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ANSWERS OF THE TEST NUMERICAL METHODS FOR DIFFERENTIAL EQUATIONS (WI3097 TU CTB2400)
Thursday August 13th 2015, 18:30-21:30

1. (a) The amplification factor can be derived as follows. Consider the test equation $y' = \lambda y$. Application of the trapezoidal rule to this equation gives:

$$w_{j+1} = w_j + \frac{\Delta t}{2}\left(\lambda w_j + \lambda w_{j+1}\right) \tag{1}$$

Rearranging of w_{j+1} and w_j in (1) yields

$$\left(1 - \frac{\Delta t}{2}\lambda\right) w_{j+1} = \left(1 + \frac{\Delta t}{2}\lambda\right) w_j.$$

It now follows that

$$w_{j+1} = \frac{1 + \frac{\Delta t}{2}\lambda}{1 - \frac{\Delta t}{2}\lambda} w_j,$$

and thus

$$Q(\Delta t\lambda) = \frac{1 + \frac{\Delta t}{2}\lambda}{1 - \frac{\Delta t}{2}\lambda}.$$

(b) The definition of the local truncation error is

$$\tau_{j+1} = \frac{y_{j+1} - Q(\Delta t\lambda) y_j}{\Delta t}.$$

The exact solution of the test equation is given by

$$y_{j+1} = e^{\Delta t\lambda} y_j.$$

Combination of these results shows that the local truncation error of the test equation is determined by the difference between the exponential function and the amplification factor $Q(\Delta t\lambda)$

$$\tau_{j+1} = \frac{e^{\Delta t\lambda} - Q(\Delta t\lambda)}{\Delta t} y_j. \tag{2}$$

The difference between the exponential function and amplification factor can be computed as follows. The Taylor series of $e^{\Delta t\lambda}$ with known point 0 is:

$$e^{\Delta t\lambda} = 1 + \lambda\Delta t + \frac{(\lambda\Delta t)^2}{2} + \mathcal{O}(\Delta t^3). \tag{3}$$

The Taylor series of $\frac{1}{1-\frac{\Delta t}{2}\lambda}$ with known point 0 is:

$$\frac{1}{1-\frac{\Delta t}{2}\lambda} = 1 + \frac{1}{2}\Delta t\lambda + \frac{1}{4}\Delta t^2\lambda^2 + \mathcal{O}(\Delta t^3). \tag{4}$$

With (4) it follows that $\frac{1+\frac{\Delta t}{2}\lambda}{1-\frac{\Delta t}{2}\lambda}$ is equal to

$$\frac{1+\frac{\Delta t}{2}\lambda}{1-\frac{\Delta t}{2}\lambda} = 1 + \Delta t\lambda + \frac{1}{2}(\Delta t\lambda)^2 + \mathcal{O}(\Delta t^3). \tag{5}$$

In order to determine $e^{\Delta t\lambda} - Q(\Delta t\lambda)$, we subtract (5) from (3). Now it follows that

$$e^{\Delta t\lambda} - Q(\Delta t\lambda) = \mathcal{O}(\Delta t^3). \tag{6}$$

The local truncation error can be found by substituting (6) into (2), which leads to

$$\tau_{j+1} = \mathcal{O}(\Delta t^2).$$

(c) Application of the trapezoidal rule to

$$y' = -2y + e^t, \text{ with } y(0) = 2,$$

and step size $\Delta t = 1$ gives:

$$w_1 = w_0 + \frac{\Delta t}{2}[-2w_0 + e^0 - 2w_1 + e].$$

Using the initial value $w_0 = y(0) = 2$ and step size $\Delta t = 1$ gives:

$$w_1 = 2 + \frac{1}{2}[-4 - 2w_1 + 1 + e].$$

This leads to

$$2w_1 = 2 + \frac{-3+e}{2} = \frac{1}{2} + \frac{e}{2}, \text{ so } w_1 = \frac{1}{4} + \frac{e}{4}.$$

(d) We use the following definition $x_1 = y$ and $x_2 = y'$. This implies that $x_1' = y' = x_2$ and $x_2' = y'' = -y' - \frac{1}{2}y = -x_2 - \frac{1}{2}x_1$. Writing this in vector notation shows that

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

so $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}$. To compute the eigenvalues we look for values of λ such that

$$|\mathbf{A} - \lambda\mathbf{I}| = 0.$$

This implies that λ is a solution of

$$\lambda^2 + \lambda + \frac{1}{2} = 0,$$

which leads to the roots:

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}i \text{ and } \lambda_2 = -\frac{1}{2} - \frac{1}{2}i.$$

(e) To investigate the stability it is sufficient that

$$|Q(\Delta t \lambda_1)| \leq 1 \text{ and } |Q(\Delta t \lambda_2)| \leq 1.$$

Since λ_1 and λ_2 are complex valued, it is sufficient to check only the first inequality. This leads to

$$\left| \frac{1 + \frac{\Delta t(-\frac{1}{2}+\frac{1}{2}i)}{2}}{1 - \frac{\Delta t(-\frac{1}{2}+\frac{1}{2}i)}{2}} \right| \leq 1,$$

which is equivalent to

$$\frac{|1 - \frac{\Delta t}{4} + \frac{\Delta t i}{4}|}{|1 + \frac{\Delta t}{4} - \frac{\Delta t i}{4}|} \leq 1.$$

Using the definition of the absolute value we arrive at the inequality

$$\frac{\sqrt{(1 - \frac{\Delta t}{4})^2 + (\frac{\Delta t}{4})^2}}{\sqrt{(1 + \frac{\Delta t}{4})^2 + (\frac{\delta t}{4})^2}} \leq 1.$$

This equality is valid for all values of Δt because

$$\sqrt{(1 - \frac{\Delta t}{4})^2 + (\frac{\Delta t}{4})^2} \leq \sqrt{(1 + \frac{\Delta t}{4})^2 + (\frac{\Delta t}{4})^2},$$

for all $\Delta t > 0$.

2. (a) The linear Lagrangian interpolatory polynomial, with nodes x_0 and x_1, is given by

$$p_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1). \tag{7}$$

This is evident from application of the given formula.

(b) The quadratic Lagrangian interpolatory polynomial with nodes x_0, x_1 and x_2 is given by

$$p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2). \tag{8}$$

This is also evident from application of the given formula.

(c) Obviously, $p_1(0) = 2$ and $p_2(0) = 2$ since the Lagrange interpolation polynomial satisfies $p_n(x_i) = f(x_i)$ for all points $x_0, x_1, \ldots, x_n$. Next, we compute $p_1(0.5)$ and $p_2(0.5)$ for both linear and quadratic Lagrangian interpolation as approximations at $x = 0.5$. For linear interpolation, we have

$$p_1(0.5) = \frac{0.5 - 0}{-1 - 0} \cdot 3 + \frac{0.5 + 1}{0 + 1} \cdot 2 = \frac{3}{2}, \tag{9}$$

and for quadratic interpolation, one obtains

$$p_2(0.5) = \frac{(0.5-0)(0.5-1)}{(-1)\cdot(-2)} \cdot 3 + \frac{(0.5+1)(0.5-1)}{1 \cdot (-1)} \cdot 2 + \frac{(0.5+1)(0.5-0)}{2 \cdot 1} \cdot 5 = 3. \tag{10}$$

(d) The method of Newton-Raphson is based on linearization around the iterate p_n. This is given by

$$L(x) = f(p_n) + (x - p_n) f'(p_n). \tag{11}$$

Next, we determine p_{n+1} such that $L(p_{n+1}) = 0$, that is

$$f(p_n) + (p_{n+1} - p_n) f'(p_n) = 0 \Leftrightarrow p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}, \qquad f'(p_n) \neq 0. \tag{12}$$

This result can also be proved graphically, see book, chapter 4.

(e) We have $f(x) = e^x - x^3$, so $f'(x) = e^x - 3x^2$ and hence

$$p_{n+1} = p_n - \frac{e^{p_n} - p_n^3}{e^{p_n} - 3p_n^2}.$$

With the initial value $p_0 = 3$, this gives

$$p_1 = 3 - \frac{e^3 - 3^3}{e^3 - 3 \cdot 3^2} = 2$$

and consequently

$$p_2 = 2 - \frac{e^2 - 2^3}{e^2 - 3 \cdot 2^2} = \frac{e^2 - 16}{e^2 - 12} \approx 1.8675$$

(f) We consider a Taylor polynomial around p_n, to express p

$$0 = f(p) = f(p_n) + (p - p_n) f'(p_n) + \frac{(p - p_n)^2}{2} f''(\xi_n), \tag{13}$$

for some ξ_n between p and p_n. Note that this form gives the exact representation. Subsequently, we consider the Newton-Raphson approximation

$$0 = L(p_{n+1}) = f(p_n) + (p_{n+1} - p_n) f'(p_n). \tag{14}$$

Subtraction of these two above equations gives

$$p_{n+1} - p = \frac{(p_n - p)^2}{2} \frac{f''(\xi_n)}{f'(p_n)}, \text{ provided that } f'(p_n) \neq 0, \tag{15}$$

and hence

$$|p_{n+1} - p| = \frac{(p_n - p)^2}{2} \left|\frac{f''(\xi_n)}{f'(p_n)}\right|, \text{ provided that } f'(p_n) \neq 0, \tag{16}$$

Using $p_n \to p$, $\xi_n \to p$ as $n \to \infty$ and continuity of $f(x)$ up to at least the second derivative, we arrive at $K = |\frac{f''(p)}{f'(p)}|$. □