On American Options Under the Variance Gamma Process

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Abstract

We consider American options in a market where the underlying asset follows a Variance Gamma process. We prove results on the continuity of the exercise boundary, on the smooth fit principle and on the behavior of the exercise boundary near maturity. We also propose a numerical method to find the American option price and the exercise boundary. It is known that the American option price satisfies a Partial Integro-Differential Equation (PIDE) in a domain with a moving boundary. We reformulate the problem as a Linear Complementarity Problem and solve it iteratively by a convenient splitting with the help of the Fast Fourier Transform. Finally, we verify the theoretical results obtained throughout a series of numerical experiments.

Keywords: Integro-differential equations, Variance Gamma, finite differences, FFT.

1 Introduction

The Variance Gamma (VG) process was first introduced in financial modeling by Madan and Seneta [18] to cope with the shortcomings of the Black-Scholes model. Empirical studies of financial time series have revealed that the normality assumption in the Black-Scholes theory cannot capture heavy tails and asymmetries present in the empirical log-returns. The empirical densities are usually too peaked compared to the normal density; a phenomenon known as excess of kurtosis. In addition, the Black-Scholes assumption on constant parameters is inconsistent since, for example, a numerical inversion of the Black-Scholes equation based on market prices from different strikes and fixed maturity, produces a so-called volatility skew or smile. In these aspects the VG modeling

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is superior to the Black-Scholes model: on one hand, it has the property that daily log-returns have long tails and on the other hand, for longer periods it approaches normality, which is also consistent with empirical studies. Moreover, by introducing extra parameters, one can control the kurtosis and asymmetry of the log-return density, and one is also able to fit the smile in the implied volatility; see [7, 18].

There exist however important drawbacks when modeling with the VG process. For example, a hedging strategy for the writer of the option that will completely remove the risk of writing the option does not exist in general, or in other words, a portfolio that replicates any contingent claim cannot be constructed.

In this paper, we prove facts about the behavior of the free boundary and in particular about the failure of the smooth fit principle for American options under the VG process; see also [1, 4, 5, 19]. Secondly, we propose a tractable numerical method based on a Linear Complementarity formulation of the free boundary value problem for the VG prices. A numerical valuation of VG American options was carried out in [14], using finite differences on a non-linear interpretation of the PIDE. Compared to [14], the method proposed here is different and more general, in the sense that it is easily extendible to other finite variation processes, whereas [14] is specially tailored for the VG process. Moreover, the method presented can naturally handle the asymptotic behavior of the free-boundary near expiry.

Another general model based on the Carr-Geman-Madan-Yor (CGMY) process is numerically solved in [19], by a combination of variational inequalities and the Galerkin method, with a convenient wavelet basis to compress the resulting full matrix. Here, a simpler implicit-explicit method is proposed, which, in combination with a fast convolution procedure based on the Fast Fourier Transform, offers an effective pricing procedure for European and American vanilla options, also applicable to the CGMY process. In a previous paper [2], we used similar ideas to numerically solve jump-diffusion European vanilla options; see also the work of d'Halluin et al. [11] for a similar treatment.

The outline of the paper is as follows. In Section 2 we offer a brief introduction into the VG market model and the option pricing problem. Section 3 contains results on the American option price, properties of the free boundary, a proof of the failure of the smooth fit principle and an asymptotic analysis of the exercise boundary near maturity. The next two sections are dedicated to the numerical valuation of a VG call option. In Section 4 we reformulate the problem as a Linear Complementarity Problem and propose a fix-point type iteration to solve it, and finally, in Section 5, we show numerical experiments that confirm the theoretical findings.

2 A market modeled by the Variance Gamma process

This section offers a brief introduction into the theory and applications of the VG process in finance. For further information, we refer the reader to [7, 18]. The VG process belongs to the family of Lévy process of infinite activity. Unlike the classical Samuelson model, or any jump-diffusion model, this process does not have any continuous component and it is of bounded variation. The process is a so-called pure jump process, and the infinite activity means that the paths jump infinitely many times, for each finite interval. Moreover, jumps that are larger than a given quantity occur only a finite number of times.

The VG process is obtained by evaluating a drifted Brownian motion at random times given by a gamma process. More precisely, consider the following drifted Brownian motion:

$$Y(t;\theta,\sigma) = \theta t + \sigma W(t), \tag{1}$$

where $\theta \in \mathbb{R}$ is the drift, $\sigma > 0$ is the volatility and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion. Let $\gamma(t,\mu,\nu)$ be a gamma process, independent of $W(t)$. By definition, the increments $\gamma(t+\Delta t,\mu,\nu) - \gamma(t,\mu,\nu)$ are independently, gamma distributed random variables with mean $\mu\Delta t$ and variance $\nu\Delta t$ on each interval of length Δt. More precisely,

$$\gamma(t+\Delta t,\mu,\nu) - \gamma(t,\mu,\nu) \sim f_\gamma(x;\mu^2\Delta t/\nu, \nu/\mu), \tag{2}$$

where f_γ is the gamma density function:

$$f_\gamma(x;a,b) = \frac{1}{b^a\Gamma(a)}x^{a-1}e^{-x/b}, \quad x > 0. \tag{3}$$

The VG process is defined by substituting t in the drifted Brownian motion (1) by the gamma process $\gamma(t,1,\nu)$, i.e.,

$$X(t;\sigma,\nu,\theta) := Y(\gamma(t,1,\nu);\theta,\sigma) = \theta\gamma(t,1,\nu) + \sigma W(\gamma(t,1,\nu)). \tag{4}$$

The three parameters determining the VG process are: the volatility σ of the Brownian motion, the variance ν of the gamma distributed time and the drift θ of the time-changed Brownian motion with drift. Following [7], the parameter θ measures the degree of skewness of the distribution and ν controls the excess of kurtosis with respect to the normal distribution. For example, for the symmetric case $\theta = 0$, and for $t = 1$, the value of kurtosis results in $3(1+\nu)$ (see [18]). Since the kurtosis of a normal distribution is 3, this result says that ν measures the degree of "peakedness" with respect to the normal distribution. A large value of ν results in fat tails, which is observed in the empirical log-returns. Heuristically, for $\nu \to 0$, the time change is close to the linear time change, so that the VG process approximates a drifted Brownian motion.

A market model

Consider a market consisting of one bank account $B(t)$, with risk-free interest rate r, and some risky asset $S(t)$. The bank account evolves as usual according to the law $dB(t) = rB(t)dt$ and the asset $\{S(t)\}_{t\geq 0}$ follows the exponential dynamics:

$$S(t) = S_0 \exp(L(t)), \tag{5}$$

where

$$L(t) = -\alpha t + X(t; \sigma, \nu, \theta). \tag{6}$$

Here $-\alpha$ is the drift of the logarithmic price of the asset. We assume also that the asset pays its owner a continuous dividend $q \geq 0$. The process $\{L(t)\}_{t\geq 0}$ is a so-called Lévy process, i.e., a process with stationary, independent increments.

Contrary to the classical Black-Scholes framework, a rational option price cannot be found by replication. We must therefore rely on the no-arbitrage assumption, which is closely related to the existence of a risk-neutral probability measure (commonly known as Equivalent Martingale Measure); see the pioneering works of Harrison and Keeps [12] and Harrison and Pliska [13], and a more complete answer in Delbaen and Schachermayer [10]. Assuming that such a measure has been chosen, it is possible to derive an Integro-Differential evolution equation satisfied by the option price.

Assume now the existence of some Equivalent Martingale Measure Q such that the discounted process $\{e^{-(r-q)t}S(t)\}_{t\geq 0}$ becomes a martingale, and suppose that the parameters σ, ν and θ are also chosen to be risk-neutral.

Having chosen a risk-neutral probability measure Q, we may write the Lévy-Khintchine representation of $L(t)$ with respect to this new measure as follows:

$$E_Q(e^{izL(t)}) = \exp\left[t\left(-i\alpha z + \int_{\mathbb{R}} (e^{izx} - 1)k(x)dx\right)\right], \tag{7}$$

where $k(x)$ is known as Lévy density.

In a risk-neutral world, it is possible to find the form of the drift α. Namely, substituting $z = -i$ in (7), and comparing the result with the so-called risk-neutrality condition

$$E_Q[S(t)] = S_0 e^{t(r-q)}, \tag{8}$$

where r and q are the risk-free interest rate and the dividend paid by the asset, respectively, one arrives at

$$\alpha = q - r - \omega, \tag{9}$$

where ω is some "compensation constant" given by

$$\omega = \int_{\mathbb{R}} (1 - e^y)k(y)dy. \tag{10}$$

Notice that the notation used here is as in [14] (the solution method however will be completely different, as mentioned in the introduction). It is possible to compute ω directly (see Remark (3.4)), or by using the characteristic function of the process $\{X(t)\}_{t\geq 0}$ (see [7]):

$$E_Q(e^{izX(t)}) = \left(1 - iz\theta\nu + z^2\sigma^2\nu/2\right)^{-t/\nu}. \tag{11}$$

Substituting $z = -i$ in this expression and using the risk-neutrality condition, one finds the following form for ω:

$$\omega = \frac{1}{\nu}\ln(1 - \theta\nu - \sigma^2\nu/2). \tag{12}$$

One important property of the VG process is that it may be written as a difference between two increasing gamma processes: the first process representing the wins and the second corresponding to the losses. This property may be readily derived by factoring the quadratic expression in z, in the right hand side of (11), and identifying in each factor the characteristic function of some scaled gamma process. Moreover, from this factorization, the following form for the Lévy density $k(x)$ follows:

$$k(x) = \begin{cases} \frac{1}{\nu}\frac{\exp(-\lambda_+|x|)}{|x|} & \text{if} \quad x > 0, \\ \frac{1}{\nu}\frac{\exp(-\lambda_-|x|)}{|x|} & \text{if} \quad x < 0, \end{cases} \tag{13}$$

and the positive parameters $\lambda_\pm$ have the form

$$\lambda_\pm^{-1} = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} \pm \frac{\theta\nu}{2}. \tag{14}$$

Note that the positive exponent λ_+ must be larger than 1 for the constant ω to be well defined.

2.1 European options in a VG market

Consider a European call option on the asset $S(t)$, with time to expiration T, and strike price K. Assuming the existence of a risk-neutral measure, we may define the price of the European call option by the formula:

$$v(t,s) = e^{-rt}E_Q\left[(sH(t) - K)^+\right], \quad 0 \leq s < \infty, \quad 0 \leq t < \infty, \tag{15}$$

where the process $H(t)$ is the price process starting at 1, or in other words, $S(t) = sH(t)$, with

$$H(t) := \exp\left[(r-q)t + \omega t + X(t)\right]. \tag{16}$$

Note that the time t here means actually the time to expiration $T - t$ in the standard framework. A formula for $v(t,s)$ involving the modified Bessel function and the degenerate hypergeometric function of two variables is shown in [7], Theorem 2. A generalized version of this formula that includes the continuous dividend is given in [14].

Estimating the risk-neutral parameters

We briefly explain here a procedure by Carr et al. [7] to estimate the risk-neutral parameters σ_{RN}, ν_{RN} and θ_{RN}.

Suppose we are given a sequence v_i of M observed option prices for European call options, for some fixed maturity T and different strike prices K_i. Let $\hat{v}_i$ be the European VG price computed with some formula for (15). It is assumed by Carr et al. that the relation between the VG prices and the observed prices is of the form

$$v_i = \hat{v}_i e^{\eta z_i - \eta^2/2}, \tag{17}$$

where z_i is a sequence of independent, normally distributed variables with zero mean and variance one. With this model for the error, the maximum likelihood estimation of the risk-neutral parameters is asymptotically equivalent to the minimization of the quantity

$$\kappa = \sqrt{\frac{1}{M}\sum_{i=1}^{M} \left[\ln(v_i) - \ln(\hat{v}_i)\right]^2}, \tag{18}$$

over the set of parameters (σ, ν, θ).

2.2 American options in a VG market

In analogy with the log-normal case, in the VG case it is not optimal to wait until maturity to exercise an American call on a dividend-paying asset. There exists some "optimal" time at which the option should be exercised.

Optimal stopping

Consider an American call option on the underlying $S(t)$, with expiry T, and strike price K. The price of the American option is defined as:

$$v(t,s) = \sup_{\tau \in \mathcal{T}_{0,t}} E_Q\left[e^{-r\tau}(sH(\tau) - K)^+\right], \tag{19}$$

where $\mathcal{T}_{0,t}$ is the set of stopping times taking values in $[0,t]$ and $H(t)$ is as in (16). We will use the stochastic representation (19) to partially prove that the exercise boundary is a continuous function.

A free boundary value problem

It is possible to prove that the function $v(t,s)$ solves a certain Partial Integro-Differential Equation on a moving domain, see e.g., [4, 14, 19]. This equation is not of second order due to the lack of a Brownian component in the VG process.

To introduce the PIDE for the option price, let us first define the so-called "continuation region"

$$\mathcal{C} = \left\{(t,s) \in (0,\infty) \times \mathbb{R}^+ \ \mid \ v(t,s) > (s-K)^+\right\}, \tag{20}$$

and the corresponding sections

$$\mathcal{C}_t = \{ s \in (0,\infty) \mid v(t,s) > (s-K)^+ \} \quad \text{for } t > 0. \tag{21}$$

In Lemma 3.3 we show that the sections $\mathcal{C}_t$ are intervals of the form $(0, c(t))$, for a certain increasing function $c(t)$, not known a-priori. The free boundary value problem for the American option price is the following:

$$\begin{aligned} & v_t + (q - r - \omega) s v_s + rv \\ & \quad - \int_{\mathbb{R}} (v(t, se^y) - v(t,s)) k(y) dy = 0 \quad \text{for } t \in (0,T], \quad s \in (0, c(t)), \end{aligned} \tag{22}$$

with initial and boundary conditions

$$v(0,s) = (s-K)^+ \quad \text{for } s \geq 0, \tag{23}$$

$$v(t,0) = 0 \quad \text{for } t \in [0,T], \tag{24}$$

$$v(t,c(t)) = c(t) - K \quad \text{for } t \in (0,T]. \tag{25}$$

The formulation of (22) is already stated forward in time. Two extra conditions on the solution must be imposed

$$\begin{aligned} & v_t + (q - r - \omega) s v_s + rv \\ & \quad - \int_{\mathbb{R}} (v(t, se^y) - v(t,s)) k(y) dy \geq 0 \quad \text{for } t \in (0,T], \quad s \in (c(t), \infty), \end{aligned} \tag{26}$$

and

$$v(t,s) \geq (s-K)^+ \quad \text{for } t \in (0,T], s \in \mathbb{R}^+. \tag{27}$$

Condition (26) is saying that the integro-differential operator is constant in sign on the exercise region. This is an important remark when reformulating this problem as a Linear Complementarity Problem. The second condition has a parallel in obstacle problems, where the obstacle in this case is the payoff function.

3 Properties of the American call price and the free boundary

In this section we provide some auxiliary results that are used to demonstrate Theorem 3.6 in the next section. As we may see, the VG American option price shares similar properties with the Black-Scholes price, but it differs in the validity of the smooth fit principle. It is known that the exercise boundary for Black-Scholes American call options is a continuous, increasing function; see e.g., [17]. We prove that this is also the case for the VG price, under a certain relation between the dividend payment and the interest rate. It is essential in our arguments to prove that the smooth fit principle fails for the VG price. This phenomenon was already noted in [19], and proved in [1, 5, 4], but only for perpetual American options under a general class of Lévy based models that did not include the Variance Gamma process. Another issue that we study is the behavior of the free boundary close to expiration. It is known from the classical Black-Scholes situation [3, 17, 23], that the free boundary behaves differently, depending on whether the interest rate r is less or greater than the dividend payment q. With strike price K, the free boundary approaches rK/q, for $q \leq r$, and K for $q > r$. This fact is different for the VG price. We prove that the boundary tends to K for $q > r + \omega_-$, where ω_- depends on the process parameters, and is a strictly positive number. If the opposite inequality occurs, we show numerically that the boundary tends to the zero of some function termed *dividend process*. Results on the asymptotic behavior of the boundary for general jump-diffusion processes were also derived in [20].

Here, we assume that:

1. There exists an optimal stopping time in the pricing formula (19).

2. The price $v(t, s)$ is continuous in both variables.

The first result concerns monotonicity of the option price.

Lemma 3.1. *The mappings $t \mapsto v(t, s)$, $s \mapsto v(t, s)$ and $s \mapsto v(t, s) - s$ are nondecreasing, nondecreasing and non-increasing respectively.*

Proof. The proof is the same as in [17], Lemma 7.4. We include it here for the sake of completeness. The first assertion follows from the fact that a stopping time in $[0, t]$ is also a stopping time in $[0, t']$, for $t \leq t'$. The second assertion is also immediate since the function $s \mapsto (sH(\tau) - K)^+$ is nondecreasing. To prove the third assertion, let $0 \leq s_1 < s_2 < \infty$ and let τ_2 be some optimal stopping time corresponding to s_2. Then

$$\begin{aligned} v(t, s_2) - v(t, s_1) &= E_Q\left[e^{-r\tau_2}(s_2 H(\tau_2) - K)^+\right] - v(t, s_1) \\ &\leq E_Q\left[e^{-r\tau_2}\left\{(s_2 H(\tau_2) - K)^+ - (s_1 H(\tau_2) - K)^+\right\}\right] \\ &\leq (s_2 - s_1) E_Q\left[e^{-r\tau_2} H(\tau_2)\right], \end{aligned}$$

where we have used the inequality $a^+ - b^+ \leq (a - b)^+$, valid for any $a, b \in \mathbb{R}$. It remains to observe that, since $\{e^{-(r-q)t} H(t)\}_{t \geq 0}$ is a Q–martingale, then $\{e^{-rt} H(t)\}_{t \geq 0}$ is a supermartingale. Hence, by Doob's Optional Sampling Theorem, $E_Q\left[e^{-r\tau_2} H(\tau_2)\right] \leq 1$.

□

In the following lemma we prove that, for positive asset values, the American option price is a strictly positive number. We follow the ideas in [20], where a similar result is proved for general jump-diffusion American options.

Lemma 3.2. *For $t > 0$ and $s > 0$, we have $v(t, s) > 0$.*

Proof. Since the American option price is larger or equal than the European price, the following inequalities hold:

$$v(t, s) \geq E_Q\left[e^{-rt}(sH(t) - K)^+\right] \geq \frac{K}{2} e^{-rt} Q([sH(t) \geq \frac{3}{2} K]). \tag{28}$$

We show now that the event $[sH(t) \geq \frac{3}{2}K]$ has positive probability of occurrence. Since the VG process may be written as a difference between two independent gamma process

$$X(t,\nu) = \gamma_+(t) - \gamma_-(t), \tag{29}$$

we may write

$$[sH(t) \geq \frac{3}{2}K] = [\gamma_+(t) \geq \ln(3K/2s) + \alpha t + \gamma_-(t)]. \tag{30}$$

Thus, by the independence of the gamma processes, one has

$$\begin{aligned} & Q([\gamma_+(t) \geq \ln(3K/2s) + \alpha t + \gamma_-(t)]) \\ = & \int_{\mathbb{R}} (1 - F_{\gamma_+})(\ln(3K/2s) + \alpha t + x) f_{\gamma_-}(x) dx > 0 \end{aligned}$$

where F_{γ_+} denotes the distribution function of $\gamma_+(t)$ and f_{γ_-} is the density function of $\gamma_-(t)$. □

The next lemma corresponds to Proposition 7.6 in [17]. It essentially explains the form of the sections $\mathcal{C}_t$ for each $t > 0$.

Lemma 3.3. *For every $t \in (0,\infty)$, there exists a number $c(t) > K$, not necessarily finite, such that $\mathcal{C}_t = (0, c(t))$. If $c(t) < \infty$, the function $t \mapsto c(t)$ is nondecreasing and left continuous.*

Proof. Let $s_2 \in \mathcal{C}_t$ and $0 < s_1 < s_2$. By the third property in Lemma 3.1

$$v(t, s_1) \geq v(t, s_2) + s_1 - s_2 > (s_2 - K)^+ + s_1 - s_2 \geq s_1 - K.$$

Now, from the positivity of $v(t, s_1)$ proved in Lemma 3.2, it follows that $v(t, s_1) > (s_1 - K)^+$, hence $s_1 \in \mathcal{C}_t$. Therefore $\mathcal{C}_t$ is some interval of the form $(0, c(t))$. For $s \leq K$, $v(t, s) > 0 = (s - K)^+$, which proves that $c(t) > K$ for $t > 0$.

Let $c(t)$ be finite. Observe that, for all $\epsilon > 0$ and $\delta > 0$

$$v(t + \epsilon, c(t) - \delta) \geq v(t, c(t) - \delta) > (c(t) - \delta - K)^+.$$

This inequality implies $c(t + \epsilon) > c(t) - \delta$. Since δ is arbitrary, we deduce that $c(t + \epsilon) \geq c(t)$, or, in other words, that $c(t)$ is a nondecreasing function.

To prove the left continuity, consider a sequence $\{t_n\}_{n=1}^{\infty}$ such that $t_n \uparrow t_0$. From the continuity of $v(t, s)$ it follows that $\mathcal{C}$ is open, and since $(t_n, c(t_n)) \notin \mathcal{C}$, then $(t_0, c(t_0^-)) \notin \mathcal{C}$, therefore $c(t_0) \leq c(t_0^-)$. The opposite inequality, $c(t_0) \geq c(t_0^-)$, also holds because of $c(t)$ being nondecreasing. □

Splitting the compensation constant ω

The role of the compensation constant ω (cf. (10)) is to render the process $\{\exp(\omega t + X(t))\}_{t \geq 0}$ into a martingale. As mentioned in the introduction, the free boundary changes its behavior, according to a certain relation between the interest rate r, the dividend q and the "negative" part of ω.

Recall that ω is given by

$$\omega = \int_{\mathbb{R}} (1 - e^y) k(y) dy.$$

Denote by ω_- and ω_+ the integral above over the negative semiaxis and the positive semiaxis, respectively, that is,

$$\omega = \omega_- + \omega_+. \tag{31}$$

We derive now a simple formula for each component ω_+ and ω_-.

Proposition 3.4. *The following expression for ω_- holds:*

$$\omega_- = \frac{1}{\nu} \ln(1 + \lambda_-^{-1}). \tag{32}$$

For $\lambda_+ > 1$ we have

$$\omega_+ = \frac{1}{\nu} \ln(1 - \lambda_+^{-1}). \tag{33}$$

Proof. We give a short sketch of the algebraic derivation. Observe that

$$\int_0^\infty \frac{(e^{xy} - 1)}{y} e^{-y} dy = \sum_{k=1}^\infty \frac{x^k}{k!} \int_0^\infty y^{k-1} e^{-y} dy = \sum_{k=1}^\infty \frac{x^k}{k} = -\ln(1 - x)$$

for $-1 < x < 1$. The left hand side defines some analytic function for $\operatorname{Re} x < 1$ (the integrand may be written as $[e^{(x-1)y} - e^{-y}]/y$) and coincides with the right hand side on the interval $(-1, 1)$. Hence, this formula may be extended to the set $\{x \in \mathbb{C} \mid \operatorname{Re} x < 1\}$.

We then have

$$\int_0^\infty \frac{e^{-y} - 1}{y} e^{-\lambda y} dy = -\ln(1 + \lambda^{-1}),$$

and (32) follows. For (33) the proof is similar. □

Note now that adding (32) and (33) we recover (12).

The smooth fit principle

The smooth fit principle (or smooth paste principle) was first introduced in the financial literature by Samuelson in [21] under the name "high contact condition". The principle essentially states that the derivative of the Black-Scholes American option price is a continuous function, also at the exercise boundary. The geometrical interpretation is that, for each fixed $t > 0$, the function $v(t, s)$ as a function of the asset value s, smoothly enters into the payoff function $g(s)$.

To prove the principle, the key idea in [17] for the Black-Scholes American put option, and in [20] for general jump-diffusion models, was to find a lower bound for the second derivative and later prove that the boundary is right and left continuous. We cannot proceed in the same way since the PIDE for the VG prices has no second order spatial derivative. However, we can give a similar argument using the first derivative, and at the same time prove that the smooth fit principle fails.

We need the following assumption:

$$q > r + \omega_-. \tag{34}$$

For the definition of ω_- see (31).The proof of the next lemma also provides a lower bound for the jump in the derivative.

Lemma 3.5. *Assume (34) and $c(T) < \infty$. Then there exists an $\epsilon > 0$, such that*

$$(1 - v_s) \geq \epsilon, \quad \forall t \in (0, T) \text{ and } \forall s \in [K, c(t)). \tag{35}$$

Proof. By Lemma 3.1, v_t is nonnegative. Hence

$$(q - r - \omega)sv_s + rv - \int_{\mathbb{R}} (v(t, se^y) - v(t, s))k(y)dy \leq 0. \tag{36}$$

Rewriting this inequality as

$$(q - r - \omega)s(1 - v_s) \geq (q - r - \omega)s + rv - \int_{\mathbb{R}} (v(t, se^y) - v(t, s))k(y)dy \tag{37}$$

and noting that, for $y < 0$, $v(t, se^y) - v(t, s) \leq 0$, and that $v \geq s - K$, we arrive at

$$(q - r - \omega)s(1 - v_s) \geq qs - \omega s - rK - \int_0^\infty (v(t, se^y) - v(t, s))k(y)dy. \tag{38}$$

As proved in Lemma 3.1, given that $se^y > s$, for $y > 0$,

$$v(t, se^y) - v(t, s) \leq s(e^y - 1), \tag{39}$$

so that the right hand side in (38) may be bounded from below by $qs - \omega_- s - rK$, which for $s \geq K$ gives

$$(q - r - \omega)s(1 - v_s) \geq K(q - r - \omega_-), \tag{40}$$

hence

$$1 - v_s \geq \frac{K}{c(T)} \frac{q - r - \omega_-}{q - r - \omega}. \tag{41}$$

□

3.1 Behavior of the exercise boundary near maturity

In this section we prove the continuity of the exercise boundary and also the analogue of the classical result by Kim [16] on the asymptotic behavior of the exercise boundary near maturity. Remarkably, the proof of the continuity of the free boundary relies on the fact that the smooth fit principle is not true. If (34) does not hold, as it may be the case for practical situations, we indicate why the boundary changes its behavior near maturity.

Theorem 3.6. *Under condition (34), the function $t \mapsto c(t)$ is continuous in $(0, T]$ and $c(0^+) = K$.*

Proof. Let $t_0 \in (0, T)$ and note that we only need to prove the inequality $c(t_0^+) \leq c(t_0)$. To achieve this, choose a sequence $t_n \downarrow t_0$, and a number $\eta > 0$ such that $c(t_n) > c(t_n) - \eta \geq c(t_0) - \eta > K$.

From the continuity condition $v(t, c(t)) = g(c(t))$, with $g(s) = s - K$. Applying now Lemma 3.5, we may write

$$\begin{aligned} v(t_n, c(t_n) - \eta) - g(c(t_n) - \eta) &= \int_{c(t_n)-\eta}^{c(t_n)} (g_s(s) - v_s(t_n, s))ds \\ &\geq \eta\epsilon. \end{aligned}$$

Letting $n \to \infty$ in this inequality, and using the continuity of v gives

$$v(t_0, c(t_0^+) - \eta) > g(c(t_0^+) - \eta),$$

or what accounts to saying that $c(t_0^+) - \eta < c(t_0)$. Since η was arbitrary, we get $c(t_0^+) \leq c(t_0)$.

Suppose now that $c(0+) > K$. Repeating the previous argument, with $t_n \downarrow 0$ and η such that $c(0+) - \eta > K$, we obtain the following contradiction:

$$c(0+) - \eta = v(0, c(0+) - \eta) > g(c(0+) - \eta) = c(0+) - \eta.$$

□

So far we have used condition (34) to describe the behavior of the boundary close to expiry. With the following analysis we intend to understand the case that (34) is not satisfied.

Proposition 3.7. *Let $0 < q \leq r + \omega_-$. Then the function*

$$\tau(s) = qs - rK - \int_{-\infty}^{\ln(K/s)} (K - se^y)k(y)dy, \quad s > 0, \tag{42}$$

is strictly increasing and has only one zero $\bar{s} \in [K, +\infty)$.

Proof. Since

$$\tau'(s) = q + \int_{-\infty}^{\ln(K/s)} e^y k(y)dy, \tag{43}$$

it follows that $\tau(s)$ is a strictly increasing smooth function.

To prove that $\tau(s)$ has one zero in the interval $[K, +\infty)$, let us verify that $\tau(s)$ changes sign in some finite subinterval of $[K, +\infty)$. At $s = K$ we have $\tau(K) = K(q - r - \omega_-) \leq 0$. We are done if we check that $\tau(s) \sim qs$, as s tends to infinity. Indeed

$$\tau(s)/s = q - rK/s - K\left[\int_{-\infty}^{\ln(K/s)} k(y)dy\right]/s + \int_{-\infty}^{\ln(K/s)} e^y k(y)dy, \qquad (44)$$

and note that $\ln(K/s) \to -\infty$, as $s \to \infty$. □

With this proposition at hand it is possible to deduce the estimate $c(0+) \geq \bar{s}$, where $\bar{s}$ is such that $\tau(\bar{s}) = 0$. To this end, we assume that the continuity of the free boundary also holds in the case $0 < q \leq r + \omega_-$. Let

$$\mathcal{L}_s v = \alpha s v_s + rv - \int_{\mathbb{R}} (v(t, se^y) - v(t, s))k(y)dy. \qquad (45)$$

In the exercise region $\{s > c(0^+)\}$ the solution is $(s-K)^+$, so that, by equation (26), $\tau(s) = \mathcal{L}_s((s-K)^+) \geq 0$. If we assume $c(0+) < \bar{s}$, then by continuity, there is a $t_0 > 0$ such that $c(t_0) < \bar{s}$. But the existence of some $s_0 \in (c(t_0), \bar{s})$ contradicts the fact that $\tau(s_0) \geq 0$, since in such a case, by the above proposition, $s_0 \geq \bar{s}$.

Note that a simple corollary of these ideas is the following: It is never optimal to exercise a VG American call option on a non-dividend paying stock. The reason is that, if $q = 0$, the function $\tau(s)$ remains negative for all $s \geq K$.

4 Numerical valuation of the American VG price

Our goal here is to solve the free boundary problem (22)-(27) numerically, when the asset pays a positive dividend.

We are interested in the effect of adding a diffusion part to the VG process: The new coefficient will then be denoted by $\bar{\sigma}$. This parameter is later used to compare numerically the regularity of the free boundary with diffusion (Generalized VG process, $\bar{\sigma} > 0$) and without diffusion (VG process, $\bar{\sigma} = 0$). The omission of this parameter is not really affecting the numerical method that we are about to explain, since a discretization of the integral term close to the singularity gives rise anyway to some artificial diffusion (which in probabilistic terms means that small jumps are approximated by a Brownian motion).

We will not work directly on (22)-(27), but rather on its logarithmic version, i.e, we change to the variable $x = \ln s$ and solve for the new function

$$u(t, x) := v(t, e^x). \qquad (46)$$

To transform equations (22)-(27) to these new variables, it is also convenient to define the "logarithmic continuation region":

$$\widetilde{\mathcal{C}} = \left\{(t, x) \in (0, \infty) \times \mathbb{R} \;\middle|\; u(t, x) > (e^x - K)^+\right\}, \qquad (47)$$

and the optimal logarithmic asset value at which the option should be exercised:

$$\widetilde{c}(t) = \sup\left\{x \in \mathbb{R} \mid u(t,x) > (e^x - K)^+\right\}, \quad t \in (0,\infty). \tag{48}$$

In Section 3 we studied some properties of the free boundary $\widetilde{c}(t)$.

We are ready now to present the formulation of (22)-(27) in the logarithmic price:

$$\begin{aligned}
u_t - \mathcal{L}u &= 0, \quad t > 0, \quad x < \widetilde{c}(t), & (49)\\
u(t,x) &= e^x - K, \quad t > 0, \quad x \geq \widetilde{c}(t), & (50)\\
u(t,x) &\geq (e^x - K)^+, \quad t > 0, \quad x \in \mathbb{R}, & (51)\\
u_t - \mathcal{L}u &\geq 0, \quad t > 0, \quad x > \widetilde{c}(t), & (52)\\
u(0,x) &= (e^x - K)^+, \quad x \in \mathbb{R}, & (53)
\end{aligned}$$

where the operator $\mathcal{L}$ is defined in the following way:

$$\begin{aligned}
\mathcal{L}\varphi :=& \frac{\bar{\sigma}^2}{2}\varphi_{xx} - (q - r + \frac{\bar{\sigma}^2}{2})\varphi_x - r\varphi \\
&+ \int_{\mathbb{R}} \left[\varphi(t,x+y) - \varphi(t,x) - (e^y - 1)\varphi_x(t,x)\right] k(y)dy.
\end{aligned} \tag{54}$$

Note that we have included a second order term.

This problem may be cast as the following so-called Linear Complementarity Problem

$$\begin{cases}
u_t - \mathcal{L}u \geq 0 & \text{in } (0,T] \times \mathbb{R}, \\
u \geq \psi & \text{in } [0,T] \times \mathbb{R}, \\
(u_t - \mathcal{L}u)\,(u - \psi) = 0 & \text{in } (0,T] \times \mathbb{R}, \\
u(0,x) = \psi(x),
\end{cases} \tag{55}$$

where the initial condition is given by

$$\psi(x) := (e^x - K)^+. \tag{56}$$

This formulation of the problem is the basis for the numerical method presented.

4.1 Discretization

We discretize the Linear Complementarity Problem (55) by finite differences. The idea of the method is to consider one part of the integral term implicitly and the remaining explicitly. The implicit part will provide a less stringent stability condition on the time step than a "fully explicit" method.

Consider a computational domain of the form $[0,T] \times [x_{min}, x_{max}]$. Let the time interval $[0,T]$ be divided into L equal parts: $0 = t_0 < t_1 < \cdots < t_L = T$, with $t_j = j\Delta t$, $j = 0, 1, \ldots, L$ and $\Delta t = T/L$. The spatial interval $[x_{min}, x_{max}]$ contains the point $\ln K$, and $x_{min} = x_0 < x_1 < \cdots < x_N = x_{max}$, with $x_i = x_{min} + ih$, $i = 0, \ldots, N$, and h is such that $h = (x_{max} - x_{min})/N$.

We split the operator $\mathcal{L}$ into a sum of two operators $\mathcal{A}$ and $\mathcal{B}$, where

$$\mathcal{A}\varphi := \frac{\bar{\sigma}^2}{2}\varphi_{xx} - r\varphi + \int\limits_{|y|\leq h} \left[\varphi(t,x+y) - \varphi(t,x) - (e^y-1)\varphi_x(t,x)\right] k(y)dy, \tag{57}$$

and

$$\mathcal{B}\varphi := -(q - r + \frac{\bar{\sigma}^2}{2})\varphi_x + \int\limits_{|y|\geq h} \left[\varphi(t,x+y) - \varphi(t,x) - (e^y-1)\varphi_x(t,x)\right] k(y)dy. \tag{58}$$

Now, define the time approximations $u^j \approx u(t_j, x)$ and consider the following implicit-explicit iteration to solve (55):

$$\begin{cases} \dfrac{u^{j+1}-u^j}{\Delta t} - \mathcal{A}u^{j+1} - \mathcal{B}u^j \geq 0. \\ u^{j+1} \geq \psi, \\ \left(\dfrac{u^{j+1}-u^j}{\Delta t} - \mathcal{A}u^{j+1} - \mathcal{B}u^j\right)(u^{j+1}-\psi) = 0, \\ u^0 = \psi. \end{cases} \tag{59}$$

This method is related to [24] for the computation of the American put for Merton's model.

Spatial discretization of $\mathcal{A}$

Before proceeding with the discretization, let us introduce a short-hand notation for the classical finite differences for the first and second order terms. Let $w_i := w(x_i)$ for $i = 0, 1, \ldots, N$, and write

$$\delta_1(w) := \frac{w_{i+1} - w_{i-1}}{2h},$$
$$\delta_2(w) := \frac{w_{i+1} - 2w_i + w_{i-1}}{h^2}.$$

The trapezoidal rule applied to the integral operator gives, for the positive interval

$$\begin{aligned} &\int_0^h \left[w(x_i+y) - w(x_i) - (e^y-1)w_x(x_i)\right] k(y)dy \\ &= \int_0^h \left[w(x_i+y) - w(x_i) - yw_x(x_i) - (e^y-1-y)w_x(x_i)\right] k(y)dy \\ &\approx \frac{h}{2}k(h)\left[w_{i+1} - w_i - h\delta_1(w)\right] - \frac{\delta_1(w)}{2}\int_0^h y^2 k(y)dy. \end{aligned}$$

Here, the term $e^y - 1 - y$ has been substituted by $y^2/2$ with an error of the order $O(y^3)$. Applying the trapezoidal rule again gives the approximation

$$\int_0^h \left[w(x_i + y) - w(x_i) - (e^y - 1)w_x(x_i)\right] k(y)dy \approx \frac{h^3}{4} k(h)[\delta_2(w) - \delta_1(w)]. \tag{60}$$

Similarly, for the negative interval we obtain

$$\int_{-h}^0 \left[w(x_i + y) - w(x_i) - (e^y - 1)w_x(x_i)\right] k(y)dy \approx \frac{h^3}{4} k(-h)[\delta_2(w) - \delta_1(w)]. \tag{61}$$

Using (60) and (61), we arrive at the following approximation for $\mathcal{A}w$ at the point x_i:

$$(\mathcal{A}w)_i \approx \frac{\bar{\sigma}^2 + \sigma^2(h)}{2}\delta_2(w) - \frac{\sigma^2(h)}{2}\delta_1(w) - rw_i, \tag{62}$$

where we have introduced the second order artificial diffusion

$$\sigma^2(h) := \frac{[k(h) + k(-h)]\, h^3}{2}. \tag{63}$$

A similar artificial diffusion appears in the discretization given in [14] on a non-linear formulation of the American option price problem. The idea of the splitting is rigorously discussed in [8], however, the authors choose a different approach that only gives first order accuracy in the solution. We show by a numerical experiment that second order in space may be achieved for a VG European put option.

Spatial discretization of $\mathcal{B}$

Away from the origin, the integral term in $\mathcal{B}$ may be split into a sum of three terms:

$$\begin{aligned} &\int\limits_{|y|\geq h} \left[w(x_i + y) - w(x_i) - (e^y - 1)w_x(x_i)\right] k(y)dy \\ &\qquad = J_i - w_i\lambda(h) + \delta_1(w)\omega(h), \end{aligned} \tag{64}$$

with the obvious notation

$$J_i = \int\limits_{|y|\geq h} w(x_i + y)k(y)dy, \tag{65}$$

$$\lambda(h) = \int\limits_{|y|\geq h} k(y)dy, \tag{66}$$

$$\omega(h) = \int\limits_{|y|\geq h} (1 - e^y)k(y)dy. \tag{67}$$

Now it is possible to write the approximations for the implicit and the explicit part of the iteration. Equations (62) applied to (59) give for the implicit part, the following result (for w equal to u^{j+1})

$$\left(\frac{1}{\Delta t}w - \mathcal{A}w\right)_i \approx \left(\frac{1}{\Delta t} + r\right) w_i - \frac{\bar{\sigma}^2 + \sigma^2(h)}{2}\delta_2(w) + \frac{\sigma^2(h)}{2}\delta_1(w). \tag{68}$$

For the explicit term we get, on applying (64) and the definition of the operator $\mathcal{B}$, that

$$\left(\frac{u^j}{\Delta t} + \mathcal{B}w\right)_i \approx \frac{1}{\Delta t}u_i^j - \left[q - r - \omega(h) + \frac{\bar{\sigma}^2}{2}\right]\delta_1(w) + J_i - \lambda(h)w_i, \tag{69}$$

for u^j instead of w. We easily see now that the entries of the tridiagonal matrix for the implicit part are given by

$$a = -\frac{P}{h^2} - \frac{\sigma^2(h)}{4h}, \tag{70}$$

$$b = \frac{1}{\Delta t} + r + \frac{2P}{h^2}, \tag{71}$$

$$c = -\frac{P}{h^2} + \frac{\sigma^2(h)}{4h}, \tag{72}$$

with the quantity $P := \frac{\bar{\sigma}^2+\sigma^2(h)}{2}$. The tridiagonal matrix whose entries are a, b and c is constant along its diagonals:

$$T = \begin{bmatrix} b & c & & & \\ a & b & c & & \\ & \ddots & \ddots & \ddots & \\ & & a & b & c \\ & & & a & b \end{bmatrix}. \tag{73}$$

One has to take into account that the right-hand side term $d^j = u^j/\Delta t + \mathcal{B}u^j$ need also be updated for the boundary condition. For American call options this is done by updating the first and the last entry in d^j, i.e.,

$$d_1^j \leftarrow 0, \quad d_{N-1}^j \leftarrow d_{N-1}^j - c(e^{x_{max}} - K). \tag{74}$$

Summarizing, the problems we wish to solve for $j = 0, 1, \ldots,$ has the following form:

$$\begin{cases} Tu^{j+1} \geq d^j, \\ u^{j+1} \geq \psi, \\ (Tu^{j+1} - d^j, u^{j+1} - \psi) = 0. \end{cases} \tag{75}$$

The matrix T is defined by (70)-(73), d^j is given by the expression (69) with the update (74) and ψ is the vector $[\psi_1, \psi_2, \ldots, \psi_{N-1}]^T$, with $\psi_i = \psi(x_i)$.

Because of the particular form of the problem and of the matrix T, the Brennan-Schwartz algorithm may be used to find the solution to this LCP. This will be explained in a later paragraph. The quantities J_i appearing in d^j will be treated separately in the next paragraph.

The integral term J

The most expensive part of the above implicit-explicit scheme lies in the computation of the numbers J_i. In his paragraph, we explain first how these entries may be computed approximately, and later we will give one method to accelerate the resulting convolutions. We also assume here that the number of spatial points N is an even number.

Let M be an integer larger than 1. The trapezoidal rule on a truncation of the integral gives:

$$\begin{aligned} J_i = \int\limits_{|y|\geq h} w(x_i+y)k(y)dy &\approx \int\limits_{h\leq |y|\leq Mh} w(x_i+y)k(y)dy \\ &\approx h\sum_{m=-M}^{M} w_{i+m}k_m\rho_m, \quad i=0,1,\ldots,N. \end{aligned} \tag{76}$$

The following notation was employed:

$$k_m = k(mh), \quad m \neq 0, \tag{77}$$

$$\rho_m = \begin{cases} 1/2 & \text{if } m \in \{-M,-1,1,M\}, \\ 1 & \text{otherwise,} \end{cases} \tag{78}$$

and for $m=0$ we have redefined $k(0)$ as 0. For indices $i+m$ such that $x_{i+m} \leq x_{min}$ (i.e., $i+m \geq 0$) or $x_{i+m} \geq x_{max}$ (i.e., $i+m \geq N$), we put $w_{i+m} := \psi_{i+m}$. In other words, we substitute w by the payoff function for points lying outside the computational domain. For the coefficients $\lambda(h)$ and $\omega(h)$ we may use the same approximation. For example

$$\lambda(h) = \int\limits_{|y|\geq h} k(y)dy \approx h\sum_{m=-M}^{M} k_m\rho_m. \tag{79}$$

The Brennan-Schwartz algorithm

The well-known Brennan–Schwartz algorithm was originally developed for American put options, for which a rigorous justification can be found in [15]. The algorithm needs to be adapted for handling American call options, as mentioned in [15]. The natural modification needed is a straightforward reordering of indices, as explained in this section.

Let a tridiagonal matrix

$$T = \begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \tag{80}$$

and vectors $d = [d_1, \dots, d_n]^T$ and $\psi = [\psi_1, \dots, \psi_n]^T$ be given. Consider the following problem: Find a vector u satisfying the system

$$\begin{cases} Tu \geq d, \\ u \geq \psi, \\ (Tu - d, u - \psi) = 0. \end{cases} \tag{81}$$

The following algorithm results for an American call:

- Step 1: Compute recursively a vector $\widetilde{b}$ as

$$\begin{aligned} \widetilde{b}_1 &= b_1, \\ \widetilde{b}_j &= b_j - a_j c_{j-1}/\widetilde{b}_{j-1}, \quad j = 2, \dots, n. \end{aligned}$$

- Step 2: Compute recursively a vector $\widetilde{d}$ as

$$\begin{aligned} \widetilde{d}_1 &= d_1, \\ \widetilde{d}_j &= d_j - a_j \widetilde{d}_{j-1}/\widetilde{b}_{j-1}, \quad j = 2, \dots, n. \end{aligned}$$

- Step 3: Compute u backwards:

$$\begin{aligned} u_n &= \max\left[\widetilde{d}_n/\widetilde{b}_n \ , \ \psi_n\right], \\ u_j &= \max\left[\left(\widetilde{d}_j - c_j u_{j+1}\right)/\widetilde{b}_j \ , \ \psi_j\right], \quad j = n-1, n-2, \dots, 1. \end{aligned}$$

We apply this algorithm using the matrix T given in (73). The algorithm for the put option is analogous to the above algorithm, but the numbering of the indices must be reversed; see [15].

Remark 4.1. A more general method by Cryer [9] allows the solution of (81) with the only requirement that T is an M-matrix. Cryer's algorithm may be used to tackle problems where the exercise boundary is not connected, as in e.g., American butterfly spread options. This algorithm only requires $O(n)$ operations.

Remark 4.2. The splitting proposed in (57)-(58) is meant to guarantee the validity of Brennan-Schwartz algorithm. However, the term containing the derivative in $\mathcal{B}$ may be included in $\mathcal{A}$ instead, with only a minor change in the entries of matrix T. The new LCP may be also solved by Brennan-Schwartz algorithm, and the solution obtained is the same, even if the sufficient conditions on the algorithm are violated. This alternative splitting seems to converge in less iterations.

Fast convolution by FFT

The Fast Fourier Transform is an algorithm that evaluates the Discrete Fourier Transform (DFT) of a vector $f = [f_0, f_2 \dots, f_{R-1}]$ in $O(R \log R)$ operations.

The Discrete Fourier Transform is defined as:

$$F_k = \sum_{n=0}^{R-1} f_n e^{-i2\pi nk/R}, \quad k = 0, 1, \ldots, R. \tag{82}$$

One of the multiple applications of the DFT is in computing convolutions. Let us first introduce the concept of circulant convolution. Let $\{x_m\}$ and $\{y_m\}$ be two sequences with period R. The convolution sequence $z := x * y$ is defined component-wise as

$$z_n = \sum_{m=0}^{R-1} x_{m-n} y_m. \tag{83}$$

We use now FFT to compute the vector $[z_0, \ldots, z_{R-1}]$. The periodic structure of x allows the derivation of the following simple relation:

$$Z_k = X_k \cdot Y_k, \tag{84}$$

where X, Y and Z denote the Discrete Fourier Transform of the sequences x, y and z respectively. That is, DFT applied to the convolution sequence is equal to the product of the transforms of the original two sequences. The vector $[z_0, \ldots, z_{R-1}]$ may be recovered by means of the Inverse Discrete Fourier Transform (IDFT):

$$z_n = \frac{1}{R} \sum_{k=0}^{R-1} Z_k e^{i2\pi kn/R}, \quad n = 0, 1, \ldots, R. \tag{85}$$

In the language of matrices, a circulant convolution may be seen as the product of a circulant matrix times a vector. For example, let $R = 3$, and use the periodicity $x_k = x_{k+R}$ to arrive at

$$\begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \end{bmatrix}. \tag{86}$$

A circulant matrix is thus a matrix in which each row is a "circular" shift of the previous row.

We are interested in the convolution (76), where the vector w is not periodic. The associated matrix is a so-called Toeplitz matrix, which by definition is a matrix that is constant along diagonals. A circulant matrix is hence a particular type of Toeplitz matrix. The next idea is to embed a Toeplitz matrix into a circulant matrix. As an example, let $M = 1$ and $N = 2$, so that the matrix-vector notation for (76) reads

$$\begin{bmatrix} w_1 & w_0 & w_{-1} \\ w_2 & w_1 & w_0 \\ w_3 & w_2 & w_1 \end{bmatrix} \begin{bmatrix} k_1/2 \\ k_0 \\ k_{-1}/2 \end{bmatrix}. \tag{87}$$

The matrix above may be embedded in a circulant matrix C of size 5 in the following way (For computational efficiency of the FFT algorithm, it is advisable to use a circulant matrix whose size is a power of 2.):

$$C = \left[\begin{array}{ccc|cc} w_1 & w_0 & w_{-1} & w_3 & w_2 \\ w_2 & w_1 & w_0 & w_{-1} & w_3 \\ w_3 & w_2 & w_1 & w_0 & w_{-1} \\ \hline w_{-1} & w_3 & w_2 & w_1 & w_0 \\ w_0 & w_{-1} & w_3 & w_2 & w_1 \end{array}\right]. \tag{88}$$

If we define the vector $\eta := [k_1/2, k_0, k_{-1}/2, 0, 0]^T$, then the product (87) is the vector consisting of the first three elements in the product $C\eta$. As explained before, a product of a circulant matrix and a vector may be efficiently done by applying the FFT algorithm.

As a summary, following the ideas explained above, it is possible to compute the convolution (76), with $M = N/2$, by "embedding" the resulting matrix into a circulant matrix. The product of a circulant matrix and a vector is carried out in three FFT operations, namely, two DFT and one IDFT.

For further details on the computation of convolutions by FFT we refer to [22].

5 Numerical experiments

Firstly, we carry out a reference experiment where we compute a European put option. The parameters are:

$$r = 0, \quad q = 0, \quad \sigma = 0.25, \quad \theta = 0, \quad \nu = 2, \quad K = 10, \quad T = 5. \tag{89}$$

It is shown that the method is second order in space by comparing the numerical solution with a numerical integration of the analytical solution from [6]. Since only an implicit Euler scheme has been implemented, we show the h^2–accuracy by dividing the time step by four and the spatial mesh-size by two. The results are summarized in Table 1.

N	M	ℓ_∞-error
50	5	0.1079
100	20	0.0326
200	80	0.0065
400	320	0.0017

Table 1: Second order convergence for a VG European put.

In the second experiment we compute the American call price using the following parameters:

$$\sigma = 0.2, \quad \theta = 0.085, \quad \nu = 1, \quad K = 1, \quad N = 2500, \quad \text{and} \quad M = 500.$$

It is assumed first that the condition $q > r + \omega_-$ is not satisfied, so $r = 0.1$ and $q = 0.1$ are chosen. Observe in Figure 1 that the principle of smooth fit does not hold, as already pointed out in [19]. The same picture 1 (right-hand) shows the smearing effect on the continuity of the derivative Δ after including a small diffusion $\bar{\sigma}$ in the VG model. For $\bar{\sigma} = 0$ the figure shows that the smearing does not appear. Related theoretical results for finite activity processes may be found in the work of Pham [20].

In the third experiment we study the behavior of the free boundary near expiry. Two cases are distinguished: a) $q \le r + \omega_-$ and b) $q > r + \omega_-$. In the first case we let $r = 0.1$ and $q = 0.1$ and find the asymptotic behavior of the free boundary to be $c(\Delta t) = 1.1246$ (see Figure 2, left). It is not difficult to verify that $\tau(1.1246) \approx 0$ (cf. (42)). In the second situation we choose $r = 0.1$ and $q = 0.21$, since $\omega_- = 0.1$. Figure 2 (right-hand) shows that $c(0+) \approx 1 = K$. This confirms the result in Theorem 3.6.

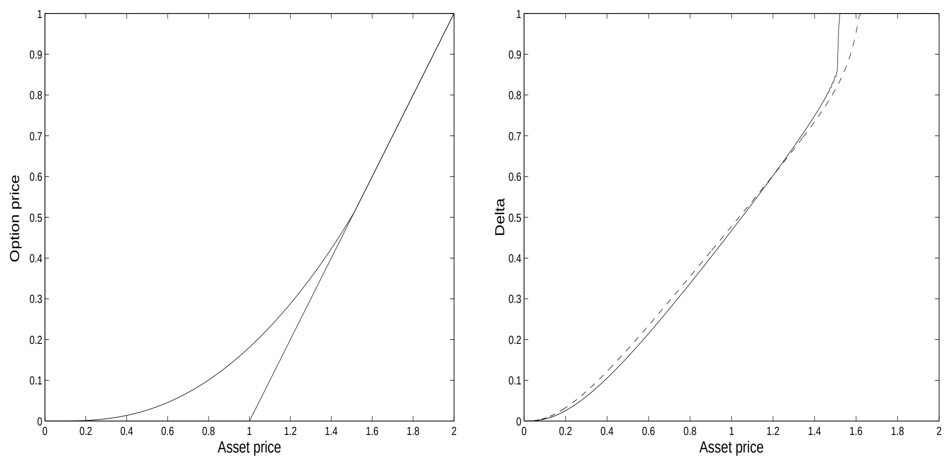

Figure 1: Left picture: VG option value and payoff function; Right picture: Delta for $\bar{\sigma} = 0.1$ (dashed line) and for $\bar{\sigma} = 0$ (continuous line). The parameters are: $r = 0.1$, $q = 0.1$, $\sigma = 0.2$, $\theta = 0.085$, $\nu = 1$, $K = 1$, $N = 2500$, $M = 500$ and $T = 9$.

6 Conclusions

In the first part of this paper it was shown that the smooth fit principle fails for the VG American call. The failure of the principle had already been pointed out in the financial literature for a large family of pure-jump processes, but to the best of our knowledge, the literature does not include the VG process. An asymptotic analysis for the free boundary near maturity is also provided together with a proof of the continuity of the exercise boundary. In the second part we proposed a numerical method to deal with the American call. This method is easy to implement and may be used for the valuation of American options under general Lévy processes, even when the Lévy measure is obtained from

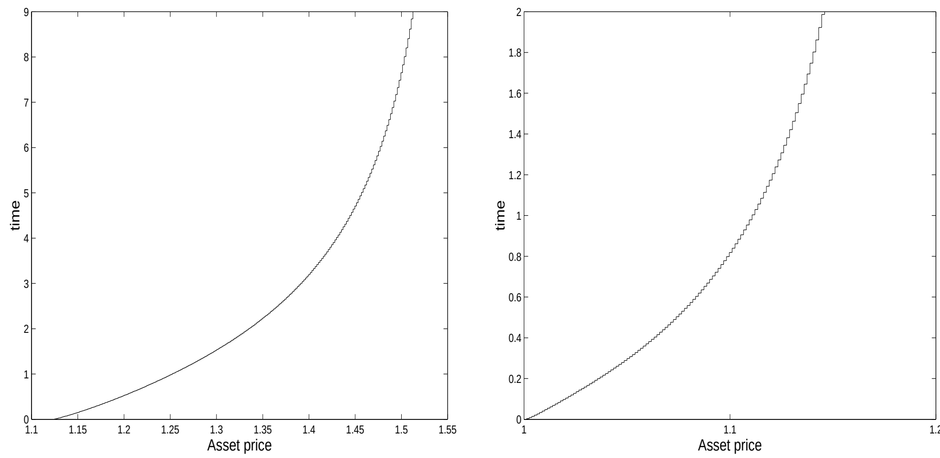

Figure 2: Left picture: Free boundary when $q < r + \omega_-$; parameters: $T = 9$, $r = 0.1$, $q = 0.1$, $\sigma = 0.2$, $\theta = 0.085$, $\nu = 1$, $K = 1$, $M = 2000$ and number of spatial points $N = 8000$. Right picture: Free boundary when $q > r + \omega_-$; parameters: $T = 2$, $r = 0.1$, $q = 0.21$ and the other parameters are the same as in the left picture.

calibration. The method does not require the knowledge of the characteristic function to find the European price, as in, e.g., [6].

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