=1,\cdots,L$ can be determined in one recursion, in which the intermediate values of V_k need to be stored for later use.*

3.3.3 Calculation of $G_k(x_1,x_2,D)$

The terms G_k in (44) are split into two parts, i.e.

$$G_k(x_1,x_2,D)=G_{k,g}(x_1,x_2,D)+G_{k,c}(x_1,x_2,D),$$

with $G_{k,g}$ from an instantaneous profit $g(x,t_m,D)$, and $G_{k,c}$ the part generated by $\phi_D^{t_m}(x,t_m)$, i.e., the continuation value from time point $t_m+\tau_R(D)$, as defined in (4).

Equations (29) and (30) can be used to compute $G_{k,g}(a,\min_{j<L}x^d(L,j),L)$ and $G_{k,g}(P_D,U_D,D), D=1,\cdots,L$, unless $P_D>ln(S_{min})$ where we use,

$$G_{k,g}(P_D,U_D,D)=D\cdot\frac{2}{b-a}(K_d\psi_k(P_D,U_D)-\chi_k(P_D,U_D)).$$

Similarly, the quantities $G_k(\max_{j<L}x^a_{Lj},b,L)$ and $G_k(W_i,Q_i,i), i=1,\cdots,L$ can be computed by (31) and (32), unless if $Q_i<ln(S_{max})$ for which we have

$$G_{k,g}(W_i,Q_i,i)=i\cdot\frac{2}{b-a}(\chi_k(W_i,Q_i)-K_a\psi_k(W_i,Q_i)).$$

Finally, the quantity $G_{k,c}(x_1,x_2,D)$ can be obtained by (34), replacing Δt and $V_j(t_{m+1})$ by $\tau_R(D)$ and $V_j(t_m+\tau_R(D))$, respectively.

Remark 3.6 (Constant recovery time)**.** *If the recovery time does not depend on D, we call the recovery time constant. This can be viewed as a special case of the pricing method discussed above. As additional profit is not related to an extra penalty, if it is profitable to exercise the swing option, we have $D_{opt}\equiv L$ from a profit maximizing point-of-view. Hence, at any point in time, we have either $D=0$, or $D=L$.*

Newton's method is now applied to determine two early-exercise points x_m^d and x_m^a, so that

$$c(x_m^d,t_m)=g(x_m^d,t_m,L)+\phi_L^{t_m}(x_m^d,t_m),$$

and

$$c(x_m^a,t_m)=g(x_m^a,t_m,L)+\phi_L^{t_m}(x_m^a,t_m),$$

with $D=L$ and $\tau_R(D)$ constant. Then $V_k(t_m)$ is split into three parts,

$$V_k(t_m)=G_k(a,x_m^d,L)+C_k(x_m^d,x_m^a,t_m)+G_k(x_m^a,b,L),$$

that can be calculated as in the case of state-dependent recovery time.

4 Numerical Results

In this section we demonstrate the performance of our pricing algorithm for swing options with constant and dynamic recovery times. The CPU used is an Intel (R) Core (TM) 2 Duo CPU E6550 2.33GHz, Cache size 4MB, and the algorithm is programmed in MATLAB 7.5. The two sub-sections to follow present results with two different types of recovery time:

- Constant recovery time is in Subsection 4.1: If $D \neq 0$, we set $\tau_R(D, t) = \frac{1}{4}$, as in [1]. In other words, the option holder needs to wait 3 months between two consecutive swing actions, independent of the time point of exercise or the size D.
- State-dependent recovery time is in Subsection 4.2: We assume $\tau_R(D, t) = D/12$ which implies that if the option holder exercises the swing option with D units, he/she has to wait D months before the option can be exercised again.

Parameter sets used for numerical examples are (unless stated otherwise):

$$\text{CGMY} \qquad C = 1, G = 5, M = 5, Y = 1.5, r = 0.05, \tag{46}$$

$$\text{OU}: \qquad \kappa = 0.301, \bar{x} = 3.150, \sigma = 0.334, r = 0.05, \tag{47}$$

where, for the OU process the value of $\bar{x}$ is under the Q-measure. The values set for the OU process is as in [1]. The values for CGMY, in particular $Y > 1$ (infinite activity jump process) are known to be particularly difficult for PIDE solvers. We will see here that these CGMY parameters do not pose any problem for the swing option COS method.

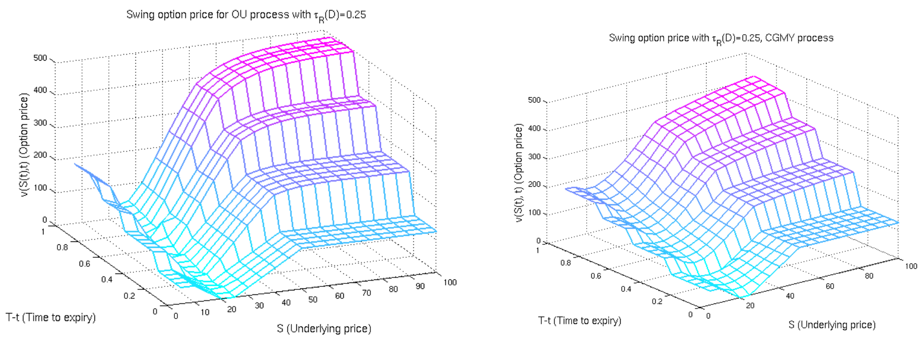
In the numerical experiments we further choose $S_{min} = 10, K_d = 20, K_a = 25, S_{max} = 50, T_0 = 0$. The choice $T_0 = 0$ does not pose any restrictions on the algorithm, as we can simply change it to any $T_0 > 0$.

4.1 Constant Recovery Time

First of all, American-style swing option values under the CGMY and OU processes, with $L = 5$, are presented in Figure 5, with as independent variables S and t; $v(S(t), t)$ is the swing option value. Jumps in the swing option values are observed at $T - t = 0.25, T - t = 0.5$ and $T - t = 0.75$. This can be explained by the fact that at these time points the maximum number of times the holder can exercise, n_s, is reduced by one. For instance, time point $T - t = 0.5$ is the last time point at which an option holder can exercise up to three times. For any $t > T - 0.5$, the holder cannot exercise more than twice.

Due to the constant recovery time, we should exercise $L = 5$ units if it is profitable to exercise. Hence for $S > 50$, with $K_a = 25$, the profit would be $L \cdot (50 - 25) = 125$. When $t \approx 0$, we have at maximum four possibilities to exercise, which is the reason for option values as high as 500 in Figure 5.

Next, we discuss the convergence behavior of the option values over N, the number of terms in the Fourier cosine series. Again the CGMY and OU processes are used, with the parameters in (46), (47). The remaining parameters are $\tau_R = 0.25, T - t = 1, \mathcal{M} = 12$; $S_0 = 8$ is set for the CGMY experiment and $S_0 = \exp(\bar{x})$ for the OU problem.



(a) OU (b) CGMY

Figure 5: American-style swing option values under the OU and CGMY processes with constant recovery time, $\tau_R(D) = 0.25$.

In Table 1 it is shown that the swing option pricing algorithm under CGMY, with the parameters chosen, takes only 0.0367 seconds to converge to one basis point. A similar convergence is observed for OU process[4] as also shown in Table 1.

N	64	96	128	160	192
CGMY option value	99.9362	53.8713	220.7021	220.7021	220.7021
CPU time (sec.)	0.0232	0.0303	0.0367	0.0467	0.0526
N	96	128	160	192	224
OU option value	51.3677	49.6984	53.4784	53.4784	53.4784
CPU time (sec.)	0.0429	0.0432	0.0493	0.0527	0.0587

Table 1: Swing option prices and CPU time under the CGMY and the OU process, with parameter sets (46), (47).

An American option can be viewed as a Bermudan option with $\mathcal{M} \to \infty$. In Table 2 the performance of two methods to approximate an American-style swing option is compared. One method is the direct approximation by means of Bermudan-style options, by increasing $\mathcal{M}$, whereas the second method is based on the repeated Richardson 4-point extrapolation technique (19) on Bermudan-style swing options with four different numbers of exercise opportunities. In Table 2, the column denoting "$P(N/2)$" gives the computed values of the Bermudan-style options with $\mathcal{M} = N/2$. For the values obtained with the Richardson extrapolation we use $\mathcal{M} = 16$ in (19) (so, $2\mathcal{M} = 32, 4\mathcal{M} = 64, 8\mathcal{M} = 128$).

The CGMY model is used here with the parameters r, C, G, M, Y, from (46) , and $T-t = 0.5, S_0 = 8, S_{min} = 10, S_{max} = 50, K_d = 20, K_a = 25$. As illustrated in Table 2, to converge to an error of $O(10^{-4})$, one would require 203 seconds with the direct approximation method, and approximately one second with the extrapolation technique.

[4]We used the reformulated characteristic function to accelerate the algorithm for the OU process, see Appendix A

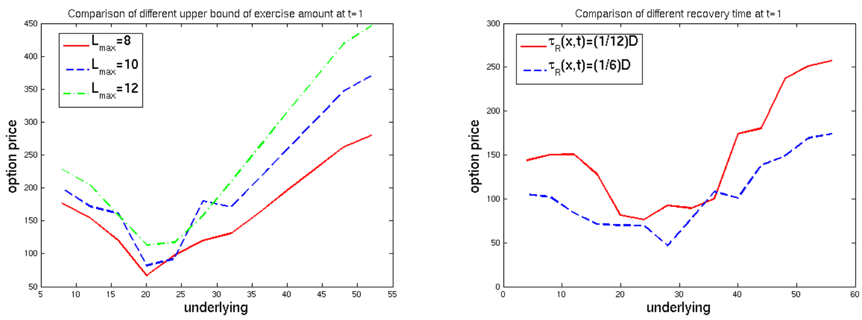
$n = \log_2 N$	$P(N/2)$		Richardson	
	option value	CPU time	option value	CPU time
7	137.423	0.27	137.395	0.59
8	137.408	0.53	137.390	0.99
9	137.399	2.00	137.390	1.79
10	137.394	8.39	137.390	3.40
11	137.392	39.55	137.390	6.68
12	137.391	203.27	137.390	13.21

Table 2: Convergence over $\mathcal{M}$ and comparison between two approximation methods for American-style swing option.

4.2 State-Dependent Recovery Time

We now consider the case where the recovery time depends on the amount D. The use the CGMY model with the parameters from (46). Figure 6a compares the swing option prices with three upper bounds of D: $L = 8, 10, 12$. A higher upper bound typically results in higher option values, because a higher upper bound implies more possibilities for an option holder at each exercise date.

In the case of a *constant recovery time* we find (not shown) that higher values of L *always* give rise to higher option values. In the case of *state-dependent recovery time*, $\tau_R(D)$, an exception is observed for the parameters under consideration, when $25 \leq S \leq 30$. In that case $L = 10$ results in higher option values than $L = 12$, see Figure 6a. In this interval, $g(x, t, D)$ is small and $\phi^t_D(x, t)$ is the dominant part of the profit. Larger D-values lead to smaller ϕ^t_D-values. However, with S further on the left side of K_d or at the right side of K_a, function $g(x, t, D)$ starts to dominate and larger L-values give higher swing option values.



(a) Varying amount L

(b) Varying recovery time $\tau_R(D)$

Figure 6: CGMY process, $T - t = 1$; Left: Different values for L, and fixed $\tau_R(D, t) = \frac{1}{12}D$; Right: Different Recovery time, and fixed $L = 5$.

Figure 6b illustrates the influence of the recovery time on the swing option value. Here we compare $\tau_R(D) = \frac{1}{12}D$ with $\tau_R(D) = \frac{1}{6}D$, which corresponds to one month (solid line) or two months (dashed line) penalty time for each unit exercised. Figure 6b shows that longer recovery time gives lower option prices. In other words, if one can wait after exercising one pays less for the swing option [5].

Table 3 shows how the option value and optimal value of D (i.e., D_{opt}) change over time. Here we take $L = 8$, and $S_0 = 8$, a case where the option is deep in-the-money. As expected, jumps in the optimal D-values are observed at $t_k^* = T - k\tau_R(1)$.

Recovery time $\tau_R(D) = \frac{1}{12}D$ implies that if we exercise k or fewer units at t_k^*, we can exercise once more before expiry T, whereas if we exercise more than k units, we cannot exercise again before T. In other words, at t_k^*, $\phi_D^t > 0$ for $D \le k$ and $\phi_D^t = 0$ otherwise.

Note that at the time points $t = T$ and $T - t = 1/24$, the optimal value equals $D_{opt} = L = 8$. For $t = T$ this is due to the arbitrage-free condition and the profit maximization principle, whereas for $T - t = 1/24$ the time left is so small that, in our setting, there is only one chance left for a swing action ($\phi_D^t = 0$ for all D, k). One should then choose the largest D-value allowed to get an optimal profit.

T-t	option value	D_{opt}	T-t	option value	D_{opt}
0	80	8	8/24	110.587	4
1/24	80	8	9/24	111.556	4
2/24	85.489	1	10/24	120.572	5
3/24	85.794	1	11/24	121.806	5
4/24	92.441	2	12/24	130.769	6
5/24	93.116	2	13/24	132.224	6
6/24	101.058	3	14/24	141.051	7
7/24	102.371	3	15/24	142.690	7

Table 3: D_{opt} over time $L = 8, S_0 = 8, \tau_R = \frac{D}{12}$.

Figure 7 shows how D_{opt} changes w.r.t. the underlying price, with $L = 8, T-t = 1, \tau_R(D) = \frac{1}{12}D$. As S goes beyond K_d and K_a, D_{opt} tends to increase, because in this region the payoff $g(x, t, D)$ dominates in the term $g(x, t, D) + \phi_D^t(x, t)$. Between $S = 20$ and $S = 25$, $D_{opt} = 0$, since $g(x, t, D) = 0$ for all $D > 0$ in this interval.

Next, the convergence of the swing option value over N, and the corresponding CPU time for the CGMY process, with $S_0 = 8, T-t = 1$ and different upper bounds L, are presented in Table 4. With $N = 256$ the swing option algorithm reaches basis point accuracy. Table 4 also illustrates that the algorithm is flexible regarding the variation in parameter L. Large L-values result in higher CPU times, since an increasing number of early-exercise points needs to be determined, and more C_k- and G_k-terms have to be computed.

In the next experiment, we use the CGMY model with $Y = 0.5$ (other parameters as in (46)). We compare for American-style swing option values, with the state-dependent recovery time, the approximation obtained by the 4–point Richardson extrapolation with the direct approximation, obtained with Bermudan option values with $\mathcal{M}$ increasing. Table 5 shows that the 4–point Richardson extrapolation is much more efficient than the direct method, and that both

[5]Similarly, smaller recovery times result in higher option prices with *constant recovery time*.

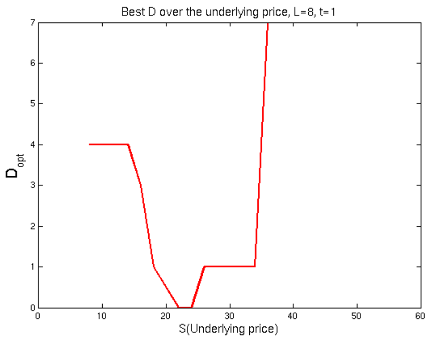


Figure 7: D_{opt} over underlying price, $L = 8, T - t = 1, \tau_R(D) = \frac{1}{12}D$

	N	128	256	512
L=2	option price	128.7532	136.8724	136.8724
	CPU time	0.0868	0.1669	0.2466
L=5	option price	138.2815	150.0041	150.0041
	CPU time	0.3943	0.6505	1.1660
L=10	option price	186.6296	199.6870	199.6870
	CPU time	1.4428	2.4115	4.3819

Table 4: Swing option values for CGMY process, dynamic recovery time, $S_0 = 8, T - t = 1$.

methods converge to the same American swing option values. Convergence of the Richardson extrapolation is already observed with $\mathcal{M}$, the number of exercise dates in (19) equal to 6. Larger values of $\mathcal{M}$ give the same extrapolation result.

Bermudan approximation			Richardson approximation		
$\mathcal{M} = N/2$	option value	CPU time	N	option value	CPU time
128	93.9501	5.7391	64	93.9710	1.6077
256	93.9710	20.1821	128	93.9707	2.3621
512	93.9707	77.0859	256	93.9707	3.9196

Table 5: Convergence over $\mathcal{M}$ and comparison between two approximation methods for American-style swing option, CGMY model, $S_0 = 10, L = 5, Y = 0.5$.

5 Conclusions

In this paper, we presented an efficient, flexible and robust pricing algorithm for swing options with early-exercise features. It performs well for different swing contracts with varying flexibility in upper bounds of exercise amount and recovery times. The algorithm is based on Fourier cosine series expansions, and can be applied to swing option pricing under different commodity processes, such as CGMY, other Lévy processes, or under the OU process. For Lévy processes the Fast Fourier Transform can be applied in the backward recursion procedure, which gives us Bermudan-style swing option prices accurate to one basis point in milli-seconds for constant recovery time, and in less than one to three seconds for dynamic recovery time with different values of L. The Richardson 4-point extrapolation technique can be used to price American-style swing options efficiently.

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References

[1] M.Dahlgren, A continuous time model to price commodity-based swing option., *Review of derivatives research,* 8,27–47, 2005.

[2] F.Fang and C.W.Oosterlee, A novel pricing method for European options based on Fourier cosine series expansions, *SIAM J. Sci Comput.*, 31:826-848, 2008.

[3] F. Fang and C.W.Oosterlee. Pricing Early-Exercise and Discrete Barrier Options by Fourier-Cosine Series Expansions. *Numerische Mathematik* 114: 27-62, 2009.

[4] C-C. Chang, S-L Chung and R.C. Stapleton, Richardson extrapolation technique for pricing American-style options. *J. Futures Markets,* 27(8): 791-817, 2007.

[5] G.E.Uhlenbeck and L.S.Ornstein, On the theory of Brownian motion. *Phys Rev,* 36:823–41, 1930.

[6] T. Björk, Arbitrage theory in continuous time. *Oxford Univ. Press*, 1998.

[7] P. Carr, H. Geman, D.B. Madan, and M. Yor The fine structure of asset returns: An empirical investigation. *Journal of Business,* Vol. 75, no. 2, 2002.

[8] G.V. Pflug, N. Broussev Electricity swing option: Behavioral models and pricing. *European journal of operational research*, 2008.

[9] J. Andreasen and M. Dahlgren At the flick of a switch. *Energy risk, 71–75*, February, 2006.

[10] P. Jaillet, E.I. Ronn, S. Tompaidis Valuation of commodity-based swing option. *Management science*, December, 2003.

[11] M. Kjaer Pricing of swing options in a mean–reverting model with jumps. *Quantitative analytics, Barclays capital*, January, 2007.

[12] A. B. Zeghal, M. Mnif Optimal multiple stopping and valuation of swing options in Lévy models. *International Journal of Theoretical and Applied Finance, 1267–1297*, 2006.

[13] R. Carmona and N. Touzi Optimal multiple stopping and valuation of swing options, *Mathematical Finance 18, 2, pp. 239-268.* , April, 2008.

[14] A. Lari–Lavassani, M. Simchi and A. Ware A discrete valuation of swing options. *Canadian applied mathematics quarterly*, Volume 9, Number 1, Spring 2001.

[15] C. Cryer, The solution of a quadratic programming problem using systematic overrelaxation. *SIAM J. Control*, 9, 385–392, 1971.

A Reformulated Characteristic Function for OU Process

For pricing Bermudan-style options under Lévy processes the Fast Fourier Transform can be applied for a highly efficient computation. This is unfortunately not the case for such options under the OU process (7), so that the resulting computational complexity is $O(\mathcal{M}-1)N^2$.

Here we present a first remedy to be able to also price Bermudan options under an OU process highly efficiently, however, only for special parameter sets. It is known from the literature that the OU process, $x(t)$, admits the solution:

$$x(t) = x_0 \mathrm{e}^{-\kappa t} + \overline{x}\left(1 - \mathrm{e}^{-\kappa t}\right) + \int_0^t \sigma \mathrm{e}^{\kappa(s-t)} \mathrm{d}W(s),$$

i.e., $x(t)$ is normally distributed, i.e.: $x(t) \sim \mathcal{N}(\mathbb{E}(x(t)), \mathbb{V}\mathrm{ar}(x(t)))$, with:

$$\mathbb{E}(x(t)|\mathcal{F}_0) = x_0 \mathrm{e}^{-\kappa t} + \overline{x}\left(1 - \mathrm{e}^{-\kappa t}\right), \tag{48}$$

$$\mathbb{V}\mathrm{ar}(x(t)|\mathcal{F}_0) = \frac{\sigma^2}{2\kappa}\left(1 - \mathrm{e}^{-2\kappa t}\right). \tag{49}$$

Therefore, we can reformulate the OU process, and define a process $y(t)$ in the following way:

$$dy(t) = \kappa(\bar{x} - x_0)e^{-\kappa t}dt + \sigma e^{-\kappa t}dW(t), y_0 = x_0$$

Then $y(t)$ is distribution-wise equal to OU process $x(t)$, whose characteristic function is

$$\varphi_Y(\omega, t) = e^{i\omega y_0 + A(\omega,t)} \tag{50}$$

where

$$A(\omega, t) = \frac{\omega}{4\kappa} e^{-2\kappa t}(1 - e^{\kappa t})(\omega\sigma^2 + e^{\kappa t}(4i(y_0 - \bar{x})\kappa + \omega\sigma^2)) \tag{51}$$

Since y_0 appears in (51), we still cannot perform the integral computations fully efficiently. However, due to the mean-reversion, the underlying will return to its long term mean, $\bar{x}$, after a certain time. In our swing option experiments, we set $S_0 = exp(\bar{x})$, which has a very favorable effect on the characteristic function. By fixing $y_0 = \bar{x}$ in (51) and applying (50), the FFT can be applied and the computational complexity is reduced to $O(\mathcal{M}-1)N \log_2 N$, like in the case of Lévy processes. The error due to the approximation is, compared to the computation with the original characteristic function of the OU process, less than a basis point.

With arbitrary initial value, y_0, the approximation is still valid and accurate, in particular for large speed of mean reversion, κ, and small volatility, σ. For other parameter sets, it is recommended to use the original OU characteristic function. Improvement of this is a topic of future research.