DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Monday January 28 2013, 18:30-21:30

1. The Modified Euler Method to integrate the initial value problem defined by $y' = f(t, y)$, $y(t_0) = y_0$, is given by

$$\begin{cases} w^*_{n+1} = w_n + hf(t_n, w_n) \\ \\ w_{n+1} = w_n + \dfrac{h}{2}\left(f(t_n, w_n) + f(t_{n+1}, w^*_{n+1})\right). \end{cases} \quad (1)$$

where h denotes the time–step and w_n represents the numerical solution at time t_n.

[a] Show that the local truncation error of the Modified Euler Method is given by $O(h^2)$. (3 pt)

The amplification factor of the Modified Euler Method is given by

$$Q(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2}.$$

[b] Derive this amplification factor for the Modified Euler Method. (1 pt)

Given the initial value problem

$$\begin{cases} \dfrac{d^2y}{dt^2} + 12\dfrac{dy}{dt} + 72y = \sin t, \\ y(0) = 1, \quad \dfrac{dy}{dt}(0) = 2. \end{cases} \quad (2)$$

[c] Show that, this problem can be written as

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -72 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}. \quad (3)$$

Give also the initial conditions for $x_1(0)$ and $x_2(0)$. (2 pt)

[d] Perform one step with the Modified Euler Method with $h = 0.1$ and $t_0 = 0$, using the given initial conditions from (2). (2 pt)

[e] Determine whether the Modified Euler Method, applied to the given initial value problem (2), is stable for $h = 0.25$. (2 pt)

[0]**please turn over, For the answers of this test we refer to: http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html**

2. In this exercise an estimate is determined for the velocity of a vehicle. The maximum speed at the road is 40 m/s. The measured positions of the vehicle are given in the table below.

t (s)	0	1	2
$f(t)$ (m)	200	215	250

(a) Give the first order backward difference formula and use this to determine an estimate of the velocity for $t = 2$ ($f'(2)$). (1 pt.)

(b) We are looking for a difference formula of the first derivative of f in $2h$ of the form: $Q(h) = \frac{\alpha_0}{h} f(0) + \frac{\alpha_1}{h} f(h) + \frac{\alpha_2}{h} f(2h)$, such that $f'(2h) - Q(h) = O(h^2)$. In the remainder of this exercise we use this formula. Show that the coefficients $\alpha_0,\ \alpha_1$ and α_2 should satisfy the next system:

$$\begin{array}{rcrcrcl} \frac{\alpha_0}{h} & + & \frac{\alpha_1}{h} & + & \frac{\alpha_2}{h} & = & 0\,, \\ -2\alpha_0 & - & \alpha_1 & & & = & 1\,, \\ 2\alpha_0 h & + & \frac{1}{2}\alpha_1 h & & & = & 0\,. \end{array}$$

(2 pt.)

(c) The solution of this system is given by $\alpha_0 = \frac{1}{2}$, $\alpha_1 = -2$ and $\alpha_2 = \frac{3}{2}$. Give for these values an expression for the rounding error $f'(2h) - Q(h)$. Use this formula to give an estimate of the velocity. (2 pt.)

(d) The maximal measuring error in the measured position is bounded by ϵ: $|f(t) - \hat{f}(t)| \leq \epsilon$. Show that this implies a measuring error in the estimate of

$$|Q(h) - \hat{Q}(h)| \leq \frac{C_1 \epsilon}{h}$$

and give C_1. (1.5 pt.)

(e) Derive the Trapezoidal Rule to approximate $\int_{x_0}^{x_1} f(x)dx$ by the use of the linear Lagrangian interpolatory polynomial. (1.5pt.)

(f) Derive that an upper bound of the truncation error of the Trapezoidal Rule applied to a single interval $[x_0, x_1]$ is given by

$$\frac{1}{12}(x_1 - x_0)^3 \max_{x \in [x_0, x_1]} |f''(x)|, \tag{4}$$

if the second order derivative of f is continuous over $[x_0, x_1]$. *Hint: The error for linear interpolation over nodes x_0 and x_1 is given by*

$$f(x) - p_1(x) = \frac{1}{2}(x - x_0)(x - x_1)f''(\chi), \ \textit{for a}\ \chi \in (x_0, x_1),$$

where $p_1(x)$ denotes the linear interpolatory polynomial. (2 pt.)