DELFT UNIVERSITY OF TECHNOLOGY
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TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU)
Thursday April 17 2014, 18:30-21:30

1. We consider the general initial value problem

$$y' = f(t, y), \qquad y(0) = y_0, \tag{1}$$

which we solve with the use of the backward Euler time integration method.

$$w_{n+1} = w_n + hf(t_{n+1}, w_{n+1}). \tag{2}$$

a Use the test equation, to demonstrate that the local truncation error of the backward Euler method is order $O(h)$. *Hint:*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, \qquad \frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots. \tag{3}$$

(3pt.)

b Use the test equation, to show that for general complex $\lambda = \mu + i\nu$, the numerical solution is stable if

$$(1 - h\mu)^2 + (h\nu)^2 \geq 1. \tag{4}$$

Sketch the stability region in the complex plane. (2pt.)

We apply the backward Euler method to the following equations

$$\begin{aligned} y_1' &= y_1(1 - (y_1 + 2y_2)), \\ y_2' &= y_2(1 - (y_1 + y_2)), \end{aligned} \tag{5}$$

subject to initial conditions, which will be specified later.

c Derive the Jacobi matrix from linearization of system (5) around $(y_1, y_2) = (0, 1)$, and give its eigenvalues. (1pt.)

d
- Determine the maximum allowable time step around $(y_1, y_2) = (0, 1)$ that warrants linear stability when for backward Euler. (1.5pt.)
- Do the same for the case in which one uses the forward Euler time integration method. (1.5pt.)

We use the forward Euler method to approximate the solution.

e Use the initial condition $(y_1(0), y_2(0)) = (0.25, 0.5)$ and time-step $h = 1$ to compute the numerical solution after one time-step. (1pt.)

[0]**please turn over, For the answers of this test we refer to:**
http://ta.twi.tudelft.nl/nw/users/vuik/wi3097/tentamen.html

2. (a) We are looking for a formula of the form:

$$Q(h) = \frac{\alpha_0}{h^2} f(0) + \frac{\alpha_{-1}}{h^2} f(-h) + \frac{\alpha_{-2}}{h^2} f(-2h),$$

such that

$$f''(0) - Q(h) = O(h).$$

Give the system of linear equations that has to be satisfied by the coefficients α_0, α_{-1} and α_{-2}. (2 pt)

(b) The solution of this system is given by $\alpha_0 = 1$, $\alpha_{-1} = -2$ and $\alpha_{-2} = 1$. Give, using these values, an expression for the truncation error $f''(0) - Q(h)$. (2 pt)

(c) Use the numbers given in Table 1. Using the Richardson method, give an

x	$f(x)$
0	0
$-\frac{1}{4}$	0.0156
$-\frac{1}{2}$	0.1250
$-\frac{3}{4}$	0.4219
-1	1.0000

Table 1: The numbers

estimate of the error: $f''(0) - Q(\frac{1}{4})$. (2 pt)

(d) The values given in the table contain rounding errors, so $|f(x) - \hat{f}(x)| \leq \epsilon$. Show that the rounding errors in the approximation satisfy the following inequality:

$$|Q(h) - \hat{Q}(h)| \leq \frac{C_1 \epsilon}{h^2}$$

and give C_1 and ϵ. (2 pt)

(e) If $f''(0) - Q(h) = 6h$, give the optimal value of h such that the total error $|f''(0) - \hat{Q}(h)|$ is minimal. (2 pt)