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Efficient Portfolio Valuation Incorporating Liquidity Risk

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According to the theory proposed by Acerbi and Scandolo (2008), an asset is described by the so-called Marginal Supply-Demand Curve (MSDC), which is a collection of bid and ask prices according to its trading volumes, and the value of a portfolio is defined in terms of commonly available market data and idiosyncratic portfolio constraints imposed by an investor holding the portfolio. Depending on the constraints, one and the same portfolio could have different values for different investors. As it turns out, within the Acerbi-Scandolo theory, portfolio valuation can be framed as a convex optimization problem. We provide useful MSDC models and show that portfolio valuation can be solved with remarkable accuracy and efficiency.

Keywords: liquidity risk; portfolio valuation; ladder MSDC; liquidation sequence; exponential MSDC; approximation

1. Introduction

According to the theory developed by Acerbi and Scandolo (2008) the value of a portfolio is determined by market data and a set of portfolio constraints. The market data is assumed to be publicly available and is the same for all investors. The market data consists of price quotes corresponding to different trading volumes. These quotes for an asset are represented in terms of a mathematical function referred to as a Marginal Supply-Demand Curve (MSDC).

The portfolio constraints may vary across different players. These idiosyncratic constraints—collectively referred to as a *liquidity policy*—refer to restrictions that any portfolio held by the investor should be prepared to satisfy. Examples of such portfolio constraints are:

- minimum cash amounts to meet short term liquidity needs;
- market or credit risk management limits;

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- capital limits.

We introduce fundamental concepts of Acerbi-Scandolo theory in Section 2. To value her portfolio, the investor will mark all the positions she could possibly unwind to satisfy the liquidity policy to the best price according to an MSDC function. As it turns out, within Acerbi and Scandolo's theory, the valuation of a portfolio of assets can be framed as a convex optimization problem. The associated constraint set is represented by a liquidity policy. Although this was already pointed out by Acerbi and Scandolo themselves, the practical implications of the theory have as yet not been well investigated. Such is the aim of the present paper.

We will study portfolio valuation under the Acerbi-Scandolo theory extensively, assuming different forms of the MSDC function. We first consider a very general setting where the MSDC is shaped as a non-increasing step function (referred to as a *ladder MSDC*) in Section 3. This corresponds to normal market situations for relatively actively traded products such as listed equities. We will present an algorithm for portfolio valuation assuming ladder MSDCs and a cash portfolio constraint. In Section 4, we will look at MSDCs which are shaped as decreasing exponential functions, which can be used to describe less liquid over-the-counter (OTC) traded products. We will also see how the exponential functions can be used as approximations of ladder MSDCs.

All numerical results are collected in Section 5. We will find that in a wide range of cases, the approximation of ladder MSDCs by exponential MSDCs appears to be accurate, suggesting that not all market price information represented in ladder MSDCs is necessary for accurate portfolio valuation. We present our conclusions in Section 6.

2. The Portfolio Theory

This section presents relevant concepts and results from Acerbi and Scandolo (2008) that will be used throughout this paper.

2.1. Asset

An **asset** is an object traded in a market and will be characterized by a Marginal Supply-Demand Curve (MSDC). This codifies available bid and ask prices corresponding to different trading volumes.

Definition 2.1: An **MSDC** is a map $m : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ satisfying the following two conditions:

(i) $m(s)$ is non-increasing, i.e., $m(s_1) \geq m(s_2)$ if $s_1 < s_2$;
(ii) $m(s)$ is càdlàg (i.e., right-continuous with left limits) for $s < 0$ and làdcàg (i.e., left-continuous with right limits) for $s > 0$.

The variable s represents the trading volume of the asset.

Condition 1 represents a no-arbitrage assumption. Condition 2 ensures that MSDCs have elegant mathematical properties. We will not heavily use this condition and we only mention it for the sake of completeness. Instead, what we will need most of the time is that an MSDC is (Riemann) integrable on its domain.

We call the limit $m^+ := \lim_{h \downarrow 0} m(h)$ the *best bid* and $m^- := \lim_{h \uparrow 0} m(h)$ the *best ask*. The *bid-ask spread*, denoted by δm, is defined as $\delta m := m^- - m^+$.

Definition 2.2: **Cash** is the asset representing the currency paid or received when trading any asset. It is characterized by a constant MSDC, $m_0(s) = 1$ (i.e., one unit) for every $s \in \mathbb{R} \setminus \{0\}$.

Cash is referred to as a *perfectly liquid* asset if the associated MSDC is constant. We call a *security* any asset whose MSDC is a positive function (e.g., a stock, a bond, a commodity) and a *swap* any asset whose MSDC can take both positive and negative values (e.g., an interest rate swap, a CDS, a repo transaction). A negative MSDC can be converted into a security by defining a new MSDC as $m^*(s) := -m(-s)$.

We presuppose one currency as the cash asset. For example, if we choose the euro as the cash asset, relative to the euro, the US dollar will be considered as an illiquid asset.

2.2. Portfolio

A portfolio is characterized by listing the holding volumes of different assets in the portfolio. Given are $N+1$ assets labeled $0, 1, \ldots, N$. We let asset 0 label the cash asset.

Definition 2.3: A **portfolio** is a vector of real numbers, $\mathbf{p} = (p_0, p_1, \ldots, p_N) \in \mathbb{R}^{N+1}$, where p_i represents the holding volume of asset i. In particular, p_0 denotes the amount of cash in the portfolio.

When we specifically want to highlight the portfolio cash we tend to write a portfolio as $\mathbf{p} = (p_0, \overrightarrow{p})$. We henceforth presuppose a set of portfolios referred to as the *portfolio space* $\mathcal{P}$. We will assume that $\mathcal{P}$ is a vector space so that it becomes meaningful to add portfolios together and to multiply portfolios by scalar numbers. Let $\mathbf{p} = (p_0, \overrightarrow{p}) \in \mathcal{P}$ and suppose we have an additional amount a of cash. We write $\mathbf{p} + a = (p_0 + a, \overrightarrow{p})$.

Definition 2.4: The **liquidation Mark-to-Market (MtM) value** $L(\mathbf{p})$ of a portfolio $\mathbf{p}$ is defined as:

$$L(\mathbf{p}) := \sum_{i=0}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x. \tag{1}$$

The liquidation MtM value can be viewed as the value of a portfolio $\mathbf{p}$ for an investor who should be able to liquidate all her positions in exchange for cash.

Definition 2.5: The **uppermost Mark-to-Market (MtM) value** $U(\mathbf{p})$ of $\mathbf{p}$ is given by

$$U(\mathbf{p}) := \sum_{i=0}^{N} m_i^{\pm} p_i = p_0 + \sum_{i=1}^{N} m_i^{\pm} p_i, \tag{2}$$

where

$$m_i^{\pm} = \begin{cases} m_i^+, \text{ if } \ p_i > 0; \\ m_i^-, \text{ if } \ p_i < 0. \end{cases} \tag{3}$$

The uppermost MtM value can be viewed as the value of a portfolio for an investor who has no cash demands. In this sense, the portfolio is unconstrained.

As MSDCs are non-increasing, $U(\mathbf{p}) \geq L(\mathbf{p})$. The difference between $U(\mathbf{p})$ and $L(\mathbf{p})$ is termed the *uppermost liquidation cost* and is defined as $C(\mathbf{p}) := U(\mathbf{p}) - L(\mathbf{p})$.

2.3. Liquidity policy

The definitions of the liquidation MtM value $L(\mathbf{p})$ and the uppermost MtM value $U(\mathbf{p})$ suggest that the value of a portfolio $\mathbf{p}$ is subject to certain cash constraints an investor should be able to meet by wholly or partly liquidating positions she has taken. These constraints are represented as a *liquidity policy.*

There could be other types of constraints besides. For example, an investor might want to impose market risk VaR limits on her positions, or credit limits, or capital constraints. All the constraints that an investor imposes can be represented as a subset of the underlying portfolio space $\mathcal{P}$. These constraints are collectively referred to as a liquidity policy. We refer to Acerbi and Finger (2010) and Weber *et al.* (2013).

Definition 2.6: A **liquidity policy** $\mathcal{L}$ is a closed and convex subset of $\mathcal{P}$ satisfying the following conditions:

(i) if $\mathbf{p} = (p_0, \vec{p}) \in \mathcal{L}$ and $a \geq 0$, then $\mathbf{p} + a = (p_0 + a, \vec{p}) \in \mathcal{L}$;
(ii) if $\mathbf{p} \in \mathcal{L}$, then $(p_0, \vec{0}) \in \mathcal{L}$.

Example 2.7: A liquidity policy setting a minimum cash requirement, c, is a *cash liquidity policy*:

$$\mathcal{L}(c) := \{\mathbf{p} \in \mathcal{P} | p_0 \geq c \geq 0\}. \tag{4}$$

An investor endorsing a cash liquidity policy should be prepared to liquidate her positions to such an extent that minimum cash level c is obtained. We will extensively use cash liquidity policies in Sections 3 and 4. We refer to Acerbi (2008) and Weber *et al.* (2013) for additional examples of liquidity policies.

2.4. Portfolio value

This section presents Acerbi and Scandolo's definition of the portfolio value function. We first need the following definition.

Definition 2.8: Let $\mathbf{p}, \mathbf{q} \in \mathcal{P}$ be portfolios. We say that $\mathbf{q}$ is **attainable** from $\mathbf{p}$ if $\mathbf{q} = \mathbf{p} - \mathbf{r} + L(\mathbf{r})$ for some $\mathbf{r} \in \mathcal{P}$. The set of all portfolios attainable from $\mathbf{p}$ is written as $\mathsf{Att}(\mathbf{p})$.

The following definition is key:

Definition 2.9: The **Mark-to-Market (MtM) value** (or the **value**, for short) of a portfolio $\mathbf{p}$ subject to a liquidity policy $\mathcal{L}$ is the value of the function $V^{\mathcal{L}} : \mathcal{P} \to \mathbb{R} \cup \{-\infty\}$ defined by

$$V^{\mathcal{L}}(\mathbf{p}) := \sup\{U(\mathbf{q}) | \mathbf{q} \in \mathsf{Att}(\mathbf{p}) \cap \mathcal{L}\}. \tag{5}$$

If $\mathsf{Att}(\mathbf{p}) \cap \mathcal{L} = \varnothing$, meaning that no portfolio attainable from $\mathbf{p}$ satisfies $\mathcal{L}$, then we stipulate the portfolio value to be $-\infty$.

Acerbi and Scandolo (2008) give the following proposition of the new portfolio value:

Proposition 2.10: *The portfolio value function* $V^{\mathcal{L}}$ *from Definition 2.9 can be alternatively defined as*

$$V^{\mathcal{L}}(\boldsymbol{p}) = \sup\{U(\boldsymbol{p} - \boldsymbol{r}) + L(\boldsymbol{r}) | \boldsymbol{r} \in \mathcal{P}, \boldsymbol{p} - \boldsymbol{r} + L(\boldsymbol{r}) \in \mathcal{L}\}. \tag{6}$$

To prove this is not very difficult; we refer to Acerbi and Scandolo (2008).

Proposition 2.10 allows us to frame the determination of the value of a portfolio as an optimization problem with explicit constraints, namely:

$$\begin{cases} \text{maximize} \quad U(\mathbf{p}-\mathbf{r})+L(\mathbf{r}); \\ \text{subject to:} \quad \mathbf{p}-\mathbf{r}+L(\mathbf{r}) \in \mathcal{L}; \\ \qquad\qquad\quad \mathbf{r} \in \mathcal{P}. \end{cases} \tag{7}$$

(We ignore the case $V^{\mathcal{L}}(\mathbf{p}) = -\infty$.) This optimization problem is convex as $\mathcal{L}$ is a convex set. Since $\mathcal{L}$ is also closed, this problem has a unique optimal value (which could be $-\infty$).

3. Portfolio Valuation Using Ladder MSDCs

In the previous section we have outlined the main concepts of Acerbi and Scandolo's portfolio theory. We discussed that portfolio valuation could be framed as a convex optimization problem (7). Convex optimization problems can often be numerically solved, see Boyd and Vandenberghe (2004).

In the present section we will provide an algorithm providing an exact global solution for problem (7) under the assumption that the MSDC for illiquid assets is piecewise constant, as such we will name them *ladder MSDCs*.

Within the Acerbi-Scandolo theory, ladder MSDCs will play a key role to model the liquidity of the assets. Equipped with the fast and accurate algorithm discussed in this section, one could solve the convex optimization problem incurred in portfolio valuation more efficiently than using conventional optimization techniques.

3.1. The optimization problem

Generally we assume a market wherein we can quote a price for each volume we wish to trade, i.e., a market of "unlimited depth". However, in a real-world market context, we will typically only be able to trade volumes within certain bounds. The upper and the lower bound of this domain represent the market depth: the upper bound represents the maximum volume we will be able to sell against prices we can quote from the market and the lower bound represents the maximum we will be able to buy against prices we will be able to quote from the market. In Weber *et al.* (2013), this set of constraints on the portfolio space is referred to as a *portfolio constraint*. In the context of limited market depth, we will need to restrict the domain to a subset of the portfolio space to solve the optimization problem of portfolio valuation.

In what follows, we still assume unlimited market depth so that we can search for the optimal solution in the whole portfolio space for simplicity, whereas the method we state below also works with limited market depth.

Reconsider problem (7). Using a cash liquidity policy $\mathcal{L}(c)$ this becomes

$$\begin{cases} \text{maximize} \quad U(\mathbf{p}-\mathbf{r})+L(\mathbf{r}); \\ \text{subject to:} \quad p_0-r_0+L(\mathbf{r}) \geq c; \\ \qquad\qquad\quad \mathbf{r} \in \mathcal{P}. \end{cases} \tag{8}$$

The inequality constraint can be replaced by the equality constraint $p_0 - r_0 + L(\mathbf{r}) = c$ without affecting the optimal value of the original problem. Furthermore, we may assume that the cash component r_0 equals 0 as it does not play a role in the optimization problem. To find the optimal solution we hence might as well solve

$$\begin{cases} \text{maximize} \quad U(\mathbf{p}-\mathbf{r}) + L(\mathbf{r}); \\ \text{subject to:} \quad L(\mathbf{r}) = c - p_0; \\ \qquad\qquad\quad \mathbf{r} \in \mathcal{P}. \end{cases} \tag{9}$$

Note that without loss of generality we may assume that $p_0 = 0$; otherwise use the cash liquidity policy $\mathcal{L}(c - p_0)$.

3.2. A calculation scheme for portfolio valuation with ladder MSDCs

In case of portfolio valuation based on ladder MSDCs we can solve the associated optimization problem (9) numerically, for example, by an interior point algorithm (see Boyd and Vandenberghe (2004)). However, the implementation of the algorithm could be computationally inefficient in the sense that several iterations might be required to bring the solution within reasonable bounds in high dimensions. In addition, the non-smoothness of the ladder MSDCs increases the difficulty of implementing conventional convex optimization algorithms.[1] Hence, the aim of this section is to provide an algorithm for problem (9) yielding an exact global optimal solution $\mathbf{r}^*$.

Unless otherwise stated, throughout the remainder of this section we use the following assumption.

Assumption 3.1: *Any investor holds a portfolio* $\mathbf{p}$ *consisting of* long *positions only and uses a cash liquidity policy* $\mathcal{L}(c)$ $(c > 0)$.

Proposition 3.2: *Under Assumption 3.1, the maximization problem (9) has the same optimal solution as the following* minimization *problem*

$$\begin{cases} \textit{minimize} \quad C(\boldsymbol{r}); \\ \textit{subject to} \quad L(\boldsymbol{r}) = c - p_0; \\ \qquad\qquad\quad \boldsymbol{r} \in \mathcal{P}. \end{cases} \tag{10}$$

Proof: Since we are prepared to liquidate our portfolio for cash under a cash policy, the portfolios $\mathbf{p}$ and $\mathbf{r}$ should have the same sign componentwise. It follows that $U(\mathbf{p}-\mathbf{r}) = U(\mathbf{p}) - U(\mathbf{r})$ by the definition of the uppermost MtM value. Consequently, the objective function of problem (7) can be rewritten as

$$U(\mathbf{p}) - U(\mathbf{r}) + L(\mathbf{r}).$$

Since, given $\mathbf{p}$, we can always determine $U(\mathbf{p})$, maximizing this function under the given constraints will yield the same optimal solution $\mathbf{r}^*$ as maximizing the following function under the same constraints:

$$-U(\mathbf{r}) + L(\mathbf{r}).$$

[1]For example, the optimality conditions in the interior point algorithm will not apply at non-smooth points of the ladder MSDC. See Boyd and Vandenberghe (2004) for more information.

Obviously, minimizing

$$U(\mathbf{r}) - L(\mathbf{r})$$

again yields the same optimal solution $\mathbf{r}^*$. Noting that $C(\mathbf{r}) = U(\mathbf{r}) - L(\mathbf{r})$ proves the result. □

Remark 1 : The following inequality holds in general:

$$U(\mathbf{p} - \mathbf{r}) \leq U(\mathbf{p}) - U(\mathbf{r}).$$

For example, we may be prepared to increase our share in several risky assets or reduce the purchase of risky assets. In situations like these, components of the original portfolio $\mathbf{p}$ and corresponding components of to-be-liquidated portfolio $\mathbf{r}$ may have different signs. Equality holds under Assumption 3.1.

Informally, Proposition 3.2 implies that to determine the value of a portfolio under a cash liquidity policy is to determine a portfolio $\mathbf{r}^*$ such that liquidating $\mathbf{r}^*$ in exchange for cash minimizes the uppermost liquidation cost $C(\mathbf{r}^*)$. This result will prove useful at a later stage.

Given that all assets are assumed to be characterized by ladder MSDCs, we can conveniently break up each and every position in our portfolio into a finite number of volumes. To each of these volumes there corresponds a definite market quote as represented by the MSDC.

The idea of the algorithm is to consider all of these portfolio bits together and to liquidate them in a systematic and orderly manner, starting with the portions which will be liquidated with the smallest cost relative to the best bid, and subsequently to the ones that can be liquidated with second smallest cost, and so on, until the cash constraint is met.

If the minimum cash requirement that the portfolio should be prepared to satisfy exceeds the liquidation MtM value of the entire portfolio, then we will never be able to meet the cash constraint; by definition, we set the portfolio value to be $-\infty$.

Alternatively, suppose that we sell off a fraction of each position against the best bid price and that the total cash we subsequently receive in return exceeds the cash constraint. Then the value of the portfolio equals the uppermost MtM value and there exist infinitely many optimal solutions.

We will now make this formal, starting with the following definition.

Definition 3.3: Given is an asset i, characterized by MSDC m_i. The **liquidity deviation** of a volume s of asset i is defined as:

$$S_i(s) := \frac{m_i^+ - m_i(s)}{m_i^+} \quad \text{for} \quad s > 0. \tag{11}$$

The liquidity deviation is the relative difference between the best bid price and the last market quote $m_i(s)$ hit for a volume s. In this sense, it measures the liquidity of asset i at s_i units traded relative to the best bid.

Given any asset, the liquidity deviation is a non-decreasing function, as the MSDC corresponding to that asset is non-increasing. For a security, the values of the liquidity deviation are in $[0, 1]$, as the lower bound of the corresponding MSDC is 0. For a swap, the values are in $[0, +\infty)$. Since the MSDC of an asset is assumed to be piecewise constant, each value of liquidity deviation corresponds to a maximum bid size.

Using the previously defined liquidity deviation, positions are liquidated in a definite order, as follows. Given a portfolio $\mathbf{r} = (r_0, r_1, \ldots, r_N)$, assume that we want to liquidate all the r_i, $i > 0$.

(a) Asset 1

Maximum Bid Size	Bid Price
200	11.65
200	11.55
200	11.45

(b) Asset 2

Maximum Bid Size	Bid Price
200	19.58
600	19.5
200	19.2

Table 1. Bid price information of assets 1 and 2

Each non-cash position r_i can be written as a sum

$$r_i = \sum_{j=1}^{J_i} r_{ij}, \quad i = 1, \dots, N, \tag{12}$$

where r_{ij} is called a *liquidation size*.

To define the liquidation size r_{ij}, consider the bid part of a ladder MSDC m_i, which is constructed by a finite number of bid prices with maximum bid sizes. For each r_i in asset i, we can identify a finite number J_i of bid prices m_{ij} with liquidation sizes r_{ij}, $j = 1, \dots, J_i$.

For the first $J_i - 1$ liquidation sizes r_{ij} ($j = 1, \dots, J_i - 1$), they are equal to the first $J_i - 1$ maximum bid sizes recognized from the market. For the J_i-th liquidation size r_{ij}, it is less than or equal to the J_i-th maximum bid size. Moreover, each liquidation size r_{ij} corresponds to each bid price m_{ij}. In particular, the first liquidation size of each asset r_{i1} corresponds to the best bid $m_i^+ = m_{i1}$.

Afterwards, the liquidity deviation for each liquidation size can be written as

$$S_{ij} = \frac{m_i^+ - m_{ij}}{m_i^+} = \frac{m_{i1} - m_{ij}}{m_{i1}}. \tag{13}$$

Now we put the liquidity deviations S_{ij} in ascending order indexed by k, and we generically refer to any term of this sequence as $S_k(\mathbf{r})$ (the addition of $\mathbf{r}$ as an extra parameter will prove convenient later on). Note that the length of the liquidation sequence equals $K = J_1 + \cdots + J_N$.

In addition, we observe that there exists a natural one-one correspondence between the sequence $\{S_k(\mathbf{r})\}$, the sequence of liquidation size $\{r_{ij}\}$ and the sequence of bid prices $\{m_{ij}\}$. Hence, while preserving these one-one correspondences, we relabel the sequences $\{r_{ij}\}$ and $\{m_{ij}\}$ as $\{r_k\}$ and $\{m_k\}$, respectively. We call the sorted index k the *liquidation sequence*, which is a permutation of the index (i, j).

Note that the first N terms of the sequence $\{m_k\}$ are the best bids m_i^+, $i = 1, \dots, N$.

To illustrate the above concepts, consider an example as follows. Given two illiquid assets, the bid part of which can be read from the market are shown in Table 1. Assume that we hold a portfolio which contains 600 units in asset 1 and 900 in asset 2. Then the liquidation sizes for the two assets are shown in Table 2 and the sorted liquidity deviations as well as the liquidation sequence are presented in Table 3.

To meet the cash constraint embodied in the cash liquidity policy we start liquidating the portfolio from $S_1(\mathbf{r})$, then $S_2(\mathbf{r})$, and so on, until we have met the cash requirement.

The liquidation sequence effectively directs the search process throughout the constraint set towards the global solution, and exactly so. This is summarized in the following theorem, which we will prove subsequently.

Proposition 3.4: *Given is a portfolio $\mathbf{p}$ such that each asset is characterized by a ladder MSDC. Under Assumption 3.1, the optimization problem (9) has the same optimal solution as*

(a) Asset 1

Liquidation Size		Bid Price	
r_{11}	200	m_{11}	11.65
r_{12}	200	m_{12}	11.55
r_{13}	200	m_{13}	11.45

(b) Asset 2

Liquidation Size		Liquidation Size	
r_{21}	200	m_{21}	19.58
r_{22}	600	m_{22}	19.5
r_{23}	100	m_{23}	19.2

Table 2. Liquidation size of our portfolio $\mathbf{r} = (0, 600, 900)$

Liquidation Sequence	Index (i, j)	Asset	Liquidation Size	Bid Price	Liquidity Deviation
1	(1, 1)	1	200	11.65	0
2	(2, 1)	2	200	19.58	0
3	(2, 2)	2	600	19.5	0.004085802
4	(1, 2)	1	200	11.55	0.008583691
5	(1, 3)	1	200	11.45	0.017167382
6	(2, 3)	2	100	19.2	0.019407559

Table 3. Liquidity deviation and liquidation sequence

the following:

$$\begin{cases} \textit{minimize} & \sum_{k=1}^{K} S_k(\boldsymbol{r}); \\ \textit{subject to:} & L(\boldsymbol{r}) = c - p_0; \\ & \boldsymbol{r} \in \mathcal{P}. \end{cases} \tag{14}$$

Loosely put, the optimal solution is the one yielding the minimum total sum of liquidity deviation. Intuitively, the proposition implies that, to meet cash demands, we should liquidate the most liquid assets as they are easier to sell off and their liquidation will incur less losses compared to more illiquid assets.

Proof: Let a portfolio $\mathbf{p} = (p_0, p_1, \ldots, p_N)$ be given and suppose we liquidate a portfolio $\mathbf{r} = (r_0, r_1, \ldots, r_N)$ to meet a liquidity policy $\mathcal{L}$. Asset i has a corresponding MSDC m_i, $i = 0, 1, \ldots, N$. For simplicity, r_0 is set to be 0.

From Proposition 3.2, the optimal solution of (9) minimizes the uppermost liquidation cost. Using that all assets are characterized by ladder MSDCs, the objective function $C(\mathbf{r})$ can be rewritten as follows:

$$\begin{aligned} C(\mathbf{r}) &= U(\mathbf{r}) - L(\mathbf{r}) \\ &= \sum_{i=1}^{N} \sum_{j=1}^{J_i} \big(m_i^+ r_{ij} - m_{ij} r_{ij}\big). \end{aligned}$$

Note that for each asset i, $m_i^+ \geq m_{ij}$ for all j. It follows that the minimum of the sum of the absolute differences between the $m_i^+ r_{ij}$ and $m_{ij} r_{ij}$ is the same as the minimum of the sum of the relative differences. Hence, to find the optimal solution we might as well minimize

$$\begin{aligned} \sum_{i=1}^{N} \sum_{j=1}^{J_i} \frac{m_i^+ r_{ij} - m_{ij} r_{ij}}{m_i^+ r_{ij}} &= \sum_{i=1}^{N} \sum_{j=1}^{J_i} \frac{m_i^+ - m_{ij}}{m_i^+} = \sum_{i=1}^{N} \sum_{j=1}^{J_i} S_{ij}(\mathbf{r}) \\ &= \sum_{k=1}^{K} S_k(\mathbf{r}). \end{aligned}$$

On the last line, note that $K = J_1 + \cdots + J_N$. □

Based on this result, we now state the algorithm for portfolio valuation assuming only ladder MSDCs under Assumption 3.1. For the sake of clarity we recall that the optimal solution $\mathbf{r}^*$ of problem (9) should satisfy $L(\mathbf{r}^*) = c - p_0$. Also, we assume that $p_0 = 0$ and $r_0 = 0$. (Otherwise, we can set the cash requirement $c = c - p_0$.) The pseudocode is summarized in Algorithm **1**.

Algorithm 1 Algorithm for portfolio valuation assuming ladder MSDCs and a cash liquidity policy $\mathcal{L}(c)$

```
Calculate:
U(p) = Σ_{i=1}^{N} m_i^+ · p_i;
L(p) = Σ_{i=1}^{N} Σ_{j=1}^{J_i} m_ij · p_ij;
V_1(p) = Σ_{i=1}^{N} m_i^+ · p_i1;
S_ij = (m_i1 − m_ij) / m_i1;
Sort the S_ij as an ascending sequence with index k. {With k running from 1 to J_1 + ··· + J_N}
if c > L(p) then
  return V^{L(c)}(p) = −∞; {There is no optimal solution satisfying the cash constraint.}
else
  if c = L(p) then
    return V^{L(c)}(p) = L(p); {The optimal solution r* = p.}
  else
    if c ≤ V_1(p) then {Liquidating p_i1 to the respective best bids meets the cash constraint.}
      return V^{L(c)}(p) = U(p); {There are infinitely many optimal solutions.}
    else
      U(r) = V_1(p);
      c = c − V_1(p);
      k = N + 1; {Start loop from the first part with non-zero liquidity deviation until c
      becomes 0.}
      while c > 0 do
        if c/m_k > p_k then
          U(r) = U(r) + m_k^+ · p_k;
          c = c − m_k · p_k;
          k = k + 1;
        else
          U(r) = U(r) + m_k^+ · c/m_k;
          c = 0;
        end if
      end while
      return V^{L(c)}(p) = U(p) − U(r) + c {Here we have L(r) = c.}
    end if
  end if
end if
```

There are generally four cases stated in Algorithm **1**:

(i) if the cash requirement c is higher than the liquidation MtM value $L(\mathbf{p})$ such that the cash liquidity policy cannot be met, then we assign $-\infty$ to the portfolio value $V^{\mathcal{L}(c)}(\mathbf{p})$ and conclude that there is no optimal solution;

(ii) if the cash requirement c is equal to the liquidation MtM value $L(\mathbf{p})$ such that we have to liquidate all parts of the portfolio, then the portfolio value $V^{\mathcal{L}(c)}(\mathbf{p})$ equals the liquidation

MtM value $L(\mathbf{p})$ and the unique optimal solution $\mathbf{r}^* = \mathbf{p}$;

(iii) if the cash requirement c is less than or equal to $V_1(\mathbf{p})$, the liquidation value of all parts of the portfolio corresponding to the best bids, then the portfolio value $V^{\mathcal{L}(c)}(\mathbf{p})$ equals $V_1(\mathbf{p})$ and there exist infinite many optimal solutions;

(iv) if the cash requirement c is higher than $V_1(\mathbf{p})$ but less than $L(\mathbf{p})$, we have to liquidate the portfolio along the liquidation sequence until the cash requirement is met and the unique optimal solution $\mathbf{r}^*$ can be found by recording the liquidation parts of corresponding assets in the calculation procedure of the algorithm.

The piecewise constant MSDCs in the convex optimization problem generally increase the difficulty of the search for the global optimal solution with standard software. With the aforementioned calculation scheme listed in Algorithm **1**, instead, we can solve the optimization problem accurately and efficiently via the liquidation sequence.

4. Portfolio Valuation Using Continuous MSDCs

There typically is no analytic solution to the convex optimization problem (9). However, it can be shown that if we model the MSDC as a continuous function we can obtain simple analytic solutions from the method of Lagrange multipliers. In Section 4.1 we will first look at continuous MSDCs without imposing any specific form for them. We will subsequently look at MSDCs shaped as exponential functions in Sections 4.2 and 4.3. Empirically, we find that exponential MSDCs can be used to model MSDCs for security-type equity assets with different caps. We then propose to use exponential MSDCs to approximate ladder MSDCs in order to improve the efficiency of portfolio valuation in Section 4.4. We will assume the cash liquidity policy in this section.

4.1. The general case

Assume N illiquid assets labeled $1, \ldots, N$ with MSDCs m_i, $i = 1, 2, \ldots, N$. Each m_i is assumed to be continuous on $\mathbb{R}$. This implies that $m_i(0)$ exists. We will exclude the point $m_i(0)$ later in this section. In addition, each m_i is assumed to be strictly decreasing. Adopting a cash liquidity policy, valuing a portfolio consisting of positions in these assets comes down to solving the optimization problem (9). The solution to this optimization problem can be analytically derived, as is shown by the following proposition proposed by Acerbi and Scandolo (2008).

Proposition 4.1: *Assuming continuous strictly decreasing MSDCs and the cash liquidity policy $\mathcal{L}(c)$, the optimal solution $\boldsymbol{r}^* = (0, \vec{r}^*)$ to optimization problem (9) is unique and given by*

$$r_i^* = \begin{cases} m_i^{-1}(\frac{m_i(0)}{1+\lambda}), & \text{if } p_0 < c; \\ 0, & \text{if } p_0 \geq c, \end{cases} \tag{15}$$

where $m_i^{-1}(\cdot)$ denotes the inverse of the MSDC function $m_i(\cdot)$, and the Lagrange multiplier λ, representing the marginal liquidation cost, can be determined from the equation $L(\boldsymbol{r}^*) = c - p_0$.

We refer to Acerbi and Scandolo (2008) for a proof.

Remark 2: Note that we can extend the above to the case where the MSDCs are not continuous at the point 0, i.e., the case where there is a positive bid-ask spread. We have to change the definition of the value at $m_i(0)$ to the limit m_i^+ in the case of long positions or to m_i^- in the case of short positions.

Obviously, by using the Lagrange multiplier method, we can generalize the case to any liquidity policy giving rise to equality constraints. When using a general liquidity policy which results in inequality constraints, we can solve the optimization problem (7) by checking the Karush-Kuhn-Tucker (KKT) conditions. In addition, the Lagrange dual method may be useful as well.

4.2. Exponential MSDCs for large- and medium-cap equities

We continue the discussion by looking at a particular example of a MSDC, i.e., the exponential MSDC. As it turns out, the exponential MSDCs form an effective model to characterize a security-type asset and to determine the portfolio value by convex optimization. We will discuss this in Section 4.4.

Many researches have shown that there is a relation between the price change and the trading volume in the market during a short time period. Cont *et al.* (2011) propose that there is a "square-root" relation between the price change and the trading volume for S&P 500 equities. A similar result can be found in Almgren *et al.* (2005), where the authors proposed a "3/5"-relation between the temporary price impact and the trade size for large-cap US equites. These parameters correspond to a medium-size price impact.

In our paper, we interpret the price change as $\log(m(s)/m^+)$, i.e., the relative change between the bid (or ask) price $m(s)$ and the best bid m^+ (or best ask m^-) over a short time period, during which an MSDC can be formed and denote trading volume to be s.

Large-and medium-cap equities listed on stock exchanges such as the London Stock Exchange and Euronext are actively traded and thus relatively liquid. From available data we observe a "square-root" relation between the bid price change and the volume over a short time, as follows.

$$\log\left(\frac{m(s)}{m^+}\right) = -k\sqrt{s} + \epsilon, \tag{16}$$

where $m^+, k > 0$ and ϵ is the noise term.

For the ask price part, where $s < 0$, then we have the following model

$$\log\left(\frac{m(s)}{m^-}\right) = k\sqrt{|s|} + \epsilon, \tag{17}$$

where $m^-, k > 0$ and ϵ is the noise term.

When skipping the noise term, we use the following exponential MSDC models to approximate the bid part of the ladder MSDC for a large- or medium-cap equity as

$$m(s) = m^+ \cdot e^{-k\sqrt{s}}, \tag{18}$$

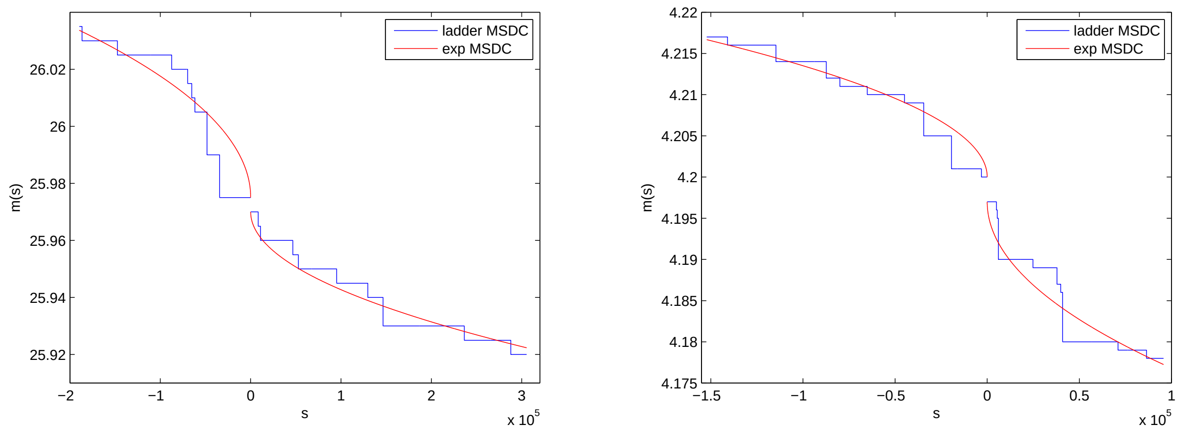
and to approximate the ask part as

$$m(s) = m^- \cdot e^{k\sqrt{|s|}}. \tag{19}$$

We give two examples of the ladder MSDCs and the above approximated exponential MSDCs by least squares regression for large- and medium-cap equities in Figure 1.

Suppose there are N (large- or medium-cap) security-type assets $1, 2, \dots, N$, the bid parts of which are characterized by

$$m_i(s) = m_i^+ \cdot e^{-k_i\sqrt{s}}. \tag{20}$$



(a) Large-cap equity (b) Medium-cap equity

Figure 1. Exponential MSDCs versus ladder MSDCs for large- and medium-cap equities

We call k_i the *liquidity risk factor* for the corresponding asset i $(i = 1, \dots, N)$, which measures the general liquidity condition of the asset i. From Proposition 4.1 and Remark **2**, we can approximate portfolio values under different liquidity policies.

As an example, assuming a portfolio with only long positions, then we have the liquidation MtM value

$$L(\mathbf{p}) = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x = p_0 + \sum_{i=1}^{N} \frac{2m_i^+}{k_i^2}\left(1 - k_i\sqrt{p_i}e^{-k_i\sqrt{p_i}} - e^{-k_i\sqrt{p_i}}\right), \tag{21}$$

and under a cash liquidity policy $\mathcal{L}(c)$ with $p_0 < c$, from Proposition 4.1, we have

$$\begin{aligned} r_i^* &= \left(\frac{\log(1+\lambda)}{k_i}\right)^2, \quad i = 1, \dots, N, \\ &\text{with} \quad \lambda = e^x - 1, \quad x > 0, \\ \text{and} \quad &\left(1 - \frac{c - p_0}{\sum_{i=1}^{N} \frac{2m_i^+}{k_i^2}}\right)e^x - x - 1 = 0. \end{aligned} \tag{22}$$

The last equation can be solved numerically by using the Newton-Raphson iteration method or the Taylor's expansion. Hence, the portfolio value under the cash liquidity $\mathcal{L}(c)$ reads

$$V^{\mathcal{L}(c)}(\mathbf{p}) = U(\mathbf{p} - \mathbf{r}^*) + L(\mathbf{r}^*) = \sum_{i=1}^{N} m_i^+ \cdot \left(p_i - \left(\frac{\log(1+\lambda)}{k_i}\right)^2\right) + c. \tag{23}$$

For large- and medium-cap assets, since they are generally very liquid to trade, the uppermost liquidation cost is usually very small.

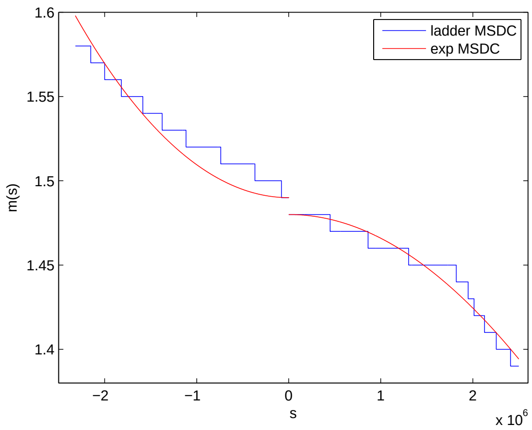


Figure 2. Exponential MSDCs versus ladder MSDCs for a small-cap equity

4.3. Exponential MSDCs for small-cap equities

On the other hand, for small-cap equities, we find there is a "square" relation between the bid price change and the volume over a short time, which implies a large price impact, as

$$\log\left(\frac{m(s)}{m^+}\right) = -ks^2 + \epsilon, \tag{24}$$

where $m^+, k > 0$ and ϵ is the noise term.

Similarly, for the ask price change we have

$$\log\left(\frac{m(s)}{m^-}\right) = k|s|^2 + \epsilon, \tag{25}$$

where $m^-, k > 0$ and ϵ is the noise term.

When skipping the noise term, we have the following exponential MSDC model to approximate the bid part of a ladder MSDC of a small-cap equity as

$$m(s) = m^+ \cdot e^{-ks^2}, \tag{26}$$

and for the ask part we have the exponential MSDC

$$m(s) = m^- \cdot e^{k|s|^2}. \tag{27}$$

Here we give an example of a ladder MSDC and the approximated exponential MSDC for a small-cap equity in Figure 2.

Suppose that there are N (small-cap) security-type assets $1, 2, \ldots, N$, whose bid parts are characterized by the following exponential MSDCs

$$m_i(s) = m_i^+ \cdot e^{-k_i s^2}, \tag{28}$$

with $m_i^+, k_i > 0$ for all $i = 1, \ldots, N$. By using a least squares approximation, we could fit the value of m_i^+ and k_i from the real data. See Section 4.4.

To illustrate this type of exponential MSDC function, we assume a portfolio with only long positions. Then the liquidation MtM value reads

$$L(\mathbf{p}) = p_0 + \sum_{i=1}^{N} \int_0^{p_i} m_i(x)\mathrm{d}x = p_0 + \frac{\sqrt{\pi}}{2} \sum_{i=1}^{N} \frac{m_i^+}{\sqrt{k_i}} \cdot \operatorname{erf}(\sqrt{k_i} p_i), \tag{29}$$

where $\operatorname{erf}(\cdot)$ is the Gauss error function, which can be numerically obtained.

For a cash liquidity policy $\mathcal{L}(c)$ with $p_0 < c$, from Proposition 4.1, we have

$$\begin{aligned} r_i^* &= \sqrt{\frac{\log(1+\lambda)}{k_i}}, \quad i = 1, \ldots, N, \\ \text{with} \quad \lambda &= e^{z^2} - 1, \\ z &= \operatorname{erf}^{-1}\left(\frac{c - p_0}{\frac{\sqrt{\pi}}{2} \sum_{i=1}^{N} \frac{m_i^+}{\sqrt{k_i}}} \right), \end{aligned} \tag{30}$$

where $\operatorname{erf}^{-1}(\cdot)$ is the inverse error function, which can also be numerically obtained. Hence,

$$V^{\mathcal{L}(c)}(\mathbf{p}) = U(\mathbf{p} - \mathbf{r}^*) + L(\mathbf{r}^*) = \sum_{i=1}^{N} m_i^+ \cdot \left(p_i - \sqrt{\frac{\log(1+\lambda)}{k_i}} \right) + c. \tag{31}$$

4.4. Approximating ladder MSDCs by exponential MSDCs

In Section 3 we have defined a fast calculation scheme for portfolio valuation with ladder MSDCs. In the real world, however, we may face a situation where to collect the price information to form a ladder MSDC is too costly, or where the information is incomplete or not available, e.g., in an over-the-counter (OTC) market.

As an order book records the trading volume, which forms the basis of MSDCs, one could model ladder MSDCs from the modeling of order book dynamics. For example, in Bouchaud *et al.* (2002), the trading volume at each bid (or ask) price in the stock order book followed a Gamma distribution. In Cont *et al.* (2010), a continuous Markov chain was used to model the evolution of the order book dynamics.

In our paper, we aim to use the basic continuous MSDC models to approximate ladder MSDCs directly, as we can then apply the Lagrange multiplier method and other convex optimization techniques to obtain analytic solutions and thus improve the efficiency.

For actively traded large- or medium-cap security-type assets, a portfolio valuation based on exponential MSDCs (20) with their analytic solutions, is significantly faster than with ladder MSDCs. For less actively traded small-cap assets, we can use the exponential MSDC model (26) to obtain portfolio values. For OTC traded assets, lacking price information, the exponential MSDC (28) for small-cap security-type asset with a large liquidity risk factor can be a first modeling attempt.

Generally, when using exponential MSDC models (28) for small-cap security-type assets, we need to estimate or model the parameters m_i^+ and k_i. The dynamics of the best bid m_i^+ can be read from market data, or modeled by asset price models (e.g., geometric Brownian motion).

If we assume that the liquidity risk factor k_i is independent of m_i^+, we can employ time series or stochastic processes to model k_i. If k_i is assumed to be correlated with m_i^+, we also need to model the correlation. Furthermore, for security-type assets traded in an OTC market, we may use the mere price information of the asset to estimate liquidity risk factors in the MSDC models (28). In particular, the liquidity risk factor may be set at a high level to represent the illiquidity of the asset.

For the approximation of ladder MSDCs of illiquid security-type assets by using small-cap exponential MSDCs (28), we assume that the portfolio consists of only long positions in N illiquid security-type assets. If we assume that the liquidity risk factor of asset i, k_i, is independent of the best bid m_i^+, then parameter k_i can be estimated from the ladder MSDC of asset i by the method of least squares as follows. Provided that m_i^+ has already been determined, we transform the exponential function as $-\log(\frac{m_i(s)}{m_i^+}) = s^2 k_i$, and estimate k_i by n discrete pairs $(s_n, -\log(\frac{m_i(s_n)}{m_i^+}))$ to minimize the merit function:

$$\sum_{j=1}^{n}(-\log(\frac{m_i(s_j)}{m_i^+}) - s_j^2 k_i)^2. \tag{32}$$

The least squares estimate of parameter k_i then reads

$$\hat{k}_i = \frac{-\sum_{j=1}^{n} s_j^2 \log(\frac{m_i(s_j)}{m_i^+})}{\sum_{j=1}^{n} s_j^4}. \tag{33}$$

5. Numerical Results

In this section we give examples for the various concepts discussed in this paper. In particular, we explain the calculation scheme for efficient portfolio valuation by means of an example. Since, for large- or medium-cap security assets, the uppermost liquidation cost is usually quite small, we will focus on relatively illiquid small-cap security assets and valuate portfolios using ladder MSDCs and exponential MSDCs.

5.1. Portfolio with four illiquid assets

The example here is based on four illiquid small-cap security-type assets. We deal with a portfolio $\mathbf{p} = (0, 3400, 2400, 3200, 2800)$ with zero cash asset and long positions in all four illiquid assets. The bid prices with liquidation sizes for the portfolio are chosen at a given time as presented in Table 4.

It is easy to calculate the uppermost MtM value $U(\mathbf{p})$ and the liquidation MtM value $L(\mathbf{p})$ from the tables, that is, $U(\mathbf{p}) = 3.01042 \times 10^5$ and $L(\mathbf{p}) = 2.73720 \times 10^5$. Hence, the uppermost liquidation cost equals $C(\mathbf{p}) = 0.27322 \times 10^5$. If the true portfolio value is equal to the liquidation MtM value, but if we would use however the uppermost MtM value instead, we would overestimate the portfolio value by as much as 10%.

For different cash requirements, we use the sorted liquidity deviations (see Table 5) to find the liquidation sequence and then calculate the portfolio values (see Figure 3). From the last row of Table 5, we can see that the liquidity deviation can be as large as 44.5% for the most illiquid part of the MSDC for asset 1, which indicates a high level of liquidity risk.

(a) Asset 1

Liquidation Size	Bid Price
200	11.65
200	11.55
200	11.45
200	11.1
200	11.05
200	11
200	10.3
500	9.3
500	6.5
1000	6.46

(b) Asset 2

Liquidation Size	Bid Price
200	19.58
600	19.5
200	19.2
200	19.15
200	19.1
200	18.6
200	18.5
200	16.85
200	16.1
200	16.05

(c) Asset 3

Liquidation Size	Bid Price
400	29.3
200	29.16
400	29.15
400	28.9
200	28
600	27.8
200	27.15
200	27
400	26
200	22

(d) Asset 4

Liquidation Size	Bid Price
200	43.1
400	42.65
200	41.9
400	41
200	40.86
200	40.4
200	39
400	37
400	36
200	35.1

Table 4. Bids of assets 1-4

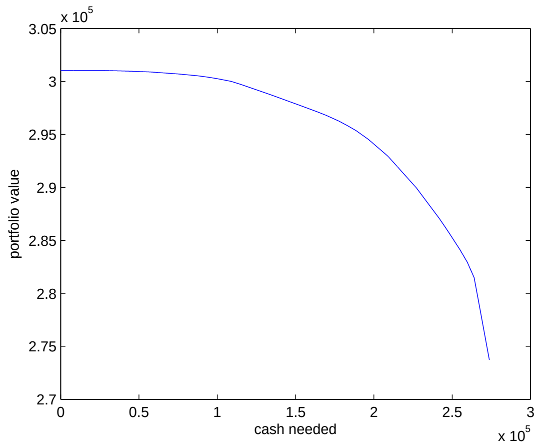


Figure 3. Portfolio value with different cash requirements

From Figure 3, we infer that the portfolio value decreases at a faster rate as we have to liquidate positions of an increasing number of illiquid assets to meet the cash requirements, which will definitely cause more significant losses during liquidation.

The calculation scheme in Algorithm **1** provides an efficient search direction to the optimal value guided by the liquidation sequence. For this four-asset example, we compare our calculation scheme with the *fmincon* function with an interior point algorithm in MATLAB for portfolio valuation. The optimization is repeated for around 2.5×10^5 different cash requirements and the total computation time is recorded.[1] The averaged time for each cash liquidity policy equals

[1]The computer used for all experiments has an Intel Core2 Duo CPU, E8600 @3.33GHz with 3.49 GB of RAM and the code is written in MATLAB R2009b.

Liquidation Sequence	Index (i, j)	Asset	Liquidation Size	Bid Price	Best Bid	Liquidity Deviation
1	(1, 1)	1	200	11.65	11.65	0
2	(2, 1)	2	200	19.58	19.58	0
3	(3, 1)	3	400	29.3	29.3	0
4	(4, 1)	4	200	43.1	43.1	0
5	(2, 2)	2	600	19.5	19.58	0.004085802
6	(3, 2)	3	200	29.16	29.3	0.004778157
7	(3, 3)	3	400	29.15	29.3	0.005119454
8	(1, 2)	1	200	11.55	11.65	0.008583691
9	(4, 2)	4	400	42.65	43.1	0.010440835
10	(3, 4)	3	400	28.9	29.3	0.013651877
11	(1, 3)	1	200	11.45	11.65	0.017167382
12	(2, 3)	2	200	19.2	19.58	0.019407559
13	(2, 4)	2	200	19.15	19.58	0.021961185
14	(2, 5)	2	200	19.1	19.58	0.024514811
15	(4, 3)	4	200	41.9	43.1	0.027842227
16	(3, 5)	3	200	28	29.3	0.044368601
17	(1, 4)	1	200	11.1	11.65	0.0472103
18	(4, 4)	4	400	41	43.1	0.048723898
19	(2, 6)	2	200	18.6	19.58	0.050051073
20	(3, 6)	3	600	27.8	29.3	0.051194539
21	(1, 5)	1	200	11.05	11.65	0.051502146
22	(4, 5)	4	200	40.86	43.1	0.051972158
23	(2, 7)	2	200	18.5	19.58	0.055158325
24	(1, 6)	1	200	11	11.65	0.055793991
25	(4, 6)	4	200	40.4	43.1	0.062645012
26	(3, 7)	3	200	27.15	29.3	0.07337884
27	(3, 8)	3	200	27	29.3	0.078498294
28	(4, 7)	4	200	39	43.1	0.09512761
29	(3, 9)	3	400	26	29.3	0.112627986
30	(1, 7)	1	200	10.3	11.65	0.115879828
31	(2, 8)	2	200	16.85	19.58	0.139427988
32	(4, 8)	4	400	37	43.1	0.141531323
33	(4, 9)	4	400	36	43.1	0.164733179
34	(2, 9)	2	200	16.1	19.58	0.17773238
35	(2, 10)	2	200	16.05	19.58	0.180286006
36	(4, 10)	4	200	35.1	43.1	0.185614849
37	(1, 8)	1	500	9.3	11.65	0.201716738
38	(3, 10)	3	200	22	29.3	0.249146758
39	(1, 9)	1	500	6.5	11.65	0.442060086
40	(1, 10)	1	1000	6.46	11.65	0.445493562

Table 5. Liquidity deviation and liquidation sequence

0.568 millisecond for our scheme, whereas *fmincon* takes 202.7 milliseconds, which implies that the time difference is a factor of 300.

Since the ascending sequence of liquidity deviations shows the illiquidity of different parts of the corresponding asset, liquidating a portfolio along the liquidation sequence will cause minimum loss of values compared to the other kinds of liquidation.

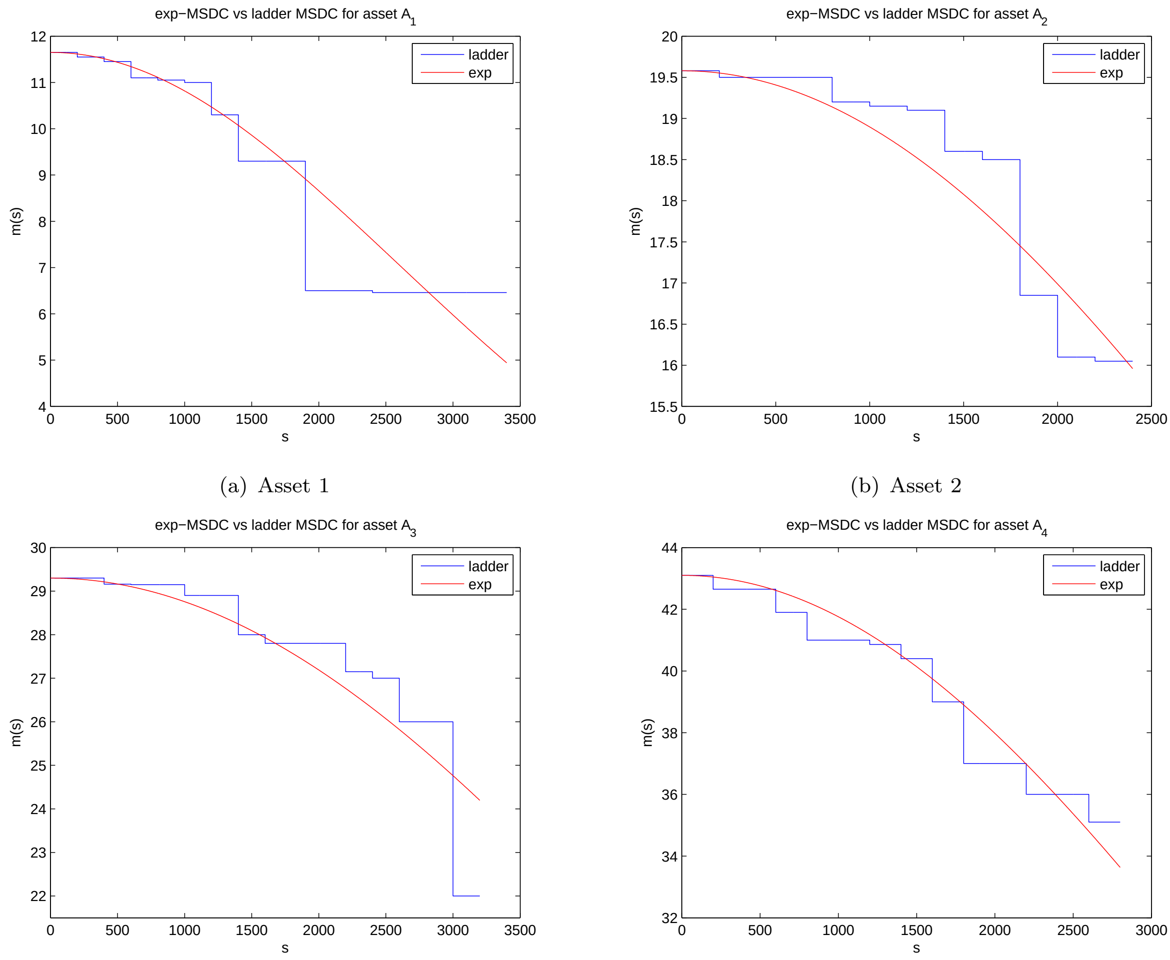
5.2. Using exponential MSDCs to approximate ladder MSDCs

For the four-asset example with the ladder MSDCs from Section 5.1, we use the exponential MSDCs (28) for small-cap equities. Figure 4 illustrates the ladder MSDCs and the corresponding exponential approximating MSDCs. The latter MSDCs are estimated by least squares (see Section 4.4).

The liquidity risk factors in the exponential MSDCs are found as $k_1 = 7.4193 \times 10^{-8}$, $k_2 = 3.5499 \times 10^{-8}$, $k_3 = 1.8691 \times 10^{-8}$ and $k_4 = 3.1634 \times 10^{-8}$. Hence, we infer that asset 1 is most illiquid and asset 3 is most liquid in general.

In Figure 5(a), we compare the portfolio values obtained by using the exponential MSDCs with the reference portfolio values by the ladder MSDCs under different cash requirements. The relative difference in the portfolio values is presented in Figure 5(b).

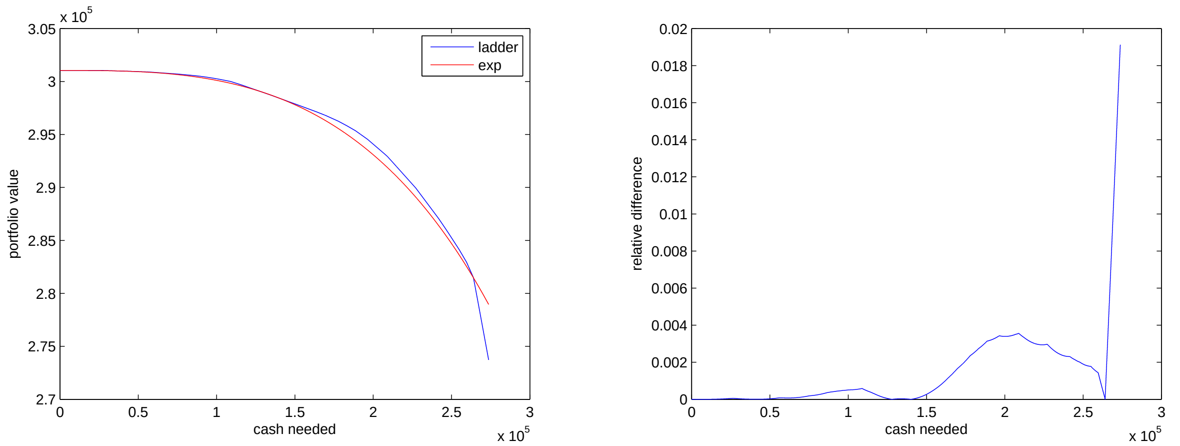
The relative difference found is at most 1.91%, so that in this example the exponential MSDCs are accurate approximations. The large approximation error lies in the tail part of the figure



(a) Asset 1 (b) Asset 2

(c) Asset 3 (d) Asset 4

Figure 4. Exponential MSDCs versus ladder MSDCs for the bid prices of assets 1-4



(a) Comparison of portfolio values (b) Relative difference in portfolio values

Figure 5. Modeling ladder MSDCs by exponential MSDCs

which caused by the illiquidity of the tail parts of assets 1 and 3. This means the exponential MSDCs may fail to approximate the tail parts of assets 1 and 3 if there are huge drops in price.

6. Conclusion

Within the theory proposed by Acerbi and Scandolo (2008) the valuation of a portfolio can be framed as a convex optimization problem. We have proposed a useful and efficient algorithm using a specific form of the market data function, i.e., all price information is represented in terms of a ladder MSDC. We have also considered approximations of ladder MSDCs by exponential functions.

As long as the portfolio is valuated by using the new models incorporating liquidity risk, one can calculate Value-at-Risk and other risk measures for risk management. Another application is in portfolio selection. Under the new portfolio theory, the procedure of portfolio selection will become a convex optimization of the allocation based on the convex optimization of portfolio valuation.

By way of future research, methods to estimate the liquidity risk factor in the exponential functions may be improved and more sophisticated models to replace the exponential functions may be considered.

Whereas in regulated markets such as stock exchanges price information is relatively easily available, bid and ask prices for assets traded in the OTC markets may not be easily obtained. Hence, it seems nontrivial to apply this portfolio theory to these type of markets. Extracting all relevant price information from OTC markets is however a challenge for all researchers.

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