DELFT UNIVERSITY OF TECHNOLOGY
Faculty of Electrical Engineering, Mathematics and Computer Science

ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday April 18 2013, 18:30-21:30

1. a The local truncation error is defined by

$$\tau_h = \frac{y_{n+1} - z_{n+1}}{h}, \tag{1}$$

where

$$z_{n+1} = y_n + hf(t_n, y_n), \tag{2}$$

for the forward Euler method. A Taylor expansion for y_{n+1} around t_n is given by

$$y_{n+1} = y_n + hy'(t_n) + \frac{h^2}{2}y''(\xi), \quad \exists\, \xi \in (t_n, t_{n+1}). \tag{3}$$

Since $y'(t_n) = f(t_n, y_n)$, we use equation (1), to get

$$\tau_h = \frac{h}{2}y''(\xi), \quad \exists\, \xi \in (t_n, t_{n+1}). \tag{4}$$

Hence, the truncation error is of first order.

b We define $y_1 := y$ and $y_2 := y'$, hence $y_1' = y_2$. Further, we use the differential equation to obtain

$$y'' + \varepsilon y' + y = y_1'' + \varepsilon y_1' + y_1 = y_2' + \varepsilon y_2 + y_1. \tag{5}$$

Hence, we obtain

$$y_2' = -y_1 - \varepsilon y_2 + sin(t). \tag{6}$$

Hence the system is given by

$$\begin{aligned} y_1' &= y_2, \\ y_2' &= -y_1 - \varepsilon y_2 + sin(t). \end{aligned} \tag{7}$$

The initial conditions are given by

$$\begin{aligned} 1 &= y(0) = y_1(0), \\ 0 &= y'(0) = y_1'(0) = y_2(0). \end{aligned} \tag{8}$$

c First, we use the test equation, $y' = \lambda y$, to analyze numerical stability. For forward Euler, we obtain

$$w_{n+1} = w_n + h\lambda w_n = Q(h\lambda)w_n, \tag{9}$$

hence the amplification factor becomes

$$Q(h\lambda) = 1 + h\lambda. \tag{10}$$

The numerical solution is stable if and only if $|Q(h\lambda)| \leq 1$. Next, we deal with the case $\varepsilon = 0$, to obtain the following system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{11}$$

This system gives the following eigenvalues $\lambda_{1,2} = \pm i$, where i is the imaginary unit. Hence, the amplification factor is given by

$$Q(h\lambda) = 1 \pm hi. \tag{12}$$

Then, it is immediately clear that $|Q(h\lambda)| > 1$ for all $h > 0$. Hence, we conclude that the forward Euler method is never stable if $\varepsilon = 0$.

d From Assignment 1.c., we know that if $\varepsilon = 0$, the eigenvalues of the system are purely imaginary. This implies that the system is analytically (zero) stable if $\varepsilon = 0$.

Nonzero values of ε give the following system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & \varepsilon \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}. \tag{13}$$

then we get the following eigenvalues $\lambda_{1,2} = \frac{\varepsilon}{2} \pm \frac{1}{2}\sqrt{\varepsilon^2 - 4}$ (real-valued), if $\varepsilon^2 - 4 \geq 0$ and $\lambda = \frac{\varepsilon}{2} \pm \frac{i}{2}\sqrt{4 - \varepsilon^2}$ (nonreal-valued) if $\varepsilon^2 - 4 < 0$. Hence, we consider two cases: real-valued and nonreal-valued eigenvalues.

Real-valued eigenvalues
In this case $|\varepsilon| \geq 2$, and $0 \leq \varepsilon^2 - 4 < \varepsilon^2$, and hence the real-valued eigenvalues have the same sign, which is determined by the sign of ε. Hence, if $\varepsilon \leq -2$, then, the system is stable. Furthermore, if $\varepsilon \geq 2$, then, the system is unstable.

Nonreal-valued eigenvalues
In this case $|\varepsilon| < 2$. The system is analytically unstable if and only if the real part of the eigenvalues is positive. Further, the real part of the eigenvalues is positive if and only if $\varepsilon > 0$. Hence, the system is analytically unstable if and only if $\varepsilon > 0$. Hence, the system is stable if and only if $(-2 <)\varepsilon \leq 0$.

From these arguments, it follows that the system is stable if and only if $\varepsilon \leq 0$.

e Since currently the discriminant, $\varepsilon^2 - 4$, is negative, the eigenvalues are nonreal. Substitution into the amplification factor yields

$$Q(h\lambda) = 1 + \frac{\varepsilon}{2}h \pm \frac{ih}{2}\sqrt{4-\varepsilon^2}. \tag{14}$$

Hence, numerical stability is warranted if

$$|Q(h\lambda)|^2 = (1 + \frac{\varepsilon}{2}h)^2 + \frac{h^2}{4}(4-\varepsilon^2) \leq 1. \tag{15}$$

Hence for stability, we have

$$1 + \varepsilon h + \frac{\varepsilon^2 h^2}{4} + h^2 - \frac{\varepsilon^2 h^2}{4} = 1 + h\varepsilon + h^2 \leq 1. \tag{16}$$

Since $h > 0$, we obtain the following stability criterion

$$h \leq -\varepsilon = |\varepsilon|. \tag{17}$$

If $\varepsilon = -2$, then both eigenvalues are real-valued and given by $\lambda_{1,2} = -1$. For this case, we obtain $Q(\lambda h) = 1 - h$, and stability is warranted if and only if $-1 \leq Q(h\lambda) \leq 1$, hence $h \leq 2(= |\varepsilon|)$.

We conclude that for $-2 \leq \varepsilon < 0$, we have a numerically stable solution if and only if $h \leq |\varepsilon|$.

2. a First we check that $y(x) = x^2$ satisfies the boundary conditions. It immediately follows that $y(0) = 0$ and using $y'(x) = 2x$, gives $y'(1) = 2$, and hence the boundary conditions are satisfied. Further, substitution of $y = x^2$, using $y''(x) = 2$, gives

$$-y'' + y' + y = -2 + 2x + x^2, \tag{18}$$

which is equal to the right-hand side of the differential equation and hence $y(x) = x^2$ satisfies the boundary value problem (the differential equation and the boundary conditions).

b Let $x_j = jh$, $x_n = 1$, hence $h = \frac{1}{n}$. We use a Taylor Series to express the relation between the differences formulae and the derivatives. Using the convention that

$y_j = y(x_j)$, gives

$$
\begin{aligned}
&-\frac{y_{j+1}-2y_j+y_{j-1}}{h^2}+\frac{y_{j+1}-y_{j-1}}{2h}+y_j = \\
&-\frac{y_j+hy'(x_j)+\frac{h^2}{2}y''(x_j)+\frac{h^3}{3!}y'''(x_j)+\frac{h^4}{4!}y''''(x_j)+O(h^5)-2y_j}{h^2}- \\
&\frac{y_j-hy'(x_j)+\frac{h^2}{2}y''(x_j)-\frac{h^3}{3!}y'''(x_j)+\frac{h^4}{4!}y''''(x_j)+O(h^5)}{h^2}+ \\
&\frac{y_j+hy'(x_j)+\frac{h^2}{2}y''(x_j)+\frac{h^3}{3!}y'''(x_j)+O(h^4)}{2h}- \\
&\frac{y_j-hy'(x_j)+\frac{h^2}{2}y''(x_j)-\frac{h^3}{3!}y'''(x_j)+O(h^4)}{2h}+y_j = \\
&-y''(x_j)+y'(x_j)+y(x_j)+\frac{h^2}{12}\left(y''''(x_j)+2y'''(x_j)\right)+O(h^3).
\end{aligned}
\tag{19}
$$

Hence the local trunction error for the discretization in the interior gives a order $O(h^2)$, where minimal third–order derivatives are involved. Further, using a virtual gridnode at $x_{n+1} = 1 + h$, gives

$$
\begin{aligned}
&\frac{y_{n+1}-y_{n-1}}{2h}=\frac{y(1)+hy'(1)+\frac{h^2}{2}y''(1)+\frac{h^3}{3!}y'''(1)+O(h^4)}{2h}- \\
&\frac{y(1)-hy'(1)+\frac{h^2}{2}y''(1)-\frac{h^3}{3!}y'''(1)+O(h^4)}{2h}=y'(1)+\frac{h^2}{6}y'''(1)+O(h^3)= \\
&2+\frac{h^2}{6}y'''(1)+O(h^3).
\end{aligned}
\tag{20}
$$

Hence, also for the differencing at $x = 1$, a local truncation error of $O(h^2)$ is obtained with derivatives of minimal third order. Hence all difference formulae give a (local) truncation error of order $O(h^2)$. Neglecting the truncation errors, and setting $f(x) = x^2 + 2x - 2$, gives the following finite difference approach for the numerical approximation w_j:

$$
\frac{-w_{j-1}+2w_j-w_{j+1}}{h^2}+\frac{w_{j+1}-w_{j-1}}{2h}+w_j=f(x_j), \qquad j=1\dots n. \tag{21}
$$

The above equation can be simplified to

$$
-\left(\frac{1}{h^2}+\frac{1}{2h}\right)w_{j-1}+\left(1+\frac{2}{h^2}\right)w_j+\left(-\frac{1}{h^2}+\frac{1}{2h}\right)w_{j+1}=f(x_j), \qquad j=1\dots n. \tag{22}
$$

Using the boundary condition $w_0 = 0$, gives for $j = 0$:

$$(1 + \frac{2}{h^2})w_1 + (-\frac{1}{h^2} + \frac{1}{2h})w_2 = f(x_1). \tag{23}$$

For $j = n$, we substitute

$$\frac{w_{n+1} - w_{n-1}}{2h} = 2 \Leftrightarrow w_{n+1} = w_{n-1} + 4h, \tag{24}$$

to obtain for $j = n$

$$-\frac{2}{h^2}w_{n-1} + (1 + \frac{2}{h^2})w_n = f(x_j) + \frac{4}{h} - 2. \tag{25}$$

Herewith, we got a discretization with local truncation errors of $O(h^2)$.

c In the previous assignment, we saw that all truncation errors are of order $O(h^2)$ with derivatives of minimal third order. Since $y(x) = x^2$ is the (only) solution to the boundary value problem considered currently, we see that all p–th order derivatives $y^{(p)}(x) = 0$, for $p \geq 3$, and hence all truncation errors are zero. Therefore, for the present boundary value problem, the current finite differences approach gives the exact solution to the boundary value problem (hence the difference between the exact solution and the numerical approximation vanishes).

d The forward difference formula, $Q(h)$, to approximate $y'(0)$ is given by

$$Q(h) = \frac{y(h) - y(0)}{h}. \tag{26}$$

For $h = 0.25$ and $h = 0.5$ from the tabular values, we, respectively, get $\tilde{Q}(0.25) = 0.252$ and $\tilde{Q}(0.5) = 0.5$. Note that the tildes indicate that we used the approximate values for $y(x)$ from Table 1.

e i Let $\tilde{y}(x_j)$, and $y(x_j)$, respectively, represent the approximate values and exact values, and let $\tilde{Q}(h)$ denote the differencing executed with the approximate values for y, then

$$|Q(h) - \tilde{Q}(h)| = |\frac{y(h) - y(0)}{h} - \frac{\tilde{y}(h) - y(0)}{h}| = \frac{|y(h) - \tilde{y}(h)|}{h} \leq \frac{\varepsilon}{h} = \frac{0.0005}{h}. \tag{27}$$

(Note that this gives an upperbound $|Q(h) - \tilde{Q}(h)| \leq 0.002$.)

ii The truncation error is given by

$$y'(0) - \frac{y(h) - y(0)}{h} = y'(0) - \frac{y(0) + hy'(0) + \frac{h^2}{2}y''(0) + O(h^3) - y(0)}{h} = -\frac{h}{2}y''(0) + O(h^2). \tag{28}$$

Hence the truncation error is of order $O(h)$.

iii The truncation error is of first order, hence for h sufficiently small, we have

$$y'(0) \approx Q(h) + Kh, \tag{29}$$

where Kh is an estimate of the error, and for $2h$, we get

$$y'(0) \approx Q(2h) + 2Kh, \tag{30}$$

Subtraction of these two equations and using the values computed earlier, gives the following estimate of the error

$$Kh \approx Q(h) - Q(2h) = 0.252 - 0.5 = -0.248. \tag{31}$$

(Not asked for: This estimate can be used to update the originally computed approximation:

$$y'(0) = Q(h) + Kh = 0.25 - 0.248 = 0.002. \tag{32}$$

It is possible to show that the discrepance with zero follows from the influence of rounding.)